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Core Mathematics 1

Edexcel AS and A-level Modular Mathematics

Greg Attwood
Alistair Macpherson
Bronwen Moran
Joe Petran
Keith Pledger
Geoff Staley
Dave Wilkins

Contents

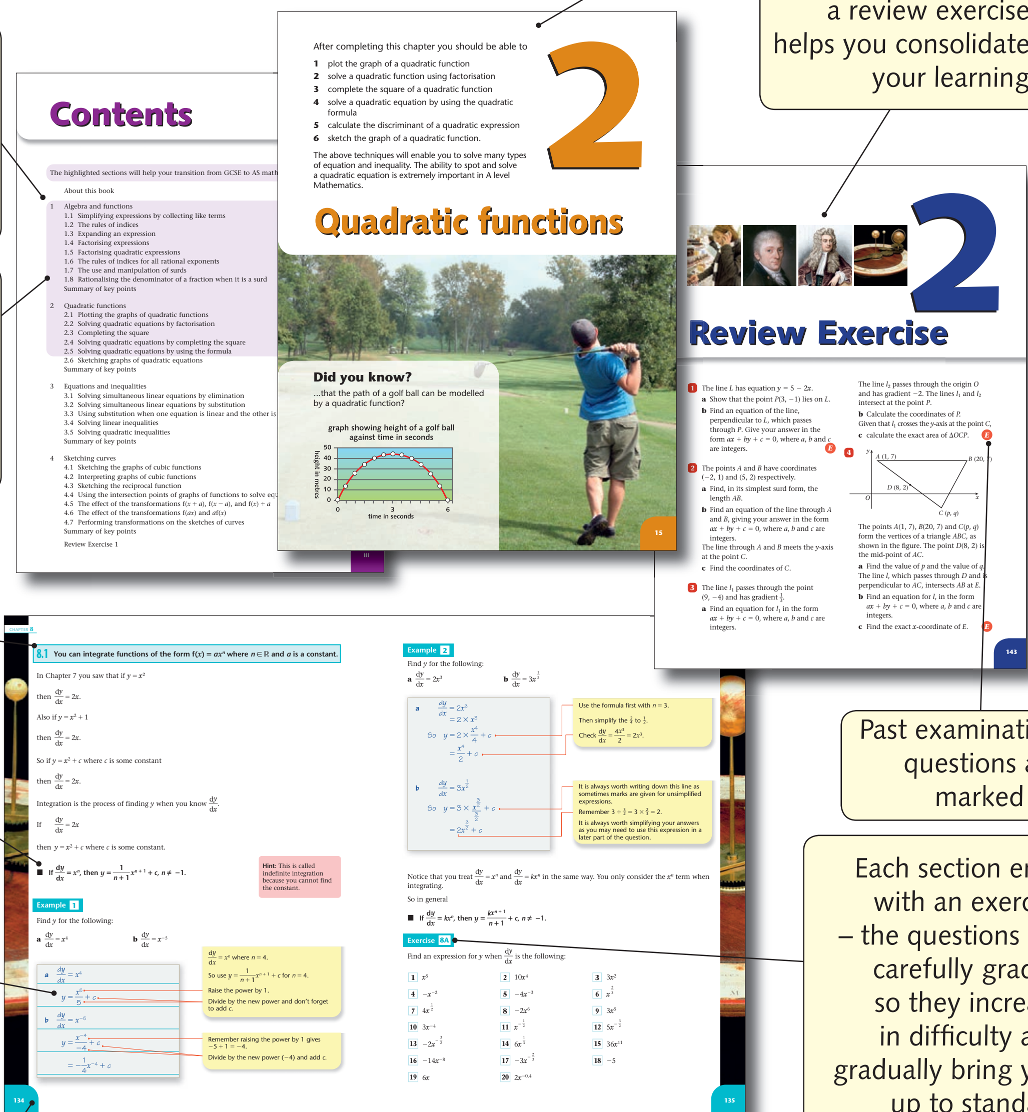
The highlighted sections will help your transition from GCSE to AS mathematics.

About this book

This book is designed to provide you with the best preparation possible for your Edexcel C1 unit examination:

- This is Edexcel's own course for the GCE specification.
- Written by a senior examining team at Edexcel: the chair of examiners, chief examiners and principal examiners.
- The LiveText CD-ROM in the back of the book contains even more resources to support you through the unit.
- A matching C1 revision guide is also available.

Finding your way around the book



Detailed contents list shows which parts of the C1 specification are covered in each section

Chapter 1 and sections 2.1 to 2.5 provide excellent transition material from your GCSE mathematics

Brief chapter overview and 'links' to underline the importance of mathematics: to the real world, to your study of further units and to your career

Every few chapters, a review exercise helps you consolidate your learning

Past examination questions are marked 'E'

Each section begins with a statement of what is covered in the section

Concise learning points

Step-by-step worked examples – they are model solutions and include examiners hints

Each section ends with an exercise – the questions are carefully graded so they increase in difficulty and gradually bring you up to standard

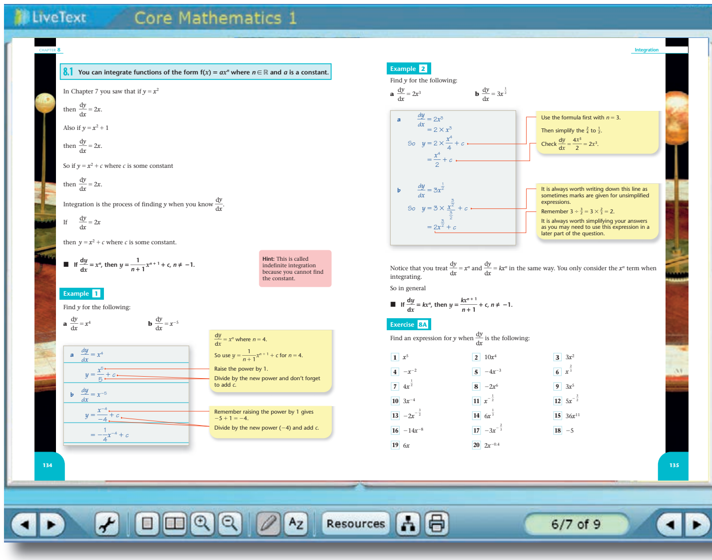
Each chapter has a different colour scheme, to help you find the right chapter quickly

Each chapter ends with a mixed exercise and a summary of key points.

At the end of the book are two exam papers: a practice paper and a full examination-style paper.

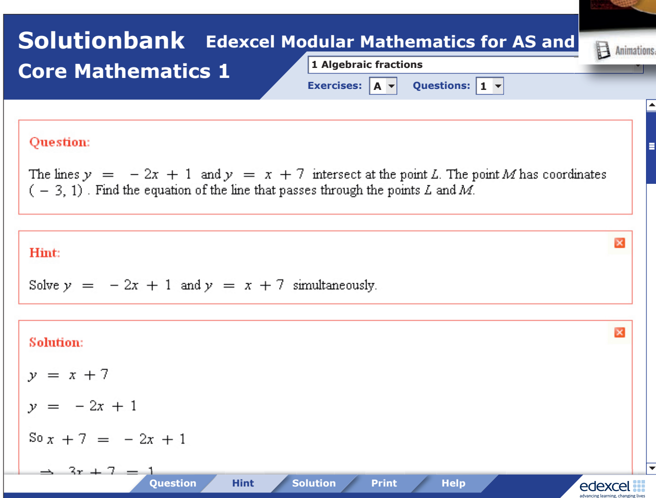
LiveText software

The LiveText software gives you additional resources: Solutionbank and Exam café. Simply turn the pages of the electronic book to the page you need, and explore!



Unique Exam café feature:

- Relax and prepare – revision planner; hints and tips; common mistakes
- Refresh your memory – revision checklist; language of the examination; glossary
- Get the result! – fully worked examination-style paper with chief examiner's commentary



Solutionbank

- Hints and solutions to every question in the textbook
- Solutions and commentary for all review exercises and the practice examination paper

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After completing this chapter you should be able to

1 simplify expressions and collect like terms

2 apply the rules of indices

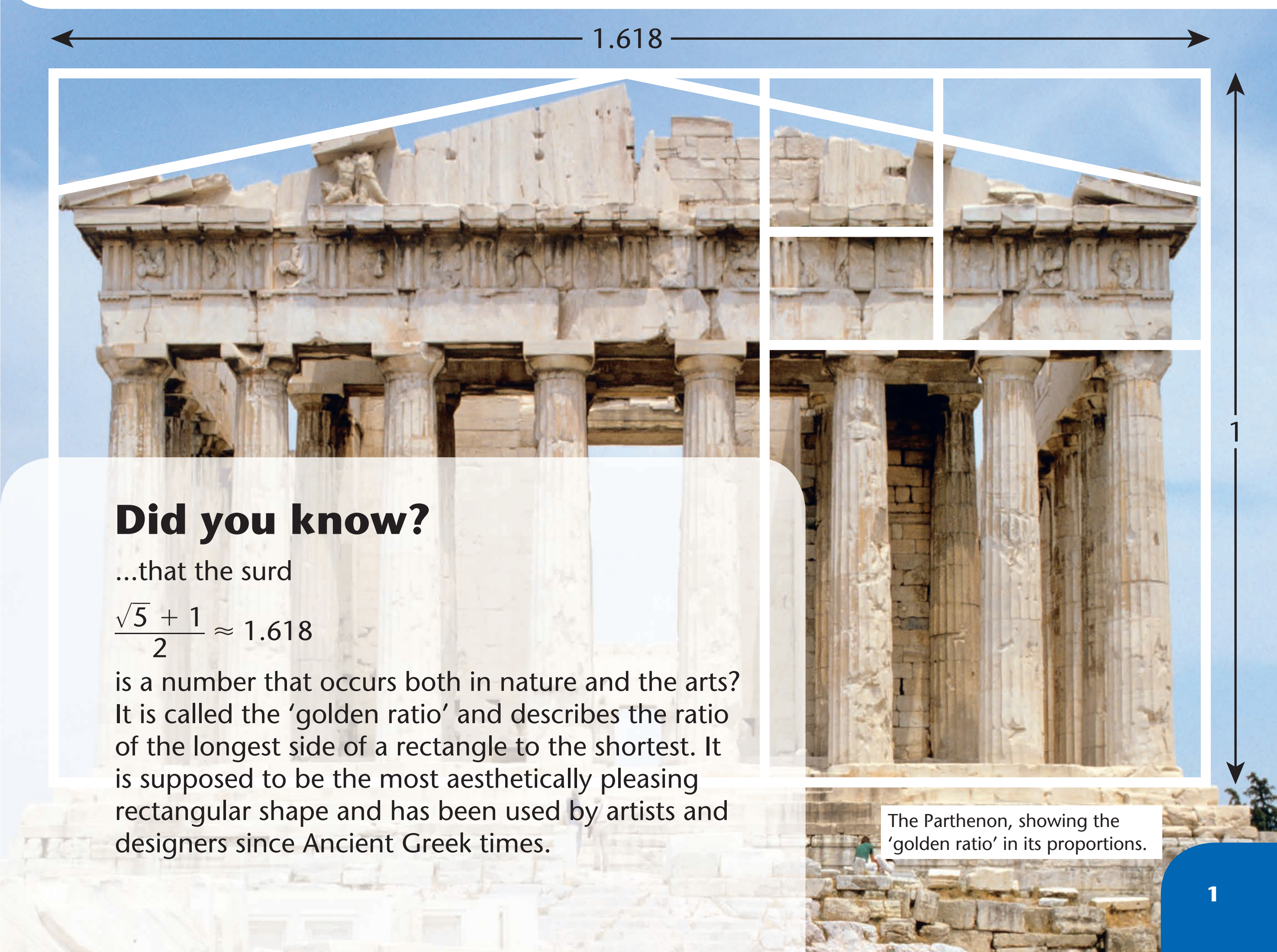
3 multiply out brackets

4 factorise expressions including quadratics

5 manipulate surds.

This chapter provides the foundations for many aspects of A level Mathematics. Factorising expressions will enable you to solve equations; it could help sketch the graph of a function. A knowledge of indices is very important when differentiating and integrating. Surds are an important way of giving exact answers to problems and you will meet them again when solving quadratic equations.

1 Algebra and functions



The Parthenon, showing the 'golden ratio' in its proportions.

Did you know?

...that the surd

$$\frac{\sqrt{5}+1}{2} \approx 1.618$$

is a number that occurs both in nature and the arts? It is called the 'golden ratio' and describes the ratio of the longest side of a rectangle to the shortest. It is supposed to be the most aesthetically pleasing rectangular shape and has been used by artists and designers since Ancient Greek times.

1.1 You can simplify expressions by collecting like terms.

Example 1

Simplify these expressions:

a $3x + 2xy + 7 - x + 3xy - 9$ **b** $3x^2 - 6x + 4 - 2x^2 + 3x - 3$

c $3(a + b^2) - 2(3a - 4b^2)$

a $3x + 2xy + 7 - x + 3xy - 9$

$= 3x - x + 2xy + 3xy + 7 - 9$

$= 2x + 5xy - 2$

Rewrite the expression with the like terms next to each other.
$+7 - 9 = -2$

b $3x^2 - 6x + 4 - 2x^2 + 3x - 3$

$= 3x^2 - 2x^2 - 6x + 3x + 4 - 3$

$= x^2 - 3x + 1$

$3x^2$ and $3x$ are not like terms:
$3x^2 = 3 \times x \times x$ $3x = 3 \times x$
$1x^2$ is written as x^2.

c $3(a + b^2) - 2(3a - 4b^2)$

$= 3a + 3b^2 - 6a + 8b^2$

$= -3a + 11b^2$

Multiply the term outside the bracket by both terms inside the bracket:
$-2 \times 3a = -6a$
$-2 \times -4b^2 = 8b^2$
So $-2(3a - 4b^2) = -6a + 8b^2$

Exercise 1A

Simplify these expressions:

1 $4x - 5y + 3x + 6y$

2 $3r + 7t - 5r + 3t$

3 $3m - 2n - p + 5m + 3n - 6p$

4 $3ab - 3ac + 3a - 7ab + 5ac$

5 $7x^2 - 2x^2 + 5x^2 - 4x^2$

6 $4m^2n + 5mn^2 - 2m^2n + mn^2 - 3mn^2$

7 $5x^2 + 4x + 1 - 3x^2 + 2x + 7$

8 $6x^2 + 5x - 12 + 3x^2 - 7x + 11$

9 $3x^2 - 5x + 2 + 3x^2 - 7x - 12$

10 $4c^2d + 5cd^2 - c^2d + 3cd^2 + 7c^2d$

11 $2x^2 + 3x + 1 + 2(3x^2 + 6)$

12 $4(a + a^2b) - 3(2a + a^2b)$

13 $2(3x^2 + 4x + 5) - 3(x^2 - 2x - 3)$

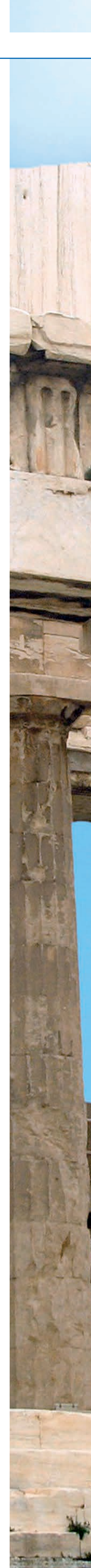
14 $7(1 - x^2) + 3(2 - 3x + 5x^2)$

15 $4(a + b + 3c) - 3a + 2c$

16 $4(c + 3d^2) - 3(2c + d^2)$

17 $5 - 3(x^2 + 2x - 5) + 3x^2$

18 $(r^2 + 3t^2 + 9) - (2r^2 + 3t^2 - 4)$



1.2 You can simplify expressions and functions by using rules of indices (powers).

- $a^m \times a^n = a^{m+n}$
  $a^m \div a^n = a^{m-n}$
  $(a^m)^n = a^{mn}$
  $a^{-m} = \dfrac{1}{a^m}$
  $a^{\frac{1}{m}} = \sqrt[m]{a}$ — The mth root of a.
  $a^{\frac{n}{m}} = \sqrt[m]{a^n}$

Example 2

Simplify these expressions:

a $x^2 \times x^5$ **b** $2r^2 \times 3r^3$ **c** $b^4 \div b^4$
d $6x^{-3} \div 3x^{-5}$ **e** $(a^3)^2 \times 2a^2$ **f** $(3x^2)^3 \div x^4$

a $x^2 \times x^5$
$= x^{2+5}$ — Use the rule $a^m \times a^n = a^{m+n}$ to simplify the index.
$= x^7$

b $2r^2 \times 3r^3$
$= 2 \times 3 \times r^2 \times r^3$
$= 6 \times r^{2+3}$ — Rewrite the expression with the numbers together and the r terms together. $2 \times 3 = 6$, $r^2 \times r^3 = r^{2+3}$
$= 6r^5$

c $b^4 \div b^4$ — Use the rule $a^m \div a^n = a^{m-n}$
$= b^{4-4}$
$= b^0 = 1$ — Any term raised to the power of zero = 1.

d $6x^{-3} \div 3x^{-5}$
$= 6 \div 3 \times x^{-3} \div x^{-5}$ — $x^{-3} \div x^{-5} = x^{-3--5} = x^2$
$= 2 \times x^2$
$= 2x^2$

e $(a^3)^2 \times 2a^2$
$= a^6 \times 2a^2$ — Use the rule $(a^m)^n = a^{mn}$ to simplify the index. $a^6 \times 2a^2 = 1 \times 2 \times a^6 \times a^2 = 2 \times a^{6+2}$
$= 2 \times a^6 \times a^2$
$= 2 \times a^{6+2}$
$= 2a^8$

f $(3x^2)^3 \div x^4$ — Use the rule $(a^m)^n = a^{mn}$ to simplify the index.
$= 27x^6 \div x^4$
$= 27 \div 1 \times x^6 \div x^4$
$= 27 \times x^{6-4}$
$= 27x^2$

Exercise 1B

Simplify these expressions:

1 $x^3 \times x^4$

2 $2x^3 \times 3x^2$

3 $4p^3 \div 2p$

4 $3x^{-4} \div x^{-2}$

5 $k^3 \div k^{-2}$

6 $(y^2)^5$

7 $10x^5 \div 2x^{-3}$

8 $(p^3)^2 \div p^4$

9 $(2a^3)^2 \div 2a^3$

10 $8p^{-4} \div 4p^3$

11 $2a^{-4} \times 3a^{-5}$

12 $21a^3b^2 \div 7ab^4$

13 $9x^2 \times 3(x^2)^3$

14 $3x^3 \times 2x^2 \times 4x^6$

15 $7a^4 \times (3a^4)^2$

16 $(4y^3)^3 \div 2y^3$

17 $2a^3 \div 3a^2 \times 6a^5$

18 $3a^4 \times 2a^5 \times a^3$

1.3 You can expand an expression by multiplying each term inside the bracket by the term outside.

Example 3

Expand these expressions, simplify if possible:

a $5(2x + 3)$

b $-3x(7x - 4)$

c $y^2(3 - 2y^3)$

d $4x(3x - 2x^2 + 5x^3)$

e $2x(5x + 3) - 5(2x + 3)$

Hint: A − sign outside a bracket changes the sign of every term inside the brackets.

a $5(2x + 3)$

$= 10x + 15$

$5 \times 2x + 5 \times 3$

b $-3x(7x - 4)$

$= -21x^2 + 12x$

$-3x \times 7x = -21x^{1+1} = -21x^2$
$-3x \times -4 = +12x$

c $y^2(3 - 2y^3)$

$= 3y^2 - 2y^5$

$y^2 \times -2y^3 = -2y^{2+3} = -2y^5$

d $4x(3x - 2x^2 + 5x^3)$

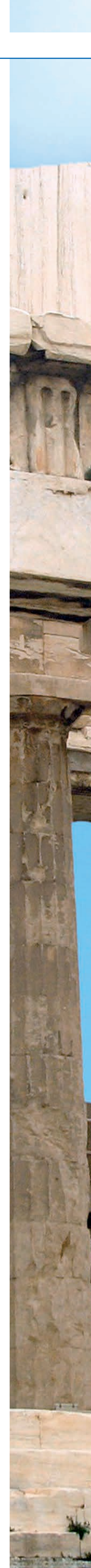
$= 12x^2 - 8x^3 + 20x^4$

e $2x(5x + 3) - 5(2x + 3)$

$= 10x^2 + 6x - 10x - 15$

$= 10x^2 - 4x - 15$

Remember a minus sign outside the brackets changes the signs within the brackets. Simplify $6x - 10x$ to give $-4x$.



Exercise 1C

Expand and simplify if possible:

1	$9(x - 2)$	**2**	$x(x + 9)$
3	$-3y(4 - 3y)$	**4**	$x(y + 5)$
5	$-x(3x + 5)$	**6**	$-5x(4x + 1)$
7	$(4x + 5)x$	**8**	$-3y(5 - 2y^2)$
9	$-2x(5x - 4)$	**10**	$(3x - 5)x^2$
11	$3(x + 2) + (x - 7)$	**12**	$5x - 6 - (3x - 2)$
13	$x(3x^2 - 2x + 5)$	**14**	$7y^2(2 - 5y + 3y^2)$
15	$-2y^2(5 - 7y + 3y^2)$	**16**	$7(x - 2) + 3(x + 4) - 6(x - 2)$
17	$5x - 3(4 - 2x) + 6$	**18**	$3x^2 - x(3 - 4x) + 7$
19	$4x(x + 3) - 2x(3x - 7)$	**20**	$3x^2(2x + 1) - 5x^2(3x - 4)$

1.4 You can factorise expressions.

■ **Factorising is the opposite of expanding expressions.**

When you have completely factorised an expression, the terms inside do not have a common factor.

Example 4

Factorise these expressions completely:

a $3x + 9$ **b** $x^2 - 5x$ **c** $8x^2 + 20x$
d $9x^2y + 15xy^2$ **e** $3x^2 - 9xy$

a $3x + 9$
$= 3(x + 3)$

3 is a common factor of $3x$ and 9.

b $x^2 - 5x$
$= x(x - 5)$

x is a common factor of x^2 and $-5x$.

c $8x^2 + 20x$
$= 4x(2x + 5)$

4 and x are common factors of $8x^2$ and $20x$. So take $4x$ outside the bracket.

d $9x^2y + 15xy^2$
$= 3xy(3x + 5y)$

3, x and y are common factors of $9x^2y$ and $15xy^2$. So take $3xy$ outside the bracket.

e $3x^2 - 9xy$
$= 3x(x - 3y)$

Exercise 1D

Factorise these expressions completely:

1	$4x + 8$	**2**	$6x - 24$
3	$20x + 15$	**4**	$2x^2 + 4$
5	$4x^2 + 20$	**6**	$6x^2 - 18x$
7	$x^2 - 7x$	**8**	$2x^2 + 4x$
9	$3x^2 - x$	**10**	$6x^2 - 2x$
11	$10y^2 - 5y$	**12**	$35x^2 - 28x$
13	$x^2 + 2x$	**14**	$3y^2 + 2y$
15	$4x^2 + 12x$	**16**	$5y^2 - 20y$
17	$9xy^2 + 12x^2y$	**18**	$6ab - 2ab^2$
19	$5x^2 - 25xy$	**20**	$12x^2y + 8xy^2$
21	$15y - 20yz^2$	**22**	$12x^2 - 30$
23	$xy^2 - x^2y$	**24**	$12y^2 - 4yx$

1.5 You can factorise quadratic expressions.

■ **A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$.**

Example 5

Factorise:

a $6x^2 + 9x$ **b** $x^2 - 5x - 6$

c $x^2 + 6x + 8$ **d** $6x^2 - 11x - 10$

e $x^2 - 25$ **f** $4x^2 - 9y^2$

a $6x^2 + 9x$

$= 3x(2x + 3)$

3 and x are common factors of $6x^2$ and $9x$. So take $3x$ outside the bracket.

b $x^2 - 5x - 6$

$ac = -6$

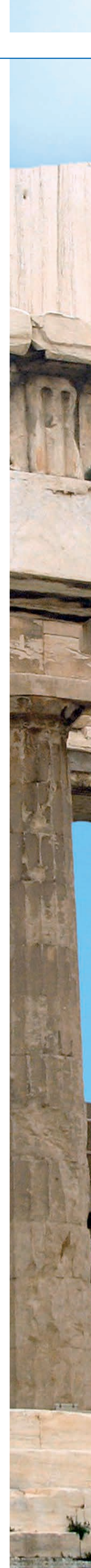
So $x^2 - 5x + 6 = x^2 + x - 6x - 6$

$= x(x + 1) - 6(x + 1)$

$= (x + 1)(x - 6)$

Here $a = 1$, $b = -5$ and $c = -6$.
You need to find two brackets that multiply together to give $x^2 - 5x - 6$. So:

1. Work out ac.
2. Work out the two factors of ac which add that give you b.
   -6 and $+1 = -5$
3. Rewrite the bx term using these two factors.
4. Factorise first two terms and last two terms.
5. $x + 1$ is a factor of both terms, so take that outside the bracket. This is now completely factorised.



c $x^2 + 6x + 8$

$= x^2 + 2x + 4x + 8$ — Since $ac = 8$ and $2 + 4 = 6 = b$, factorise.

$= x(x + 2) + 4(x + 2)$ — $x + 2$ is a factor so you can factorise again.

$= (x + 2)(x + 4)$

d $6x^2 - 11x - 10$

$= 6x^2 - 15x + 4x - 10$ — $ac = -60$ and $4 - 15 = -11 = b$.

$= 3x(2x - 5) + 2(2x - 5)$ — Factorise.

$= (2x - 5)(3x + 2)$ — Factorise $(2x - 5)$.

e $x^2 - 25$ — This is called the difference of two squares as the two terms are x^2 and 5^2.

$= x^2 - 5^2$

$= (x + 5)(x - 5)$ — The two x terms, $5x$ and $-5x$, cancel each other out.

f $4x^2 - 9y^2$ — This is the same as $(2x)^2 - (3y)^2$.

$= 2^2x^2 - 3^2y^2$

$= (2x + 3y)(2x - 3y)$

■ $x^2 - y^2 = (x + y)(x - y)$
This is called the difference of two squares.

Exercise 1E

Factorise:

1. $x^2 + 4x$
2. $2x^2 + 6x$
3. $x^2 + 11x + 24$
4. $x^2 + 8x + 12$
5. $x^2 + 3x - 40$
6. $x^2 - 8x + 12$
7. $x^2 + 5x + 6$
8. $x^2 - 2x - 24$
9. $x^2 - 3x - 10$
10. $x^2 + x - 20$
11. $2x^2 + 5x + 2$
12. $3x^2 + 10x - 8$
13. $5x^2 - 16x + 3$
14. $6x^2 - 8x - 8$
15. $2x^2 + 7x - 15$
16. $2x^4 + 14x^2 + 24$
17. $x^2 - 4$
18. $x^2 - 49$
19. $4x^2 - 25$
20. $9x^2 - 25y^2$
21. $36x^2 - 4$
22. $2x^2 - 50$
23. $6x^2 - 10x + 4$
24. $15x^2 + 42x - 9$

Hints:
Question 14 – Take 2 out as a common factor first.
Question 16 – let $y = x^2$.

1.6 You can extend the rules of indices to all rational exponents.

■ $a^m \times a^n = a^{m+n}$
$a^m \div a^n = a^{m-n}$
$(a^m)^n = a^{mn}$
$a^{\frac{1}{m}} = \sqrt[m]{a}$
$a^{\frac{n}{m}} = \sqrt[m]{a^n}$
$a^{-m} = \frac{1}{a^m}$
$a^0 = 1$

Hint: Rational numbers can be written as $\frac{a}{b}$ where a and b are both integers, e.g. -3.5, $1\frac{1}{4}$, 0.9, 7, $0.1\dot{3}$

Example 6

Simplify:

a $x^4 \div x^{-3}$

b $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$

c $(x^3)^{\frac{2}{3}}$

d $2x^{1.5} \div 4x^{-0.25}$

a $x^4 \div x^{-3}$
$= x^{4--3}$ — Use the rule $a^m \div a^n = a^{m-n}$. Remember $- + - = +$.
$= x^7$

b $x^{\frac{1}{2}} \times x^{\frac{3}{2}}$ — This could also be written as $\sqrt{x}$.
$= x^{\frac{1}{2}+\frac{3}{2}}$ — Use the rule $a^m \times a^n = a^{m+n}$.
$= x^2$

c $(x^3)^{\frac{2}{3}}$ — Use the rule $(a^m)^n = a^{mn}$.
$= x^{3\times\frac{2}{3}}$
$= x^2$

d $2x^{1.5} \div 4x^{-0.25}$
$= \frac{1}{2}x^{1.5--0.25}$ — Use the rule $a^m \div a^n = a^{m-n}$. $2 \div 4 = \frac{1}{2}$. $1.5 - -0.25 = 1.75$
$= \frac{1}{2}x^{1.75}$

Example 7

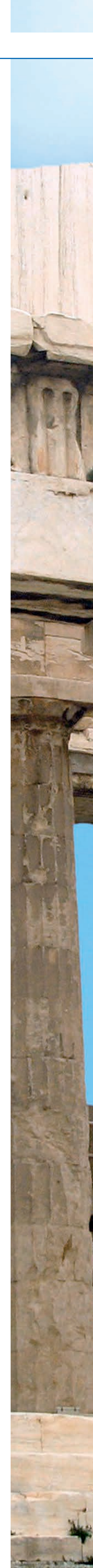
Evaluate:

a $9^{\frac{1}{2}}$

b $64^{\frac{1}{3}}$

c $49^{\frac{3}{2}}$

d $25^{-\frac{3}{2}}$



a $9^{\frac{1}{2}}$

$= \sqrt{9}$ — Using $a^{\frac{1}{m}} = \sqrt[m]{a}$.

$= \pm 3$ — When you take a square root, the answer can be positive or negative as $+ \times + = +$ and $- \times - = +$.

b $64^{\frac{1}{3}}$

$= \sqrt[3]{64}$ — This means the cube root of 64.

$= 4$ — As $4 \times 4 \times 4 = 64$.

c $49^{\frac{3}{2}}$

$= (\sqrt{49})^3$ — Using $a^{\frac{n}{m}} = \sqrt[m]{a^n}$. This means the square root of 49, cubed.

$= \pm 343$

d $25^{-\frac{3}{2}}$

$= \dfrac{1}{25^{\frac{3}{2}}}$ — Using $a^{-m} = \dfrac{1}{a^m}$.

$= \dfrac{1}{(\pm\sqrt{25})^3}$ — $\sqrt{25} = \pm 5$

$= \dfrac{1}{(\pm 5)^3}$

$= \pm\dfrac{1}{125}$

Exercise 1F

1 Simplify:

a $x^3 \div x^{-2}$ **b** $x^5 \div x^7$ **c** $x^{\frac{3}{2}} \times x^{\frac{5}{2}}$

d $(x^2)^{\frac{3}{2}}$ **e** $(x^3)^{\frac{5}{3}}$ **f** $3x^{0.5} \times 4x^{-0.5}$

g $9x^{\frac{2}{3}} \div 3x^{\frac{1}{6}}$ **h** $5x^{1\frac{2}{5}} \div x^{\frac{2}{5}}$ **i** $3x^4 \times 2x^{-5}$

2 Evaluate:

a $25^{\frac{1}{2}}$ **b** $81^{\frac{1}{2}}$ **c** $27^{\frac{1}{3}}$

d 4^{-2} **e** $9^{-\frac{1}{2}}$ **f** $(-5)^{-3}$

g $(\frac{3}{4})^0$ **h** $1296^{\frac{1}{4}}$ **i** $(1\frac{9}{16})^{\frac{3}{2}}$

j $(\frac{27}{8})^{\frac{2}{3}}$ **k** $(\frac{6}{5})^{-1}$ **l** $(\frac{343}{512})^{-\frac{2}{3}}$

1.7 You can write a number exactly using surds, e.g. $\sqrt{2}$, $\sqrt{3} - 5$, $\sqrt{19}$.
You cannot evaluate surds exactly because they give never-ending, non-repeating decimal fractions, e.g. $\sqrt{2} = 1.414\,213\,562\ldots$
The square root of a prime number is a surd.

■ You can manipulate surds using these rules:

$$\sqrt{(ab)} = \sqrt{a} \times \sqrt{b}$$

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

Example 8

Simplify:

a $\sqrt{12}$ **b** $\dfrac{\sqrt{20}}{2}$ **c** $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

a $\sqrt{12}$

$= \sqrt{(4 \times 3)}$

$= \sqrt{4} \times \sqrt{3}$

$= 2\sqrt{3}$

Use the rule $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$.
$\sqrt{4} = 2$

b $\dfrac{\sqrt{20}}{2}$

$= \dfrac{\sqrt{4} \times \sqrt{5}}{2}$

$= \dfrac{2 \times \sqrt{5}}{2}$

$= \sqrt{5}$

$\sqrt{20} = \sqrt{4} \times \sqrt{5}$

$\sqrt{4} = 2$

Cancel by 2.

c $5\sqrt{6} - 2\sqrt{24} + \sqrt{294}$

$= 5\sqrt{6} - 2\sqrt{6}\sqrt{4} + \sqrt{6} \times \sqrt{49}$

$= \sqrt{6}(5 - 2\sqrt{4} + \sqrt{49})$

$= \sqrt{6}(5 - 2 \times 2 + 7)$

$= \sqrt{6}(8)$

$= 8\sqrt{6}$

$\sqrt{6}$ is a common factor.
Work out the square roots $\sqrt{4}$ and $\sqrt{49}$.
$5 - 4 + 7 = 8$

Exercise 1G

Simplify:

1 $\sqrt{28}$

2 $\sqrt{72}$

3 $\sqrt{50}$

4 $\sqrt{32}$

5 $\sqrt{90}$

6 $\frac{\sqrt{12}}{2}$

7 $\frac{\sqrt{27}}{3}$

8 $\sqrt{20} + \sqrt{80}$

9 $\sqrt{200} + \sqrt{18} - \sqrt{72}$

10 $\sqrt{175} + \sqrt{63} + 2\sqrt{28}$

11 $1\sqrt{28} - 2\sqrt{63} + \sqrt{7}$

12 $\sqrt{80} - 2\sqrt{20} + 3\sqrt{45}$

13 $3\sqrt{80} - 2\sqrt{20} + 5\sqrt{45}$

14 $\frac{\sqrt{44}}{\sqrt{11}}$

15 $\sqrt{12} + 3\sqrt{48} + \sqrt{75}$

1.8 You rationalise the denominator of a fraction when it is a surd.

■ **The rules to rationalise surds are:**

- **Fractions in the form $\frac{1}{\sqrt{a}}$, multiply the top and bottom by $\sqrt{a}$.**
- **Fractions in the form $\frac{1}{a+\sqrt{b}}$, multiply the top and bottom by $a - \sqrt{b}$.**
- **Fractions in the form $\frac{1}{a-\sqrt{b}}$, multiply the top and bottom by $a + \sqrt{b}$.**

Example 9

Rationalise the denominator of:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

a $\frac{1}{\sqrt{3}}$

$= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$

$= \frac{\sqrt{3}}{3}$

Multiply the top and bottom by $\sqrt{3}$.
$\sqrt{3} \times \sqrt{3} = (\sqrt{3})^2 = 3$

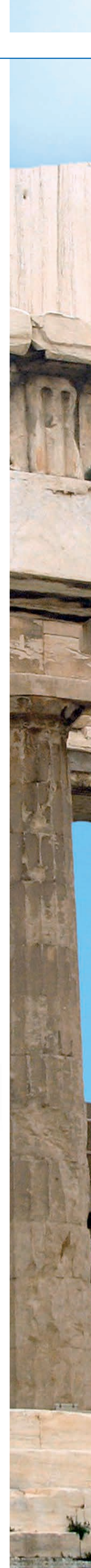
b $\frac{1}{3+\sqrt{2}}$

$= \frac{1 \times (3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}$

$= \frac{3-\sqrt{2}}{9-3\sqrt{2}+3\sqrt{2}-2}$

$= \frac{3-\sqrt{2}}{7}$

Multiply top and bottom by $(3 - \sqrt{2})$.
$\sqrt{2} \times \sqrt{2} = 2$
$9 - 2 = 7$, $-3\sqrt{2} + 3\sqrt{2} = 0$



c $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$

$= \dfrac{(\sqrt{5}+\sqrt{2})(\sqrt{5}+\sqrt{2})}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})}$

$= \dfrac{5+\sqrt{5}\sqrt{2}+\sqrt{2}\sqrt{5}+2}{5-2}$

$= \dfrac{7+2\sqrt{10}}{3}$

Multiply top and bottom by $\sqrt{5}+\sqrt{2}$.
$-\sqrt{2}\sqrt{5}$ and $\sqrt{5}\sqrt{2}$ cancel each other out.
$\sqrt{5}\sqrt{2}=\sqrt{10}$

Exercise 1H

Rationalise the denominators and simplify:

1 $\dfrac{1}{\sqrt{5}}$ **2** $\dfrac{1}{\sqrt{11}}$ **3** $\dfrac{1}{\sqrt{2}}$

4 $\dfrac{\sqrt{3}}{\sqrt{15}}$ **5** $\dfrac{\sqrt{12}}{\sqrt{48}}$ **6** $\dfrac{\sqrt{5}}{\sqrt{80}}$

7 $\dfrac{\sqrt{12}}{\sqrt{156}}$ **8** $\dfrac{\sqrt{7}}{\sqrt{63}}$ **9** $\dfrac{1}{1+\sqrt{3}}$

10 $\dfrac{1}{2+\sqrt{5}}$ **11** $\dfrac{1}{3-\sqrt{7}}$ **12** $\dfrac{4}{3-\sqrt{5}}$

13 $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **14** $\dfrac{3-\sqrt{2}}{4-\sqrt{5}}$ **15** $\dfrac{5}{2+\sqrt{5}}$

16 $\dfrac{5\sqrt{2}}{\sqrt{8}-\sqrt{7}}$ **17** $\dfrac{11}{3+\sqrt{11}}$ **18** $\dfrac{\sqrt{3}-\sqrt{7}}{\sqrt{3}+\sqrt{7}}$

19 $\dfrac{\sqrt{17}-\sqrt{11}}{\sqrt{17}+\sqrt{11}}$ **20** $\dfrac{\sqrt{41}+\sqrt{29}}{\sqrt{41}-\sqrt{29}}$ **21** $\dfrac{\sqrt{2}-\sqrt{3}}{\sqrt{3}-\sqrt{2}}$

Mixed exercise 1I

1 Simplify:

a $y^3 \times y^5$ **b** $3x^2 \times 2x^5$

c $(4x^2)^3 \div 2x^5$ **d** $4b^2 \times 3b^3 \times b^4$

2 Expand the brackets:

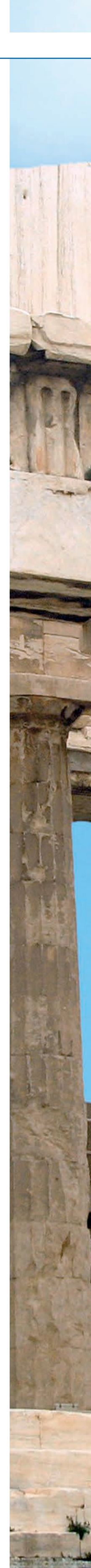
a $3(5y + 4)$ **b** $5x^2(3 - 5x + 2x^2)$

c $5x(2x + 3) - 2x(1 - 3x)$ **d** $3x^2(1 + 3x) - 2x(3x - 2)$

3 Factorise these expressions completely:

a $3x^2 + 4x$ **b** $4y^2 + 10y$

c $x^2 + xy + xy^2$ **d** $8xy^2 + 10x^2y$



4 Factorise:

a $x^2 + 3x + 2$

b $3x^2 + 6x$

c $x^2 - 2x - 35$

d $2x^2 - x - 3$

e $5x^2 - 13x - 6$

f $6 - 5x - x^2$

5 Simplify:

a $9x^3 \div 3x^{-3}$

b $(4^{\frac{3}{2}})^{\frac{1}{3}}$

c $3x^{-2} \times 2x^4$

d $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$

6 Evaluate:

a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$

b $\left(\frac{225}{289}\right)^{\frac{3}{2}}$

7 Simplify:

a $\frac{3}{\sqrt{63}}$

b $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$

8 Rationalise:

a $\frac{1}{\sqrt{3}}$

b $\frac{1}{\sqrt{2} - 1}$

c $\frac{3}{\sqrt{3} - 2}$

d $\frac{\sqrt{23} - \sqrt{37}}{\sqrt{23} + \sqrt{37}}$

Summary of key points

1 You can simplify expressions by collecting like terms.

2 You can simplify expressions by using rules of indices (powers).

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$a^{-m} = \dfrac{1}{a^m}$

$a^{\frac{1}{m}} = \sqrt[m]{a}$

$a^{\frac{n}{m}} = \sqrt[m]{a^n}$

$(a^m)^n = a^{mn}$

$a^0 = 1$

3 You can expand an expression by multiplying each term inside the bracket by the term outside.

4 Factorising expressions is the opposite of expanding expressions.

5 A quadratic expression has the form $ax^2 + bx + c$, where a, b, c are constants and $a \neq 0$.

6 $x^2 - y^2 = (x + y)(x - y)$
This is called a difference of squares.

7 You can write a number exactly using surds.

8 The square root of a prime number is a surd.

9 You can manipulate surds using the rules:

$\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

$\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

10 The rules to rationalise surds are:

- Fractions in the form $\dfrac{1}{\sqrt{a}}$, multiply the top and bottom by $\sqrt{a}$.
- Fractions in the form $\dfrac{1}{a + \sqrt{b}}$, multiply the top and bottom by $a - \sqrt{b}$.
- Fractions in the form $\dfrac{1}{a - \sqrt{b}}$, multiply the top and bottom by $a + \sqrt{b}$.

After completing this chapter you should be able to

1. plot the graph of a quadratic function
2. solve a quadratic function using factorisation
3. complete the square of a quadratic function
4. solve a quadratic equation by using the quadratic formula
5. calculate the discriminant of a quadratic expression
6. sketch the graph of a quadratic function.

The above techniques will enable you to solve many types of equation and inequality. The ability to spot and solve a quadratic equation is extremely important in A level Mathematics.

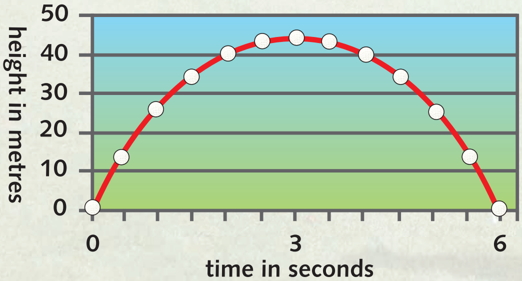
2

Quadratic functions

Did you know?

...that the path of a golf ball can be modelled by a quadratic function?

graph showing height of a golf ball against time in seconds



2.1 You need to be able to plot graphs of quadratic equations.

■ **The general form of a quadratic equation is**

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$.

This could also be written as $f(x) = ax^2 + bx + c$.

Example 1

a Draw the graph with equation $y = x^2 - 3x - 4$ for values of x from -2 to $+5$.

b Write down the minimum value of y and the value of x for this point.

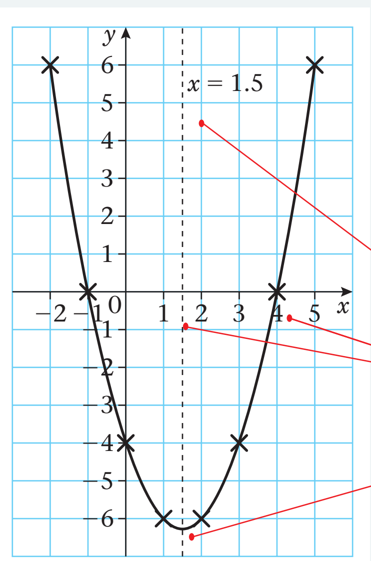
c Label the line of symmetry.

a

x	-2	-1	0	1	2	3	4	5
x^2	4	1	0	1	4	9	16	25
$-3x$	$+6$	$+3$	0	-3	-6	-9	-12	-15
-4	-4	-4	-4	-4	-4	-4	-4	-4
y	6	0	-4	-6	-6	-4	0	6

① First draw a table of values. Remember any number squared is positive.

② Look at the table to determine the extent of the y-axis. Use values of y from -6 to $+6$.



③ Plot the points and then join all the points together with a smooth curve.

The general shape of the curve is a ∪, it is called a parabola.

This is the line of symmetry. It is always half-way between the x-axis crossing points. It has equation $x = 1.5$.

This is the minimum.

b Minimum value is $y = -6.3$ when $x = 1.5$.

c See graph.

Exercise 2A

Draw graphs with the following equations, taking values of x from -4 to $+4$.

For each graph write down the equation of the line of symmetry.

1 $y = x^2 - 3$

2 $y = x^2 + 5$

3 $y = \frac{1}{2}x^2$

4 $y = -x^2$

5 $y = (x - 1)^2$

6 $y = x^2 + 3x + 2$

7 $y = 2x^2 + 3x - 5$

8 $y = x^2 + 2x - 6$

9 $y = (2x + 1)^2$

Hint: The general shape for question **4** is an upside down ∪-shape. i.e. ∩.

2.2 You can solve quadratic equations using factorisation.

Quadratic equations have two solutions or roots. (In some cases the two roots are equal.) To solve a quadratic equation, put it in the form $ax^2 + bx + c = 0$.

Example 2

Solve the equation $x^2 = 9x$

$x^2 = 9x$

$x^2 - 9x = 0$ — Rearrange in the form $ax^2 + bx + c = 0$.

$x(x - 9) = 0$ — Factorise by x (factorising is in Chapter 1). Then either part of the product could be zero.

Then either $x = 0$

or $x - 9 = 0 \Rightarrow x = 9$

So $x = 0$ or $x = 9$ are the two solutions of the equation $x^2 = 9x$.

A quadratic equation has two solutions (roots). In some cases the two roots are equal.

Example 3

Solve the equation $x^2 - 2x - 15 = 0$

$x^2 - 2x - 15 = 0$

$(x + 3)(x - 5) = 0$ — Factorise.

Then either $x + 3 = 0 \Rightarrow x = -3$

or $x - 5 = 0 \Rightarrow x = 5$

The solutions are $x = -3$ or $x = 5$.

Example 4

Solve the equation $6x^2 + 13x - 5 = 0$

$6x^2 + 13x - 5 = 0$

$(3x - 1)(2x + 5) = 0$

Then either $3x - 1 = 0 \Rightarrow x = \frac{1}{3}$

or $2x + 5 = 0 \Rightarrow x = -\frac{5}{2}$

The solutions are $x = \frac{1}{3}$ or $x = -\frac{5}{2}$.

Factorise.

The solutions can be fractions or any other type of number.

Example 5

Solve the equation $x^2 - 5x + 18 = 2 + 3x$

$x^2 - 5x + 18 = 2 + 3x$

$x^2 - 8x + 16 = 0$

$(x - 4)(x - 4) = 0$

Then either $x - 4 = 0 \Rightarrow x = 4$

or $x - 4 = 0 \Rightarrow x = 4$

$\Rightarrow x = 4$

Rearrange in the form $ax^2 + bx + c = 0$.

Factorise.

Here $x = 4$ is the only solution, i.e. the two roots are equal.

Example 6

Solve the equation $(2x - 3)^2 = 25$

$(2x - 3)^2 = 25$

$2x - 3 = \pm 5$

$2x = 3 \pm 5$

Then either $2x = 3 + 5 \Rightarrow x = 4$

or $2x = 3 - 5 \Rightarrow x = -1$

The solutions are $x = 4$ or $x = -1$.

This is a special case.

Take the square root of both sides.

Remember $\sqrt{25} = +5$ or -5.

Add 3 to both sides.

Example 7

Solve the equation $(x - 3)^2 = 7$

$(x - 3)^2 = 7$

$x - 3 = \pm\sqrt{7}$

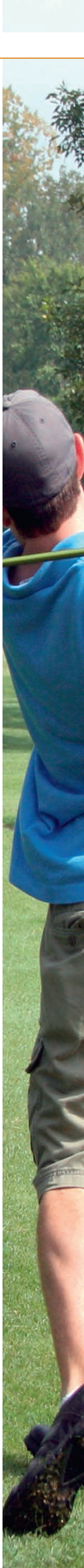
$x = +3 \pm \sqrt{7}$

Then either $x = 3 + \sqrt{7}$

or $x = 3 - \sqrt{7}$

The solutions are $x = 3 + \sqrt{7}$ or $x = 3 - \sqrt{7}$.

Square root. (If you do not have a calculator, leave this in surd form.)



Exercise 2B

Solve the following equations:

1 $x^2 = 4x$

2 $x^2 = 25x$

3 $3x^2 = 6x$

4 $5x^2 = 30x$

5 $x^2 + 3x + 2 = 0$

6 $x^2 + 5x + 4 = 0$

7 $x^2 + 7x + 10 = 0$

8 $x^2 - x - 6 = 0$

9 $x^2 - 8x + 15 = 0$

10 $x^2 - 9x + 20 = 0$

11 $x^2 - 5x - 6 = 0$

12 $x^2 - 4x - 12 = 0$

13 $2x^2 + 7x + 3 = 0$

14 $6x^2 - 7x - 3 = 0$

15 $6x^2 - 5x - 6 = 0$

16 $4x^2 - 16x + 15 = 0$

17 $3x^2 + 5x = 2$

18 $(2x - 3)^2 = 9$

19 $(x - 7)^2 = 36$

20 $2x^2 = 8$

21 $3x^2 = 5$

22 $(x - 3)^2 = 13$

23 $(3x - 1)^2 = 11$

24 $5x^2 - 10x^2 = -7 + x + x^2$

25 $6x^2 - 7 = 11x$

26 $4x^2 + 17x = 6x - 2x^2$

2.3 You can write quadratic expressions in another form by completing the square.

$$x^2 + 2bx + b^2 = (x + b)^2$$
$$x^2 - 2bx + b^2 = (x - b)^2$$

These are both perfect squares.

To complete the square of the function $x^2 + 2bx$ you need a further term b^2. So the completed square form is

$$x^2 + 2bx = (x + b)^2 - b^2$$

Similarly

$$x^2 - 2bx = (x - b)^2 - b^2$$

Example 8

Complete the square for the expression $x^2 + 8x$

$$x^2 + 8x$$
$$= (x + 4)^2 - 4^2$$
$$= (x + 4)^2 - 16$$

$2b = 8$, so $b = 4$

In general

■ **Completing the square:** $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

Example 9

Complete the square for the expressions

a $x^2 + 12x$ **b** $2x^2 - 10x$

a $x^2 + 12x$

$= (x + 6)^2 - 6^2$

$= (x + 6)^2 - 36$

$2b = 12$, so $b = 6$

b $2x^2 - 10x$

$= 2(x^2 - 5x)$

$= 2[(x - \frac{5}{2})^2 - (\frac{5}{2})^2]$

$= 2(x - \frac{5}{2})^2 - \frac{25}{2}$

Here the coefficient of x^2 is 2.
So take out the coefficient of x^2.
Complete the square on $(x^2 - 5x)$.
Use $b = -5$.

Exercise 2C

Complete the square for the expressions:

1 $x^2 + 4x$	**2** $x^2 - 6x$	**3** $x^2 - 16x$	**4** $x^2 + x$
5 $x^2 - 14x$	**6** $2x^2 + 16x$	**7** $3x^2 - 24x$	**8** $2x^2 - 4x$
9 $5x^2 + 20x$	**10** $2x^2 - 5x$	**11** $3x^2 + 9x$	**12** $3x^2 - x$

2.4 You can solve quadratic equations by completing the square.

Example 10

Solve the equation $x^2 + 8x + 10 = 0$ by completing the square.

$x^2 + 8x + 10 = 0$ — Check coefficient of $x^2 = 1$.

$x^2 + 8x = -10$ — Subtract 10 to get LHS in the form $ax^2 + b$.

$(x + 4)^2 - 4^2 = -10$ — Complete the square for $(x^2 + 8x)$.

$(x + 4)^2 = -10 + 16$ — Add 4^2 to both sides.

$(x + 4)^2 = 6$

$(x + 4) = \pm\sqrt{6}$ — Square root both sides.

$x = -4 \pm \sqrt{6}$ — Subtract 4 from both sides.

Then the solutions (roots) of

$x^2 + 8x + 10 = 0$ are either

$x = -4 + \sqrt{6}$ or $x = -4 - \sqrt{6}$.

Leave your answer in surd form as this is a non-calculator question.