MATH 10021

Core Mathematics I

http://www.math.kent.edu/ebooks/10021/CMI.pdf

Department of Mathematical Sciences
Kent State University
July 23, 2010

Contents

Chapter 1

Real Numbers and Their Operations

1.1 Introduction

Welcome to Core Mathematics I (Math 10021) or Core Mathematics I and II (Math 10006)! Beginnings are always difficult, as you and your fellow students may have a wide range of backgrounds in previous math courses, and may differ greatly in how long it has been since you last did math. So, to start, we want to suggest some ground rules, refresh your memory on some basics, and demonstrate what you are expected to know already. Let's start with the ground rules.

First, this course will be easier if you keep up with the material. Math often builds upon itself, so if you don't understand one section, the next section may be impossible. To avoid this, practice, practice, practice, and get help from a tutor or the instructor immediately if you are having trouble. This book has many problems after each section with answers to all in the back, so feel free to do more than what is assigned to be collected! The student who shows up an hour prior to the exam and tells the instructor that they need to go over everything is going to be sorely disappointed.

Second, please do NOT ask your instructor "When am I ever going to use this?", as this may lead him or her to develop nervous tics or to begin muttering under his or her breath. This is college, and the student's role at college is to acquire knowledge. Some of the material in this book may have obvious applications in your future job, some may have surprising applications, and some you may never use again. ALL OF IT is required to pass this course!

Finally, you may already know, or will soon learn, that you will be expected to do the first exam *without* a calculator. The idea is that the material covered will just be basic operations on real numbers, and you should know the mechanics behind all of these things. Then, later, if you wish to use a calculator to save yourself some time and effort (and to have less chance of making a simple error), that is fine. The calculator, however, should NOT be an arcane and mysterious device which is relied upon like a crutch. Letting it do your busy work is one thing; letting it do your thinking is quite another. The dangers of letting machines do all of our thinking for us is aptly illustrated in any of the Terminator or Matrix movies!

With all that being said, let us now move on to numbers. When you first learned to count, you probably counted on your fingers and used the numbers one through ten. In set notation, this group of numbers would be written $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Notice that for set notation we just listed all the numbers, separated by commas, inside curly braces. If there are a lot of numbers in the set, sometimes we use the notation "...", called periods of ellipsis, to take the place of some of the numbers. For example, the set of numbers one through one hundred could be written $\{1, 2, \ldots, 99, 100\}$. When periods of ellipsis are used, the author assumes the reader can see a pattern in the numbers in the set and can figure out what is missing. If the periods of ellipsis occur after the last listed number, this means that the numbers go on forever. An example of such a set of numbers is the first listed in this book that is important enough to have an official name (three names actually!):

$$\textbf{natural numbers} = \textbf{counting numbers} = \textbf{positive integers} = \{1, 2, 3, \ldots\}.$$

These numbers are the ones for which you probably have the best "feel". For example, you can imagine seven pennies or eighty-two cows, although you probably get a little bit hazier for the bigger numbers. You will have heard of a million or billion, and possibly even a trillion (like the national deficit in dollars), but do you know the names of larger numbers? Try the following example:

Example 1. *Which of the following are natural numbers: (a) a quintillion, (b) a zillion, (c) a bazillion, (d) an octillion, (e) a google?*

> **Solution 1.** *Did you say (a), (d), and (e)?*
> *(a) a quintillion = 1,000,000,000,000,000,000*
> *(b) a zillion is a real word (in the dictionary), but it represents an indeterminately large number (i.e. a whole heck of a lot)*
> *(c) a bazillion is not in the dictionary; it is slang for a zillion*
> *(d) an octillion = 1,000,000,000,000,000,000,000,000,000*
> *(e) a google = one followed by one hundred zeros. That's right! Before it was ever the name of an internet search engine, it was the name of a large natural number.*

The next number we would like to review is zero, another number of which you should have a good understanding. For example, if you have four quarters in your pocket and your brother takes all four to buy a bottle of iced tea, you are left with no quarters, or zero quarters. Once we add zero into the number set, the set gets a new name:

whole numbers = nonnegative integers $= \{0, 1, 2, 3, \ldots\}$.

IMPORTANT NOTATION: Prior to this course, you should have seen how to apply the four basic mathematical operations (addition, subtraction, multiplication or division) to whole numbers. This book will insert the word *old-fashioned* before any such operation whenever both the numbers AND the result are whole numbers. For example, you are doing old-fashioned subtraction when you are subtracting two whole numbers and the bigger number is first. Note, even though we will illustrate how to do old-fashioned addition, subtraction, multiplication, and division without a calculator, the use or lack of use of a calculator plays no role in our using the adjective old-fashioned. The terminology is just to indicate the most basic use of the operation. Once we mix in negative numbers, fractions, etc., there will be new rules which will have to be applied. Note that all four of these operations require a number (or expression) *both* before *and* after the operation sign. Therefore, later when we see an expression like "-2", we will know that the "$-$" sign here does NOT mean subtract, as there is no number before it.

Addition

For addition, we will use the standard symbol of "+". The two numbers being added are called the addends, while the answer is called the sum. A property of addition which will be useful to know is that when you add two numbers, the order in which you add them doesn't matter. Thus:

$3{+}5 = 5{+}3.$

This is the commutative property of addition which we will study in more detail in section 1.13; for now, you just want to know that you can switch the order. To do old-fashioned addition without a calculator, you should have seen the tower method. When using the tower method, make sure that the place-values of the two numbers are aligned, i.e. the one's digit of the top number is over the one's digit of the bottom number, the ten's digit of the top number is over the ten's digit of the bottom number, etc.

Example 2. *Add* $357 + 4,282$.

Solution 2. *Scratch work:*

$$\begin{array}{ccccc} & & 1 & & \\ & & 3 & 5 & 7 \\ + & 4 & 2 & 8 & 2 \\ \hline & 4 & 6 & 3 & 9 \end{array}$$

So: $357 + 4,282 = \mathbf{4,639}$.

Notice that when we added the five to the eight and got thirteen, we carried the one to the next column. It is not unusual for you to have to carry when using the vertical tower method. Some other comments on the last example: First, it will be a good practice to have a separate line for your solution versus your actual work space, especially later when you may need to adjust the sign of your solution. Second, you should clearly indicate your solution for your instructor. We will bold-face our solutions, while you may wish to circle or box-in yours. Finally, it is always a good idea to check your work if there is time. Here, we encourage you to check this answer.

You can even see that the commutative property of addition holds if you check by putting the 4,282 on top of the 357.

Example 3. *Add* $43,578 + 7,694$.

Solution 3. *Scratch work:*

$$\begin{array}{cccccc} & 1 & 1 & 1 & 1 & \\ & 4 & 3 & 5 & 7 & 8 \\ + & & 7 & 6 & 9 & 4 \\ \hline & 5 & 1 & 2 & 7 & 2 \end{array}$$

So: $43,578 + 7,694 = \mathbf{51,272}$.

Subtraction

We will use the standard symbol "−" for subtraction. Addition and subtraction are inverse operations, and this is probably how you first learned to subtract. For example, $9 - 5 = 4$ because $4 + 5 = 9$. This also means that addition and subtraction can undo each other. For example, you have 4 one-dollar bills, and your sister gives you 5 more, so now you have 9 one-dollar bills ($4 + 5 = 9$). Later, she takes the 5 one-dollar bills back, so you are back to 4 ($9 - 5 = 4$). You will see this concept of two operations which undo each other recur over and over again in mathematics.

To do old-fashioned subtraction of large numbers without a calculator, we will again use a tower method, and again we will make sure to align the place-values of the two numbers.

Example 4. *Subtract* $1978 - 322$.

Solution 4. *Scratch work:*

$$\begin{array}{ccccc} & 1 & 9 & 7 & 8 \\ - & & 3 & 2 & 2 \\ \hline & 1 & 6 & 5 & 6 \end{array}$$

So: $1978 - 322 = \mathbf{1656}$.

Now consider $1321 - 567$. When we line these numbers up according to place value, we see that we would like to take 7 away from 1 in the ones place. This cannot happen. Therefore, we need to borrow from the next column to the left, the tens. This is often referred to as "borrowing 10", but what you are borrowing ten of depends on the situation. It is similar to how you can exchange 1 ten-dollar bill for 10 one-dollar bills, 1 one-dollar bill for 10 dimes, and 1 dime for 10 pennies. Our number system uses a base of 10, so each column is worth ten times more than the last. Therefore, in the following example, we will first borrow 1 ten to get 10 more ones, then 1 hundred to get 10 more tens, then 1 thousand to get 10 more hundreds:

Example 5. *Subtract* $1321 - 567$.

Solution 5. *Scratch work:*

$$\begin{array}{rrrrr} & & & 1 & 11 \\ & 1 & 3 & \cancel{2} & \cancel{1} \\ - & & 5 & 6 & 7 \\ \hline & & & & 4 \end{array} \Rightarrow \begin{array}{rrrrr} & & 2 & 11 & 11 \\ & 1 & \cancel{3} & \cancel{2} & \cancel{1} \\ - & & 5 & 6 & 7 \\ \hline & & & 5 & 4 \end{array} \Rightarrow \begin{array}{rrrrr} & 0 & 12 & 11 & 11 \\ & \cancel{1} & \cancel{3} & \cancel{2} & \cancel{1} \\ - & & 5 & 6 & 7 \\ \hline & & 7 & 5 & 4 \end{array}$$

So: $1321 - 567 = \mathbf{754}$.

Let us do one more subtraction example.

Example 6. *Subtract* $13200 - 4154$.

Solution 6. *Scratch work:*

$$\begin{array}{rrrrrr} & & & 1 & 10 & \\ & 1 & 3 & \cancel{2} & \cancel{0} & 0 \\ - & & 4 & 1 & 5 & 4 \\ \hline & & & & & \end{array} \Rightarrow \begin{array}{rrrrrr} & & & 1 & 9 & 10 \\ & 1 & 3 & \cancel{2} & \cancel{0} & \cancel{0} \\ - & & 4 & 1 & 5 & 4 \\ \hline & & & 0 & 4 & 6 \end{array} \Rightarrow \begin{array}{rrrrrr} & 0 & 13 & 1 & 9 & 10 \\ & \cancel{1} & \cancel{3} & \cancel{2} & \cancel{0} & \cancel{0} \\ - & & 4 & 1 & 5 & 4 \\ \hline & & 9 & 0 & 4 & 6 \end{array}$$

So: $13200 - 4154 = \mathbf{9046}$.

In the last example, we were originally frustrated in our attempt to borrow from the ten's place since there was a zero there, so we had to first borrow from the hundred's place.

Multiplication

In multiplication, the two numbers being multiplied are called the factors, and the result is called the product. You may have seen many different notations for multiplication: $\times, \cdot, *, ()$. The "*" is mostly used in computer science, and we will not use it in this book. The "$\times$" sign is the one you probably learned originally, and it is the sign most often used for multiplication in applications (like on your calculator), but there is a problem with it. In an algebra course like this, we will eventually be using the letter x (variable), and the two are easily confused. Therefore, we will mostly use the "$\cdot$" notation for multiplication. The one exception will be when we use the tower method to multiply without a calculator. Here, we will use the "$\times$" sign, as a sloppy "$\cdot$" could be mistaken for a decimal point. Parentheses also show multiplication, so that whenever the left parenthesis "(" is preceded by *a number* <u>or</u> *a right parenthesis* ")", this means multiply.

Example 7. *Multiply:*
$(4)(3) = 12$
$2(7) = 14.$

This idea may be extended to any grouping symbol (we will see brackets "[...]" and absolute value "$|\ldots|$" later). One curious fact about multiplication is that it is sometimes *hidden*. You will see several cases in the Core Math courses where different expressions are right next to each other with no obvious operation indicated, and you will need to understand that you are supposed to multiply. You could treat parentheses like this if it helps. So the last example could have been written:

Example 8. *Multiply:*
$(4)(3) = (4) \cdot (3) = 4 \cdot 3 = 12$
$2(7) = 2 \cdot (7) = 2 \cdot 7 = 14.$

Multiplication is defined as repeated addition. Therefore, multiplying four times three is the same as adding three to itself four times:

Example 9. *The following are equal:*
$4 \cdot 3 = 3 + 3 + 3 + 3 = 12.$

Multiplication is commutative just like addition, so that when you are multiplying two numbers, you can multiply in either order:

Example 10. *The following are equal:*
$4 \cdot 3 = 3 + 3 + 3 + 3 = 12 = 4 + 4 + 4 = 3 \cdot 4.$

Even though you could do any old-fashioned multiplication problem through repeated addition, this would get tedious. Instead, you should know your multiplication table for multiplying any single-digit whole number times any other single-digit whole number. This is probably already the case, but just in case you have forgotten (or are helpless without your calculator...), we will include it next.

Basic Multiplication Table

·	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	18	27	36	45	54	63	72	81

The first test will be much easier if you take the time to memorize this table as soon as possible. For multiplying larger numbers without a calculator, we will use a tower method. To refresh your memory on how this works, let us look at an example.

Example 11. *Multiply* $152 \cdot 8$.

Solution 11. *Scratch work:*

$$\begin{array}{cccc} & 4 & 1 & \\ & 1 & 5 & 2 \\ \times & & & 8 \\ \hline 1 & 2 & 1 & 6 \end{array}$$

So $152 \cdot 8 = \mathbf{1216}$.

Notice that when we multiplied the 8 times the 2, we got 16, wrote down the 6 and carried the 1 to the next column. Then, when we multiplied the 8 times the 5 and got 40, we added the 1 to get 41, and etc. If the bottom number has two-digits, first multiply the one's digit by the entire top number as in the last example. Then *cancel or erase* all of your previous carries, *put* a zero below the rightmost digit of your previous product (well, that might be old-school; some people just leave this spot blank), *multiply* the ten's digit by the entire top number, and *add* the two products together. For example:

Example 12. *Multiply* $327 \cdot 47$.

Solution 12. *Scratch work:*

$$\begin{array}{cccc} & 1 & 4 & \\ & 3 & 2 & 7 \\ \times & & 4 & 7 \\ \hline 2 & 2 & 8 & 9 \end{array} \Rightarrow \begin{array}{cccc} & 3 & 2 & 7 \\ \times & & 4 & 7 \\ \hline 2 & 2 & 8 & 9 \\ & & & 0 \end{array} \Rightarrow \begin{array}{ccccc} & & 1 & 2 & \\ & & 3 & 2 & 7 \\ \times & & & 4 & 7 \\ \hline & 2 & 2 & 8 & 9 \\ 1 & 3 & 0 & 8 & 0 \\ \hline 1 & 5 & 3 & 6 & 9 \end{array}$$

So $327 \cdot 47 = \mathbf{15,369}$.

The zero we added in prior to multiplying the second time accounts for the fact that we were multiplying by 40 rather than 4. If the bottom number has three digits, start as in the last example, but do not add the two products together. Instead, *cancel or erase* all of your previous carries, *put* a zero below the rightmost digit AND the ten's digit of your previous product, *multiply* the hundred's digit of the bottom number by the entire top number, and *add* the three products together. For example:

Example 13. *Multiply* $85 \cdot 324$.

Solution 13. *Scratch work:*

$$
\begin{array}{cccc}
 & & 2 & \\
 & & 8 & 5 \\
\times & 3 & 2 & 4 \\
\hline
 & 3 & 4 & 0
\end{array}
\Rightarrow
\begin{array}{ccccc}
 & & & 1 & \\
 & & & 8 & 5 \\
\times & & 3 & 2 & 4 \\
\hline
 & & 3 & 4 & 0 \\
 & 1 & 7 & 0 & 0
\end{array}
\Rightarrow
\begin{array}{ccccc}
 & & & 1 & \\
 & & & 8 & 5 \\
\times & & 3 & 2 & 4 \\
\hline
 & & 3 & 4 & 0 \\
 & 1 & 7 & 0 & 0 \\
2 & 5 & 5 & 0 & 0 \\
\hline
2 & 7 & 5 & 4 & 0
\end{array}
$$

So $85 \cdot 324 = \mathbf{27,540}$.

Recall that multiplication is commutative, so some people may prefer to always put the number with the least digits on bottom. In that case, the last example becomes:

Example 14. *Multiply* $85 \cdot 324$.

Solution 14. $85 \cdot 324 = 324 \cdot 85$
Scratch work:

$$
\begin{array}{ccccc}
 & & 1 & 2 & \\
 & & 3 & 2 & 4 \\
\times & & & 8 & 5 \\
\hline
 & 1 & 6 & 2 & 0
\end{array}
\Rightarrow
\begin{array}{ccccc}
 & & 1 & 3 & \\
 & & 3 & 2 & 4 \\
\times & & & 8 & 5 \\
\hline
 & 1 & 6 & 2 & 0 \\
2 & 5 & 9 & 2 & 0 \\
\hline
2 & 7 & 5 & 4 & 0
\end{array}
$$

So $85 \cdot 324 = \mathbf{27,540}$.

Division

Division is represented by the symbols "÷" or "/". The "/" is used most often in computer science, and we will not be using it in this book. We will see later that the fraction bar also shows division.

Just as subtraction is the inverse operation of addition, division is the inverse operation of multiplication. Thus, $15 \div 5 = 3$ because $3 \cdot 5 = 15$. This means that multiplication and division should be able to undo each other, but here we must be careful, as this isn't quite as straight-forward as it was with addition-subtraction. For example, while it is still true that if you multiply a number by 2, you could then get back the original number by then dividing by 2:

as $8 \cdot 2 = 16$ and $16 \div 2 = 8$.

You must be more careful when 0 is involved. Let's examine what happens when you are dividing into 0 (0 is in front of ÷ sign), when you are dividing by 0 (0 is after ÷ sign), and when you are doing both.

Example 15. *Find $0 \div 8$ by treating division as the inverse operation of multiplication.*

Solution 15. $0 \div 8 = \_\_\_\_\_ \quad \Rightarrow \quad \_\_\_\_\_ \cdot 8 = 0.$
What number can be placed in the blank to make this true? Well, $0 \cdot 8 = 0$ (and this is the only number which could be placed in the blank to make a true statement), so:
$0 \div 8 = \mathbf{0}$.

This is only one example, but we will spoil the suspense and go ahead and state this more generally.

$$\boxed{\textbf{Let a be any number other than 0, then: } \mathbf{0 \div a = 0}.}$$

This means that division *into* 0 (except by itself), is as trivially easy as multiplication by 0 or 1 or division *by* 1. What about division by 0?

Example 16. *Find* $8 \div 0$ *by treating division as the inverse operation of multiplication.*

Solution 16. $8 \div 0 = \underline{\qquad} \quad \Rightarrow \quad \underline{\qquad} \cdot 0 = 8.$
What number can be placed in the blank to make this true? The answer is ***no*** *number, as any number times 0 is 0, not 8. We will call division by 0, undefined.*
Therefore, $8 \div 0 =$***undefined****.*

Thus, division into 0 makes the division very easy; division by zero makes the division impossible. What happens when these two rules collide?

Example 17. *Find* $0 \div 0$ *by treating division as the inverse operation of multiplication.*

Solution 17. $0 \div 0 = \underline{\qquad} \quad \Rightarrow \quad \underline{\qquad} \cdot 0 = 0.$
What number can be placed in the blank to make this true? The answer, of course, is 0! Or 1, ...er, or 2, or 16, or 137,.... While this division is no longer impossible, it is actually too possible! Instead of getting one nice answer like we would desire when dividing two numbers, we get way too many options. Therefore, we will label this as undefined as well, but not for the same reason as before.
So, $0 \div 0 =$***undefined****.*

Together, these last two examples illustrate another important point.

$$\boxed{\textbf{For any number a:} \quad \mathbf{a \div 0 = undefined.}}$$

To do old-fashioned division of large numbers without a calculator, you need to remember how to do long division.

Example 18. *Divide* $74,635 \div 23$.

Solution 18.

$$
\begin{array}{rrl}
 & 3245 & \\
23 & \overline{)74635} & 23 \textit{ goes into } 74 \textit{ three times} \\
 & \underline{69} & 23 \cdot 3 = 69 \\
 & 56 & \textit{Subtract then bring down the } 6 \textit{ from the } 74635 \\
 & \underline{46} & 23 \textit{ goes into } 56 \textit{ two times; } 23 \cdot 2 = 46 \\
 & 103 & \textit{Subtract then bring down the } 3 \textit{ from the } 74635 \\
 & \underline{92} & 23 \textit{ goes into } 103 \textit{ four times; } 23 \cdot 4 = 92 \\
 & 115 & \textit{Subtract then bring down the } 5 \textit{ from the } 74635 \\
 & \underline{115} & 23 \textit{ goes into } 115 \textit{ five times; } 23 \cdot 5 = 115 \\
 & 0 & \textit{Subtracting there is a remainder of zero.}
\end{array}
$$

Thus, $74,635 \div 23 = \mathbf{3,245}$.

Don't forget that whenever you place a number on top of the division bar, you immediately multiply this number by the divisor (number in front). At this point, you may check to make sure you put the proper number up top. For example, had we thought that 23 went into 74 four times at the beginning of the last example, we would have multiplied $23 \cdot 4 = 92$, and then we would have noticed that 92 is bigger than 74. That is a clear indication that the number you are trying is too big. On the other hand, had we thought 23 went into 74 two times, we would have multiplied $23 \cdot 2 = 46$ and subtracted $74 - 46 = 28$. This difference is larger than the divisor (the 23), so we know that we are trying too small a number. Occasionally, after bringing down the next number, the new number is still too small for the divisor to divide into. This is fine, it just means that we need to enter a 0 on top. To illustrate this, let us do a few more examples.

Example 19. *Divide* $8,736 \div 42$.

Solution 19.

$$\begin{array}{rrl}
 & 208 & \\
42 & \overline{)8736} & \textit{42 goes into 87 two times} \\
 & \underline{84} & 42 \cdot 2 = 84 \\
 & 33 & \textit{Subtract then bring down the 3 from the 8736} \\
 & \underline{0} & \textit{42 goes into 33 zero times; } 42 \cdot 0 = 0 \\
 & 336 & \textit{Subtract then bring down the 6 from the 8736} \\
 & \underline{336} & \textit{42 goes into 336 eight times; } 42 \cdot 8 = 336 \\
 & 0 & \textit{Subtracting there is a remainder of zero.}
\end{array}$$

Thus, $8,736 \div 42 = \mathbf{208}$.

Example 20. *Divide* $68,068 \div 34$.

Solution 20.

$$\begin{array}{rrl}
 & 2002 & \\
34 & \overline{)68068} & \textit{34 goes into 68 two times} \\
 & \underline{68} & 34 \cdot 2 = 68 \\
 & 00 & \textit{Subtract then bring down the 0 from the 68068} \\
 & \underline{0} & \textit{34 goes into 0 zero times; } 34 \cdot 0 = 0 \\
 & 06 & \textit{Subtract then bring down the 6 from the 68068} \\
 & \underline{0} & \textit{34 goes into 6 zero times; } 34 \cdot 0 = 0 \\
 & 68 & \textit{Subtract then bring down the 8 from the 68068} \\
 & \underline{68} & \textit{34 goes into 68 two times; } 34 \cdot 2 = 68 \\
 & 0 & \textit{Subtracting there is a remainder of zero.}
\end{array}$$

Thus, $68,068 \div 34 = \mathbf{2,002}$.

1.2 Integers, Absolute Values and Opposites

Negative Integers - what are they?

Many people dislike negative numbers, because they believe (mistakenly) that there is no good physical meaning for them. The idea being that while you can picture what three pennies represent, and what five pennies represent, how do you picture negative two pennies? It doesn't seem to help if we tell you that three minus five is equal to negative two, because if you have three pennies and try to take away five of them, nothing obvious springs to mind. This just means, however, that we will have to develop a new way of doing subtraction (next section), and a new way to think about numbers rather than just a collection of objects.

One way in which negative numbers may arise is when you have assigned a zero, and it is possible to go below this number as in the following examples.

Example 1. *In America, we mostly use the Fahrenheit scale to measure temperature, and we have thermometers which can measure the changes. Winter in Ohio can get very cold, so what do we do if the temperature is* $0°F$*, and then it gets colder? Wouldn't it be confusing if we kept calling the temperature* $0°F$ *just because we don't feel as though we can take anymore away?*

Example 2. *Elevation is measured in height above sea level. This would seem to make sense because any land next to the ocean which is lower would be underwater (and hence not land). Inland, though, the elevation could drop lower (e.g. Death Valley, California).*

In both of these examples, we could have defined our scales to avoid negative numbers, but then typical values would have become unwieldy. For example, there is a Rankine scale where a change of one degree Rankine is the same as one degree Fahrenheit, yet whose zero is defined to be absolute zero (the coldest temperature possible). Some typical values you may encounter: water freezes at $492°R$, water boils at $672°R$, and a pleasant summer day would be $540°R$ (=$80°F$). As for elevation, we could measure distance from the center of the Earth, but since the radius of the Earth is approximately 4000 miles, the elevation of sea level would then become approximately 21,000,000 feet!

Negative numbers can have a more physical meaning. Often there may be other, equivalent, terminology used instead, but the idea is still there.

Example 3. *A company is keeping track of how much profit it makes each month. One month, due to a large number of renovations being made, its revenue (amount of money taken in) is \$2000 less than its expenditures. Therefore the company's profit is negative \$2000, or \$2000 in the red.*

Example 4. *You own a store which sells Item X. At the end of the month, you restock your shelves, and you like to order enough to have 50 Item X in stock at the beginning of the next month. One month there is an unusual demand for Item X, and 73 customers request one. Selling to the first fifty is no problem, but what do you do then? You could just explain to those customers that you are sold out, and to try back again next month after you get another shipment, but you might lose their business. Instead, you could take their order and either have Item X shipped directly to them or promise to set aside their order and have them called when it arrives. If you choose this second method, you are choosing to use negative numbers. After all, you certainly do not want to order just 50 more Item X's - you have essentially already sold 23 of them, and even a more normal demand month could run you out of your product. Rather, you should think of your inventory as negative 23 so that when you reorder, you will order 23 for those already demanded, and then another 50 so that you begin the next month as usual.*

It was in commerce, like this last example, where negative numbers first appeared. Notice that in neither of these last two examples would redefining zero make any sense - i.e. you would not want to define losing one million dollars to be "\$0 profit".

Another physical meaning for negative numbers occur whenever you have two characteristics which cancel each other when combined. The most common example of this is electric charge.

Example 5. *A sodium ion (Na^+) with a charge of positive one will form an ionic bond with a chlorine ion (Cl^-) with a charge of negative one. The resulting sodium chloride molecule (salt) will be neutral (charge equal to zero).*

Other examples of this type of characteristic in nature would be matter versus antimatter, and up versus down spin in elementary particles. It is useful to remember that when you combine (i.e. add) a positive one with a negative one, they make zero.

Finally, keep in mind that negative numbers may just mean that an implication or direction is the *wrong way.* This was the case in the previous example involving the company with the negative \$2000 profit. For that month, the company did not *make* \$2000, rather they had to *pay it out.* The following example also illustrates this point.

Example 6. *Slick and the Kid are playing poker. The Kid consistently loses so they are keeping track of how much he owes Slick every night, intending to settle up at the end of the week. After one night, he owes Slick \$40; after two nights, \$60; etc. After the sixth night, the Kid owes Slick \$270. On the seventh and final night, Lady Luck smiles on the Kid, and he wins \$300. Slick, getting ready to settle up, stacks \$270 worth of chips on the table. "Well, Kid," Slick says, "this represents the amount of money you owed me going into tonight. Let me settle up with you by removing your winnings from this debt." After he counts out all \$270, Slick frowns. "Hmmm, there doesn't seem to be any more to take away, so I guess we are even." The Kid, who only knows old-fashioned subtraction, nods and says "Yup.", thereby ensuring that he would always find people willing to play him in poker.*

Had that have been you instead of the Kid, you would surely have noticed that Slick should still owe you \$30. This goes to show that you already know how to do fancier subtraction than just the old-fashioned take-away kind! By the way, while you might have said that Slick owes the Kid \$30, you could also say that the Kid owes Slick negative \$30. The negative here would indicate that the implication (in this case, who owes whom) is the wrong way around.

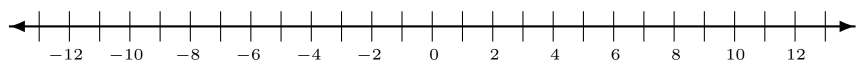
Negative Integers - notation

To represent a negative, we will use the "$-$" sign. That is right, the same sign we use for subtraction. Similarly, if we wanted to emphasize that a number is positive, we could place a "+" before it, such as +8 for positive eight, but the "+" sign is traditionally left off. Therefore, in some of the future problems, we may temporarily put a "+" sign to show positive in our scratch work, but any final answer which is positive will not have the "+".

Now that we have negative integers, we can define the set of all integers:

$$\textbf{integers} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

The periods of ellipsis at the beginning show that there are an infinite number of numbers before the negative three. Sometimes people like to illustrate the set of integers by making a number line.



Note that we only labeled every other tick mark, but this was done only for space considerations - you could label every one if you would like. Also note that the number line does not stop at -13 and 13 but continues on in both directions as shown by the arrows on the ends. The spacing between the numbers should be uniform - although the units may be feet, inches, centimeters, miles, whatever. We will refer to the spacing as steps, marked off by someone with a uniform stride length, but again any unit may be substituted.

The number line may be thought of as a straight railroad track which runs east - west, with you standing at 0. We could give you directions to any point on the track by telling you how many steps to take to the east or how many to take to the west, but the property of a negative sign of implying the "other" direction may be exploited. Instead of having two directions (east versus west), we only need one (in this case just east), and a negative number means go the other way. Thus, with you standing at 0, the number 5 would be five steps to the east, while the number -7 is seven steps to the west.

Less than, Greater than, Less than or equal to, Greater than or equal to

Whenever you are working with a set of numbers, it may be necessary to compare two numbers to see which is the bigger or the smaller. With whole numbers, this was fairly simple, and you should have no problem stating which of the two numbers 5 and 12 is the greater. With the introduction of negative numbers, this issue may seem to be a bit cloudy. For example, which of the numbers 5 and -12 do you think is the greater? To extend the same concept of greater or lesser that we use with whole numbers to the set of integers, we will use the following definitions.

Definition: For any two integers (this definition will extend to all real numbers once we learn what they are), the number which is further to the left on the number line is said to be <u>less than</u>, $<$, the other.

Definition: For any two integers (this definition will extend to all real numbers once we learn what they are), the number which is further to the right on the number line is said to be <u>greater than</u>, $>$, the other.

Let us look at an example.

Example 7. *Insert a $<$ or $>$ between each of the following pair of numbers to make a true statement. Then write out what each statement means in words.*

1. $2 \_\_\_ 7$
2. $-3 \_\_\_ 6$
3. $5 \_\_\_ -12$
4. $0 \_\_\_ -3$
5. $-2 \_\_\_ -9$
6. $-15 \_\_\_ -8$

Solution 7. *The solutions are:*

1. $2 < 7$
   Two is less than seven.

2. $-3 < 6$
   Negative three is less than six.
3. $5 > -12$
   Five is greater than negative twelve.
4. $0 > -3$
   Zero is greater than negative three.
5. $-2 > -9$
   Negative two is greater than negative nine.
6. $-15 < -8$
   Negative fifteen is less than negative eight.

If you are having trouble remembering which sign is which, there are several different mnemonics people use. Some people think of the signs as greedy mouths which want to eat the larger number; some people think of the signs as arrows which the larger number (the bully) is shooting at the smaller.

In addition to these basic signs, there are two more which are a combination of these and an equals sign.

Definition: The symbol $\leq$, read as <u>less than or equal to</u>, states that the number to its left is EITHER less than OR equal to the number on its right.

Definition: The symbol $\geq$, read as <u>greater than or equal to</u>, states that the number to its left is EITHER greater than OR equal to the number on its right.

Example 8. *Insert a $\leq$ or $\geq$ between each of the following pair of numbers to make a true statement. Then write out what each statement means in words.*

1. $5 \underline{\quad} 6$
2. $-2 \underline{\quad} -10$
3. $3 \underline{\quad} 3$
4. $0 \underline{\quad} -4$

5. $-11 \_\_\_ -9$

6. $-5 \_\_\_ -8$

Solution 8. *The solutions are:*

1. $5 \leq 6$
   Five is less than or equal to six.
2. $-2 \geq -10$
   Negative is greater than or equal to negative ten.
3. *Here either of the two signs would be correct:*
   $3 \leq 3$
   Three is less than or equal to three.
   OR
   $3 \geq 3$
   Three is greater than or equal to three.
4. $0 \geq -4$
   Zero is greater than or equal to negative four.
5. $-11 \leq -9$
   Negative eleven is less than or equal to negative nine.
6. $-5 \geq -8$
   Negative five is greater than or equal to negative eight.

Absolute Value

We have just learned that $-124 < 2$, since -124 is much further to the left on the number line, but sometimes we are not interested in whether the number is positive or negative, just how far away from zero the number is. For example, if you are on stranded on a railroad track and one town is two miles to the east and another is one hundred twenty-four miles to the west, you would prefer to walk to the east. The fact that you were defining east as the positive direction makes no difference in this case. Mathematically, we will call this concept the absolute value.

Definition: The absolute value of a number a, written $|a|$, is the distance the number is from zero on the number line.

This means that the absolute value of a positive number or zero (i.e. a whole number) is itself, while the absolute value of a negative number is the same number without the negative sign.

Example 9. *Find the absolute value of the following numbers:*

1. 16
2. 209
3. 0
4. -4
5. -82

Solution 9. *Evaluating the absolute values:*

1. $|16| = \mathbf{16}$.
2. $|209| = \mathbf{209}$.
3. $|0| = \mathbf{0}$.
4. $|-4| = \mathbf{4}$.
5. $|-82| = \mathbf{82}$.

The absolute value bars represent *an operation*, and just as you would not want to leave an answer as $5+3$ but would add and write 8 instead (except in very particular instances), you will always want to take the absolute value of the number inside.

Example 10. *Simplify.*

1. $|18|$
2. $|-31|$
3. $|-1|$
4. $|-22|$

Solution 10. *We get:*

1. $|18| = \mathbf{18}$.
2. $|-31| = \mathbf{31}$.
3. $|-1| = \mathbf{1}$.
4. $|-22| = \mathbf{22}$.

Opposites

Definition: Two numbers on the number line which are the same distance from zero, but on opposite sides, are called opposites.

Example 11. *The following numbers are opposites:*

1. 2 *and* -2.
2. -7 *and* 7.
3. 3 *and* -3.
4. -1 *and* 1.
5. 0 *and* 0.

Notice in the last example that zero is considered its own opposite.

If we wish you to take the opposite of a number, we will indicate this with the symbol "$-$". This is the third use you have seen for this same symbol, so you need some rules to let you know how to interpret the sign.

Rules for determining meaning of $-$ sign:

1. Subtraction is an operation which requires something (for right now we will examine the easiest case where the something is a number) both immediately before and after it. If you want to compare mathematical operations to English verbs (as they are telling you to do something), then the operation of subtraction requires both a subject and a direct object.

2. Negative is not an operation; it simply modifies the number after it. Using the English grammar comparison again, it would be an adjective. There may be nothing in front of it, but if there is a number or expression in front, there must be an operation between the two.

3. The opposite of is an operation, but unlike subtraction, it requires only something (again, to start we will only use a number) immediately after it. Like a negative, there may be nothing in front of it, but if there is a number or expression in front, there must be another operation in between. The difference between this and a negative sign, is that if we wish to take the opposite of a number, we will use parentheses.

This may become more clear with some examples.

Example 12. *State in words what the following expressions represent, then simplify if possible.*

1. $73 - 45$
2. -7
3. $-(-2)$
4. $17 - 8$
5. $-(3)$

Solution 12. *We have:*

1. $73 - 45$
   Seventy-three minus forty-five; this is subtraction.
   $73 - 45 = \mathbf{28}$*, (do scratch work if necessary).*

2. -7
   Negative seven, no operation being done here, just a single number being listed. Thus, simplified answer is just $-\mathbf{7}$*.*

3. $-(-2)$
   The opposite of negative two. Neither sign can represent subtraction as there is no number or expression in front (no subject). Recalling that a negative is always immediately followed by a number, the first sign cannot be a negative (as there is a parenthesis and a "$-$" sign before it and the two), while the second sign is followed by a number. To simplify, we need to take the opposite of negative two, which recall is the number which is the same distance away from zero (two steps), but on the other side. Thus:
   $-(-2) = \mathbf{2}$.

4. $17 - 8$
   Seventeen minus eight, good old-fashioned subtraction.
   $17 - 8 = \mathbf{9}$.

5. $-(3)$
   The opposite of three OR the opposite of positive three. Obviously not subtraction (no subject) and not immediately followed by a number (the parenthesis is in between). Being an operation, we need to simplify.
   $-(3) = \mathbf{-3}$.

Please note that when we want you to take the opposite of a number, we will usually put that number in parentheses. For negative numbers, this avoids the unsightly notation of "$--$". For positive numbers, you could interpret something like -7 from (2) of the last example as "the opposite of seven", but since this is an operation, you would then need to proceed to take the opposite of seven to get that the opposite of seven is equal to negative seven ($-7 = -7$), which seems silly to write like this. Rather, if we wish to emphasize the idea of taking the opposite of a positive number, we will use parentheses like in (5) of last example. The only possible exception to the parentheses use is zero. Both -0 and $-(0)$ are read as "the opposite of zero". This is due to zero being neither positive nor negative, so "negative zero" would make no sense.

Example 13. *Simplify.*

1. $-(13)$

2. $-(-6)$
3. $-(-87)$
4. $-(32)$
5. -0

Solution 13. *The solutions are:*

1. $-(13) = \mathbf{-13}$.
   The opposite of (positive) thirteen is negative thirteen.

2. $-(-6) = \mathbf{6}$.
   The opposite of negative six is (positive) six.

3. $-(-87) = \mathbf{87}$.
   The opposite of negative eighty-seven is (positive) eighty-seven.

4. $-(32) = \mathbf{-32}$.
   The opposite of (positive) thirty-two is negative thirty-two.

5. $-0 = \mathbf{0}$.
   The opposite of zero is zero.

Combining the concepts

If both operations of *taking the opposite of* and *absolute value* are being applied to a number, apply the operation nearest the number first, and work your way out.

Example 14. *Simplify.*

1. $-|-6|$
2. $-|6|$

3. $|-(-4)|$
4. $-|35|$
5. $|-0|$
6. $-|0|$

Solution 14. *We get:*

1. $-|-6|$ *is the opposite of the absolute value of negative six.*

$$\begin{aligned} -|-6| &= -(6) && \text{as the absolute value of negative six is positive six,} \\ &= -6 && \text{as the opposite of positive six is negative six.} \end{aligned}$$

Thus, $-|-6| = \mathbf{-6}$.

2. $-|6|$ *is the opposite of the absolute value of six.*

$$\begin{aligned} -|6| &= -(6) && \text{as the absolute value of six is positive six,} \\ &= -6 && \text{as the opposite of positive six is negative six.} \end{aligned}$$

Thus, $-|6| = \mathbf{-6}$.

3. $|-(-4)|$ *is the absolute value of the opposite of negative four.*

$$\begin{aligned} |-(-4)| &= |4| && \text{as the opposite of negative four is positive four,} \\ &= 4 && \text{as the absolute value of positive four is positive four.} \end{aligned}$$

Thus, $|-(-4)| = \mathbf{4}$.

4. $-|35|$ *is the opposite of the absolute value of thirty-five.*

$$\begin{aligned} -|35| &= -(35) && \text{as the absolute value of thirty-five is thirty-five,} \\ &= -35 && \text{as the opposite of thirty-five is negative thirty-five.} \end{aligned}$$

Thus, $-|35| = \mathbf{-35}$.

5. $|-0|$ *is the absolute value of the opposite of zero.*

$$\begin{aligned} |-0| &= |0| && \text{as the opposite of zero is zero,} \\ &= 0 && \text{as the absolute value of zero is zero.} \end{aligned}$$

Thus, $|-0| = \mathbf{0}$.

6. *$-|0|$ is the opposite of the absolute value of zero.*

$$\begin{aligned} -|0| &= -(0) && \textit{as the absolute value of zero is zero,} \\ &= 0 && \textit{as the opposite of zero is zero.} \end{aligned}$$

Thus, $-|0| = \mathbf{0}$.

You may also wish to compare numbers which involve *taking the opposite* or *absolute value*. In this case, you should always simplify the numbers before trying to compare them.

Example 15. *Insert a $\leq$ or $>$ between each of the following pair of numbers to make a true statement.*

1. $-(-2)$ ___ $|-5|$
2. $-(-3)$ ___ $-|7|$
3. $|-8|$ ___ $|-(-8)|$
4. $-(4)$ ___ $-(-4)$
5. -0 ___ $|0|$

Solution 15. *Simplifying the numbers first gives:*

1. $-(-2)$ ___ $|-5|$
   Left side: $-(-2) = 2$.
   Right side: $|-5| = 5$.
   Since $2 \leq 5$, *we have* $-(-\mathbf{2}) \ \leq \ |-\mathbf{5}|$.
2. $-(-3)$ ___ $-|7|$
   Left side: $-(-3) = 3$.
   Right side: $-|7| = -(7) = -7$.
   Since $3 > -7$, *we have* $-(-\mathbf{3}) \ > \ -|\mathbf{7}|$.
3. $|-8|$ ___ $|-(-8)|$
   Left side: $|-8| = 8$.
   Right side: $|-(-8)| = |8| = 8$.
   Since $8 \leq 8$ *(as equality is included with the $\leq$ sign), we have* $|-\mathbf{8}| \ \leq \ |-(-\mathbf{8})|$.

4. $-(4)$ ___ $-(-4)$
   Left side: $-(4) = -4$.
   Right side: $-(-4) = 4$.
   Since $-4 \leq 4$, *we have* $-(\mathbf{4}) \leq -(-\mathbf{4})$.
5. -0 ___ $|0|$
   Left side: $-0 = 0$.
   Right side: $|0| = 0$.
   Since $0 \leq 0$ *(as equality is included with the* $\leq$ *sign), we have* $-\mathbf{0} \leq |\mathbf{0}|$.

SECTION 1.2 EXERCISES

(Answers are found on page 369.)

Insert a $\leq$ *or* $>$ *between each of the following pairs of numbers to make a true statement.*

1. -8 ___ 4
2. 22 ___ 0
3. -5 ___ -5
4. -18 ___ -7
5. 56 ___ 43
6. -19 ___ -19
7. 33 ___ 54
8. -14 ___ -4
9. 197 ___ 197
10. -3 ___ -5

Insert a $<$ *or* $\geq$ *between each of the following pairs of numbers to make a true statement.*

11. 12 ___ -12
12. 7 ___ 7
13. -3 ___ -9
14. 15 ___ 13
15. 0 ___ 10
16. -11 ___ 1
17. 25 ___ 52
18. 34 ___ 34
19. -9 ___ -10
20. -2 ___ -1

Find the absolute value of the following numbers.

21. 206
22. -34
23. -16
24. 0
25. 28
26. 17
27. 45
28. -91
29. 73
30. -22

Find the opposite of the following numbers.

31. 52
32. -17
33. -8
34. 0
35. 102
36. -1
37. -13
38. 37
39. 22
40. -9

Simplify.

41. $|-2|$
42. $-(-2)$
43. $-(9)$
44. $-|-7|$
45. $-|5|$
46. $|-(-3)|$
47. $|-4|$
48. $-(18)$
49. $-|-6|$
50. $-(-32)$

Insert a $\leq$ or $>$ between each of the following pairs of numbers to make a true statement.

51. $-(-5)$ ___ $-|-5|$
52. $|-3|$ ___ $|-4|$
53. -0 ___ $|0|$
54. $-(-8)$ ___ -7
55. $-|-6|$ ___ $-|6|$
56. $|-12|$ ___ $-|12|$
57. $|-3|$ ___ $|3|$
58. $-(-23)$ ___ $-(-20)$
59. $|0|$ ___ $|-1|$
60. $-|13|$ ___ $-|-2|$

Insert a $<$ or $\geq$ between each of the following pairs of numbers to make a true statement.

61. $-(-14)$ ___ -12
62. $|-45|$ ___ $-(-45)$
63. -0 ___ $|-0|$
64. 17 ___ $|-17|$
65. $-|-8|$ ___ $|-(-8)|$
66. -1 ___ $-(-1)$
67. $|28|$ ___ $|-32|$
68. $|-12|$ ___ $|-48|$
69. $-(-11)$ ___ $-(-10)$
70. $|0|$ ___ $|-109|$

1.3 Integer Addition and Subtraction

Integer Addition - an easy method?

Instead of always adding two positive numbers as you may be more familiar with, we now want to involve negative integers as well. Thus, we want to be able to add quantities like:
$4+7$ (this is good old-fashioned addition),
$-3+4$,
$5+(-8)$, or
$-2+(-9)$.

Note that when the number after the plus sign is a negative number, we will place this number in parentheses. This will be true whenever a negative number follows a $+, -, \cdot$, or $\div$ as we will see later. So, how can we do such addition?

One method for adding integers involves using a number line. To use this method, first you need a (sufficiently long) number line. You may refer back to the number line we presented back on page 22, or you may draw your own. To add two numbers, treat the first number as the starting point on the number line, and then use the following rule:

When you are adding a:

1. **positive number - move (step) to the right on the number line.**
2. **negative number - move (step) to the left on the number line.**

So, for example, to add $-5+4$, we will start at negative five on the number line, and take four steps to the right. Try this on your number line, and you should get an answer of $-5+4=\mathbf{-1}$. While this method is easy to do, it does have some problems. First, your number line must be long enough that both the numbers being added (the addends) and the answer (the sum) all appear on it - try adding $53+28$ with a number line, for example! Next, while this method helps you to visualize how to add integers, it does not work nearly as well once we start using fractions and decimals. Therefore, we would rather encourage you to memorize some simple rules which work well for integers and fractions and decimals. However, this number line method is good if you are having trouble right from the start, and it may be used as a rough check of your answer, even when we get to fractions and decimals.

To do the rough check, just use the above rules and the commutative property of addition to get bounds for your answer. To illustrate all of this, let us look at the four examples we presented at the beginning.

Example 1. *Add* $4 + 7$*, and do a rough check of your answer.*

> **Solution 1.** *This is old-fashioned addition, so no special rules should be needed, but we will practice using a number line. Start at positive four and take seven steps to the right (as we are adding a positive number). You should get that:*
> $4 + 7 = \mathbf{11}$.
> *Rough check:*
> *Written as* $4 + 7$*, we are starting at positive four and moving to the right, so the answer should be greater than four. Is* $11 > 4$*? Yes,* ✓.
> *Written as* $7+4$*, we are starting at positive seven and moving to the right, so the answer should be greater than seven. Is* $11 > 7$*? Yes,* ✓.

We still recommend a true check of your work, if there is time, but you may find the rough check useful as well. While the rough check looks like a lot of work, the hope is that as you practice more and more of these problems, you will be able to do this quickly in your head. In this case, with both addends being positive, we were always adding a positive number (and so moving to the right on the number line). This was the basis for the old adage, "when you add two numbers, you can't get less than what you started with." While this was true back when you were learning old-fashioned addition, we will soon see that when one of the addends is negative, it will no longer be true.

Example 2. *Add* $-3 + 4$*, and do a rough check of your answer.*

Solution 2. *Start at negative three and take four steps to the right. You should get that:*
$-3 + 4 = \mathbf{1}$.
Rough check:
Written as $-3 + 4$, *we are starting at negative three and moving to the right, so the answer should be greater than negative three. Is* $1 > -3$*? Yes,* ✓.
Written as $4 + (-3)$, *we are starting at positive four and moving to the left (as here we are adding a negative number), so the answer should be less than four. Is* $1 < 4$*? Yes,* ✓.

Example 3. *Add* $5 + (-8)$, *and do a rough check of your answer.*

Solution 3. *Start at positive five and take eight steps to the left. You should get that:*
$5 + (-8) = \mathbf{-3}$.
Rough check:
Written as $5 + (-8)$, *we are starting at positive five and moving to the left, so the answer should be less than five. Is* $-3 < 5$*? Yes,* ✓.
Written as $-8 + 5$, *we are starting at negative eight and moving to the right, so the answer should be greater than negative eight. Is* $-3 > -8$*? Yes,* ✓.

Notice in the last two examples that when we are adding a positive number to a negative number, the sum may be positive or it may be negative.

Example 4. *Add* $-2 + (-9)$, *and do a rough check of your answer.*

Solution 4. *Start at negative two and take nine steps to the left. You should get that:*
$-2 + (-9) = \mathbf{-11}$.
Rough check:

Written as $-2+(-9)$, we are starting at negative two and moving to the left, so the answer should be less than negative two. Is $-11 < -2$? Yes, ✓.
Written as $-9+(-2)$, we are starting at negative nine and moving to the left, so the answer should be less than negative nine. Is $-11 < -9$? Yes, ✓.

Integer Addition - the rules

To add two integers, you may do as is indicated in the following rules. We encourage you to memorize these rules *as soon as possible*, as next section we will be presenting different rules for multiplication, and it is very easy to mix the two up if you have not learned them both well.

Rules for adding two integers

To add two integers, determine the sign of the two numbers being added (the addends).

- If one or both of the addends are zero, the addition is trivially easy as any number plus zero is itself.
- If the addends have the same sign (both positive or both negative):
  1. Take the absolute value of both addends.
  2. Add the two positive numbers like normal (i.e. do old-fashioned addition).
  3. The sum should have the same sign as the two addends originally had.
- If the addends have the opposite sign (one positive and one negative number):
  1. Take the absolute value of both addends.
  2. Subtract the smaller number from the bigger (i.e. do old-fashioned subtraction).
  3. The answer should have the same sign as the original sign of the addend which was larger in absolute value (the number you subtracted from in your scratch work).

Let us start by redoing the first four examples using these rules.

Example 5. *Add* $4 + 7$.

Solution 5. *This is old-fashioned addition, so no special rules should be needed, but we will practice using the new ones. Cover up the* $+$ *sign with your finger to see that we are adding two positive numbers (positive four and positive seven). So, the rules say:*

1. *Take the absolute value of both numbers so we have positive four and positive seven.*
2. *Add these numbers like normal.*
   Scratch work:
   $4 + 7 = 11$.
3. *The sum should have the same sign as the two addends had originally* $\longrightarrow$ *positive.*

Therefore, $4 + 7 = \mathbf{11}$.
Rough check:
Written as $4 + 7$*, we are starting at positive four and moving to the right, so the answer should be greater than four. Is* $11 > 4$*? Yes,* $\checkmark$.
Written as $7+4$*, we are starting at positive seven and moving to the right, so the answer should be greater than seven. Is* $11 > 7$*? Yes,* $\checkmark$.

Example 6. *Add* $-3 + 4$.

Solution 6. *Looking at the two addends, we see that one is negative and one is positive. In this case the rules state:*

1. *Take the absolute value of both numbers so we have positive three and positive four.*

2. *Subtract bigger minus smaller.*
   Scratch work:
   $4 - 3 = 1$.

3. *The sum should have the same sign as the addend which was larger in absolute value. Well,* $4 > 3$ *and the four's original sign was positive, so the sum will be positive.*

Therefore, $-3 + 4 = \mathbf{1}$.
Rough check:
Written as $-3+4$*, we are starting at negative three and moving to the right, so the answer should be greater than negative three. Is* $1 > -3$*? Yes,* $\checkmark$.
Written as $4+(-3)$*, we are starting at positive four and moving to the left (as here we are adding a negative number), so the answer should be less than four. Is* $1 < 4$*? Yes,* $\checkmark$.

Example 7. *Add* $5 + (-8)$.

Solution 7. *Looking at the two addends, we see that one is positive and one is negative. In this case the rules state:*

1. *Take the absolute value of both numbers so we have positive five and positive eight.*
2. *Subtract bigger minus smaller.*
   Scratch work:
   $8 - 5 = 3$.

3. *The sum should have the same sign as the addend which was larger in absolute value. Well,* $8 > 5$ *and the eight's original sign was negative, so the sum will be negative.*

Therefore, $5 + (-8) = -\mathbf{3}$.
Rough check:
Written as $5+(-8)$*, we are starting at positive five and moving to the left, so the answer should be less than positive five. Is* $-3 < 5$*? Yes,* $\checkmark$.

Written as $-8+5$*, we are starting at negative eight and moving to the right, so the answer should be greater than negative eight. Is* $-3 > -8$*? Yes,* ✓.

Example 8. *Add* $-2+(-9)$.

Solution 8. *Both the numbers being added are negative, so the rules state:*

1. *Take the absolute value of both numbers so we have positive two and positive nine.*
2. *Add these numbers like normal.*
   Scratch work:
   $2+9=11$.
3. *The sum should have the same sign as the two addends had originally* $\longrightarrow$ *negative.*

Therefore, $-2+(-9) = \mathbf{-11}$.
Rough check:
Written as $-2+(-9)$*, we are starting at negative two and moving to the left, so the answer should be less than negative two. Is* $-11 < -2$*? Yes,* ✓.
Written as $-9+(-2)$*, we are starting at negative nine and moving to the left, so the answer should be less than negative nine. Is* $-11 < -9$*? Yes,* ✓.

We want to mention again that you should keep your scratch work separate from your final answer, as the scratch work will always give a positive result (or possibly zero), and then you may need to adjust the sign for the final answer. This will be even more apparent when the addition involves larger numbers which you will need to add using the vertical tower method.

Example 9. *Add* $247+(-195)$.

Solution 9. *We are adding one number of each sign, so the rules state:*

1. *Take the absolute value of both numbers so we have positive* 247 *and positive* 195.
2. *Subtract larger minus smaller.*
   Scratch work:

$$\begin{array}{cccc} & 2 & 4 & 7 \\ - & 1 & 9 & 5 \\ \hline & & & 2 \end{array} \Rightarrow \begin{array}{cccc} & 1 & 14 & \\ & \cancel{2} & \cancel{4} & 7 \\ - & 1 & 9 & 5 \\ \hline & & 5 & 2 \end{array}$$

3. *The larger number in absolute value is the* 247 *(*$247 > 195$*), and it was originally positive, so the sum is positive.*

Therefore, $247 + (-195) = \mathbf{52}$.
Rough check:
Written as $247 + (-195)$*, we are starting at positive* 247 *and moving to the left, so the answer should be less than* 247*. Is* $52 < 247$*? Yes,* ✓.
Written as $-195+247$*, we are starting at* -195 *and moving to the right, so the answer should be greater than* -195*. Is* $52 > -195$*? Yes,* ✓.

Example 10. *Add* $-341 + (-894)$.

Solution 10. *Both the numbers being added are negative, so the rules state:*

1. *Take the absolute value of both numbers so we have* 341 *and* 894.
2. *Add these numbers like normal.*
   Scratch work:

$$\begin{array}{ccccc} & & 1 & & \\ & & 3 & 4 & 1 \\ + & & 8 & 9 & 4 \\ \hline & 1 & 2 & 3 & 5 \end{array}$$

3. *The sum should have the same sign as the two addends had originally* $\longrightarrow$ *negative.*

Therefore, $-341 + (-894) = \mathbf{-1,235}$.
Rough check:
Written as $-341 + (-894)$*, we are starting at* -341 *and moving to the left, so the answer should be less than* -341*. Is* $-1235 < -341$*? Yes,* ✓.
Written as $-894 + (-341)$*, we are starting at* -894 *and moving to the left, so the answer should be less than* -894*. Is* $-1235 < -894$*? Yes,* ✓.

Example 11. *Add* $48 + (-573)$.

Solution 11. *We are adding one number of each sign, so the rules state:*

1. *Take the absolute value of both numbers so we have* 48 *and positive* 573.
2. *Subtract larger minus smaller.*
   Scratch work:

$$\begin{array}{rccc} & 5 & 7 & 3 \\ - & & 4 & 8 \\ \hline \end{array} \Rightarrow \begin{array}{rccc} & & 6 & 13 \\ & 5 & \cancel{7} & \cancel{3} \\ - & & 4 & 8 \\ \hline & 5 & 2 & 5 \end{array}$$

3. *The larger number in absolute value is the* 573 *(*$573 > 48$*), and it was originally negative, so the sum is negative.*

Therefore, $48 + (-573) = \mathbf{-525}$.
Rough check:
Written as $48 + (-573)$*, we are starting at* 48 *and moving to the left, so the answer should be less than* 48*. Is* $-525 < 48$*? Yes,*

$\checkmark$.
Written as $-573 + 48$, we are starting at -573 and moving to the right, so the answer should be greater than -573. Is $-525 > -573$? Yes, $\checkmark$.

Example 12. *Add $-97 + (-266)$.*

Solution 12. *Both the numbers being added are negative, so the rules state:*

1. *Take the absolute value of both numbers so we have 97 and 266.*
2. *Add these numbers like normal.*
   Scratch work:

$$\begin{array}{rccc} & 1 & 1 & \\ & 2 & 6 & 6 \\ + & & 9 & 7 \\ \hline & 3 & 6 & 3 \end{array}$$

3. *The sum should have the same sign as the two addends had originally $\longrightarrow$ negative.*

Therefore, $-97 + (-266) = \mathbf{-363}$.
Rough check:
Written as $-97+(-266)$, we are starting at -97 and moving to the left, so the answer should be less than -97. Is $-363 < -97$? Yes, $\checkmark$.
Written as $-266 + (-97)$, we are starting at -266 and moving to the left, so the answer should be less than -266. Is $-363 < -266$? Yes, $\checkmark$.

Integer Subtraction - the rules

You could memorize a whole new list of rules similar to the rules for integer addition on page 39 to do integer subtraction, but this tends to get tedious. Instead, we will present the following simple rule which lets you change subtraction to addition.

Rule to change subtraction to addition:
You may ALWAYS change subtraction to addition by completing the following two steps.

1. Change the minus sign to a plus sign.
2. Change the number which used to follow the minus sign to its opposite.

Let us look at an example.

Example 13. *Change the following subtraction problems into addition problems. You do not need to do the actual subtraction or addition yet.*

1. $8 - 3$
2. $-5 - 19$
3. $4 - (-13)$
4. $-8 - (-16)$
5. $7 - 0$
6. $5 - 42$

Solution 13. *Applying the above rule to the subtraction problems gives:*

1. $8 - 3 = \mathbf{8} + (-\mathbf{3})$.
2. $-5 - 19 = -\mathbf{5} + (-\mathbf{19})$.
3. $4 - (-13) = \mathbf{4} + \mathbf{13}$.
4. $-8 - (-16) = -\mathbf{8} + \mathbf{16}$.

5. $7 - 0 = \mathbf{7} + \mathbf{0}$.
6. $5 - 42 = \mathbf{5} + (-\mathbf{42})$.

Notice that the number in front of the minus sign does not change sign, only the number after it. Also, you probably will not want to change from subtraction to addition if the subtraction is straightforward (old-fashioned), only when the subtraction involves one or two negative numbers or two nonnegative numbers being subtracted smaller minus larger. Let us see some subtraction examples.

Example 14. *Subtract* $8 - 3$.

> **Solution 14.** *This is old-fashioned subtraction so we do not need to change to addition.*
> *Therefore,* $8 - 3 = \mathbf{5}$.

Example 15. *Subtract* $-5 - 19$.

> **Solution 15.** *The first number is negative, so this is NOT good old-fashioned subtraction. Let us change to addition and apply the addition rules.*
> $-5 - 19 = -5 + (-19)$.
> *Both numbers in the addition are negative, so the rules state:*
>
> 1. *Take the absolute value of both numbers to get* 5 *and* 19.
> 2. *Add like normal. Scratch work:*
> $5 + 19 = 24$.
> 3. *The sum has the same sign as the two addends did originally* $\longrightarrow$ *negative.*
>
> *Therefore,* $-5 - 19 = -\mathbf{24}$.

Example 16. *Subtract* $4 - (-13)$.

Solution 16. *Not old-fashioned subtraction so let us change to addition to solve.*
$4 - (-13) = 4 + 13$.
This is old-fashioned addition, so we do not need to refer back to the new addition rules, $4 + 13 = 17$.
Therefore, $4 - (-13) = \mathbf{17}$.

Example 17. *Subtract* $-8 - (-16)$.

Solution 17. *Not old-fashioned subtraction, so change to addition.*
$-8 - (-16) = -8 + 16$.
We are adding one of each sign so:

1. *Taking absolute value of both numbers gives us* 8 *and* 16.
2. *Subtracting bigger minus smaller gives* $16 - 8 = 8$.
3. *The larger of the two numbers in absolute value is the* 16, *and the* 16 *was originally positive (make sure you go back to the addition problem - NOT all the way back to the subtraction problem!!), so the sum should be positive.*

Therefore, $-8 - (-16) = \mathbf{8}$.

Example 18. *Subtract* $7 - 0$.

Solution 18. *Old-fashioned subtraction - nothing fancy needs to be done.*
Therefore, $7 - 0 = \mathbf{7}$.

Example 19. *Subtract* $5 - 42$.

Solution 19. *Both numbers are positive, but the smaller number is in front, so this is not old-fashioned subtraction. Changing to addition gives:*
$5 - 42 = 5 + (-42)$.
We are adding a positive number to a negative number, so the addition rules state:

1. *Take the absolute value of both numbers to get* 5 *and* 42.
2. *Subtract larger minus smaller* $\longrightarrow 42 - 5 = 37$.
3. *The larger of the two numbers in absolute value is the* 42, *and it was originally negative (again, going back to the addition problem, not to the original subtraction one), so the answer should be negative.*

Therefore, $5 - 42 = -\mathbf{37}$.

Note that we did not do even a rough check in the last examples. If the subtraction was changed to addition, the rough check can still be made, same as before. If it was old-fashioned subtraction and you do not change to addition, the same rough check can not be made (subtraction is not commutative like addition), but you should have a good feel for what kind of answers you can get - the answer must be positive (or zero) and less than or equal to the number in front of the minus sign. We encourage you to go back to the last example and roughly check all the solutions to make sure they are reasonable.

Take your time, learn the rules, and do plenty of problems for practice. We will end with a few more examples, some of which may require more scratch work than the examples we have done so far.

Example 20. *Subtract* $-438 - 256$.

Solution 20. *Not old-fashioned subtraction, so we will start by changing to addition.*
$-438 - 256 = -438 + (-256)$.
We are adding two negative numbers so the addition rules state:

1. *Take the absolute value of both numbers to get* 438 *and* 256.
2. *Add like normal. Scratch work:*

$$\begin{array}{cccc} & & 1 & \\ & 4 & 3 & 8 \\ + & 2 & 5 & 6 \\ \hline & 6 & 9 & 4 \end{array}$$

3. *The sum should have the same sign as the two addends did originally* $\longrightarrow$ *negative.*

So, $-438 - 256 = -\mathbf{694}$
Rough check:
Written as $-438 + (-256)$, *we are starting at* -438 *and moving to the left, so the answer should be less than* -438. *Is* $-694 < -438$? *Yes,* ✓.
Written as $-256 + (-438)$, *we are starting at* -256 *and moving to the left, so the answer should be less than* -256. *Is* $-694 < -256$? *Yes,* ✓.

Example 21. *Subtract* $698 - 3,021$.

Solution 21. *Both numbers are positive, but the smaller one is in front of the minus sign - not old-fashioned subtraction. Changing to addition gives:*
$698 - 3021 = 698 + (-3021)$.
To add one positive to one negative, the addition rules state:

1. *Take the absolute value of both numbers to get* 698 *and* 3021.
2. *Subtract bigger minus smaller. Scratch work:*

$$\begin{array}{rcccc} & & & 1 & 11 \\ & 3 & 0 & \cancel{2} & \cancel{1} \\ - & & 6 & 9 & 8 \\ \hline & & & & 3 \end{array} \Rightarrow \begin{array}{rcccc} & 2 & 10 & 1 & 11 \\ & \cancel{3} & \cancel{0} & \cancel{2} & \cancel{1} \\ - & & 6 & 9 & 8 \\ \hline & & & & 3 \end{array} \Rightarrow \begin{array}{rcccc} & & 9 & 11 & \\ & 2 & \cancel{10} & \cancel{1} & 11 \\ & \cancel{3} & \cancel{0} & \cancel{2} & \cancel{1} \\ - & & 6 & 9 & 8 \\ \hline & 2 & 3 & 2 & 3 \end{array}$$

3. *The larger of the two numbers in absolute value is the* 3021*, and it was negative in the addition problem, so the answer should be negative.*

Thus, $698 - 3,021 = \mathbf{-2,323}$.
Rough check:
Written as $698 + (-3021)$*, we are starting at* 698 *and moving to the left, so the answer should be less than* 698*. Is* $-2323 < 698$*? Yes,* ✓.
Written as $-3021 + 698$*, we are starting at* -3021 *and moving to the right, so the answer should be greater than* -3021*. Is* $-2323 > -3021$*? Yes,* ✓.

Example 22. *Subtract* $421 - 354$.

Solution 22. *This is old-fashioned subtraction (both positive numbers and the larger in front of the minus sign), so no special rules are needed.*
Scratch work:

$$\begin{array}{rccc} & & 1 & 11 \\ & 4 & \cancel{2} & \cancel{1} \\ - & 3 & 5 & 4 \\ \hline & & & 7 \end{array} \Rightarrow \begin{array}{rccc} & & 11 & \\ & 3 & \cancel{1} & 11 \\ & \cancel{4} & \cancel{2} & \cancel{1} \\ - & 3 & 5 & 4 \\ \hline & & 6 & 7 \end{array}$$

So, $421 - 354 = \mathbf{67}$.

Rough check:
Since we never changed the subtraction problem to an addition

problem, we can not do the same rough check (unless you wish to change it to addition now). Instead, for old-fashioned subtraction the answer should be nonnegative, $67 \geq 0 \quad \checkmark$*; and less than or equal to* 421*,* $67 \leq 421 \quad \checkmark$*.*

Example 23. *Subtract* $0 - 302$.

Solution 23. *Both numbers are nonnegative, but the smaller number is first, so this is not old-fashioned subtraction. Thus, we will start by changing this to an addition problem.*
$0 - 302 = 0 + (-302)$.
Addition with zero is trivial,
$0 + (-302) = -302$.
Thus, $0 - 302 = \mathbf{-302}$.

Learning and being able to apply the addition and subtraction rules is absolutely necessary to do well in this course. The following shortcuts MAY be used if desired, or you may just do as in the previous examples.

<u>Subtraction shortcut 1:</u> Minus a negative is the same as plus a positive.

Many students will get into a habit of just crossing the minus signs to make plus signs in this instance. For example:

Example 24. *Subtract* $3 - (-18)$.

Solution 24. *Using the shortcut, we get:*

$$\begin{aligned} 3 - (-18) &= 3 + (+18) \\ &= 3 + 18 \\ &= 21. \end{aligned}$$

So, $3 - (-18) = \mathbf{21}$.

You could also redo examples 16 and 17 using this shortcut.

<u>Subtraction shortcut 2:</u> When you are subtracting two nonnegative integers, but the smaller number is first, subtract the other way, and just make your answer negative.

Example 25. *Subtract* $5 - 16$.

> **Solution 25.** *Using the new shortcut, we will subtract the other way (making it an old-fashioned subtraction problem):*
> $16 - 5 = 11$.
> *And now we just need to make the answer negative.*
> *Thus,* $5 - 16 = \mathbf{-11}$.

You could also redo examples 19, 21, and 23 using this shortcut.

SECTION 1.3 EXERCISES

(Answers are found on page 370.)

Add or subtract as indicated.

1. $5 + (-8)$
2. $-14 + (-7)$
3. $-3 + 9$
4. $-18 + (-11)$
5. $6 + (-1)$
6. $2 - 8$
7. $-11 - 3$
8. $7 - (-5)$
9. $-4 - (-5)$
10. $17 - (-2)$
11. $54 - 18$
12. $-36 + (-22)$
13. $-125 + 67$
14. $-82 - (-19)$
15. $14 - (-18)$
16. $63 + (-19)$

17. $0 - 15$
18. $52 - 65$
19. $-21 - 15$
20. $-16 - (-34)$
21. $-111 + 74$
22. $15 - 25$
23. $-23 - 32$
24. $91 + (-3)$
25. $-18 + (-17)$
26. $71 - 39$
27. $-11 + (-22)$
28. $-2 - (-37)$
29. $123 - 138$
30. $-84 + 82$
31. $-20 + (-16)$
32. $11 - (-5)$
33. $-34 - 30$
34. $27 + (-24)$
35. $8 - 12$
36. $83 - 100$
37. $74 - 24$
38. $-12 + 33$
39. $-49 + 15$
40. $-7 - 4$

1.4 Integer Multiplication and Division

The rules

To multiply or divide two integers together, apply the following rules.

Rules for multiplying or dividing two integers

The basic rule is the same regardless of whether we are multiplying or dividing. The only deviation arises when zero is involved.

- If you are multiplying two integers and one of the factors is zero, the answer (product) is **0**.
- If you are dividing zero by anything other than itself, the answer is **0** (see page 15).
- If you are dividing by zero, the answer is **undefined** (see page 16).
- Otherwise:
  1. Take the absolute value of both numbers, and multiply or divide as normal, using the tower method or long division if necessary.
  2. The sign of the answer is:
     (a) positive if the two numbers being multiplied or divided have the same signs (i.e. both positive or both negative).
     (b) negative if the two numbers being multiplied or divided have opposite signs (i.e. one positive and one negative).

Note that determining the proper numerical answer (step 1), and determining the proper sign (step 2) are totally unconnected. If you would prefer, you may switch the order, and figure out the answer's sign first; we will show examples both ways. This was also true to some extent with the new addition rules, although whether your scratch work involved addition or subtraction depended upon the signs of the addends.

Also notice that there truly is a difference between these multiplication and division rules and the addition rules of last section. For a simple example, recall that two negative numbers being *added* yield a negative answer (adding two numbers of the same sign, the answer has the common sign), while two negative numbers being multiplied or divided yield a positive answer (see above rule).

Example 1. *Multiply* $(-3)(5)$.

Solution 1. *To multiply, the rules say:*

1. *Take the absolute value of each number to get* 3 *and* 5*, and* $3 \cdot 5 = 15$.
2. *There was originally one negative factor and one positive factor, so the answer should be negative.*

So, $(-3)(5) = \mathbf{-15}$.

Example 2. *Multiply* $(4)(-7)$.

Solution 2. *To multiply, the rules state:*

1. *Take the absolute value of each number to get* 4 *and* 7*, and* $4 \cdot 7 = 28$.
2. *There was one positive factor and one negative factor, so the answer should be negative.*

So, $(4)(-7) = \mathbf{-28}$.

Example 3. *Multiply* $(0)(-2)$.

Solution 3. *We are multiplying and zero is one of the factors so the answer is* 0.
Thus, $(0)(-2) = \mathbf{0}$.

Example 4. *Divide* $(-48) \div (-6)$.

Solution 4. *Following our new rules, we get:*

1. *The absolute values are* 48 *and* 6, *and* $48 \div 6 = 8$.
2. *There were two negative numbers in the division problem, so the answer should be positive.*

So, $(-48) \div (-6) = \mathbf{8}$.

Example 5. *Divide* $(-212) \div (0)$.

Solution 5. *Division by zero is always undefined. Therefore,* $(-212) \div (0) =$ ***undefined****.*

Now we will do some examples where the scratch work is more difficult, and we will also demonstrate figuring out the sign of the answer first, in case you decide you would prefer that instead.

Example 6. *Multiply* $(-38)(-25)$.

Solution 6. *We will choose to find the sign of our answer first this time. By the sign formula (step 2) in our new multiplication rules, we know that:*
$(-38)(-25) = +\_\_\_$,
where the "+" sign emphasizes that our answer will be positive. Recall that in our final answer, we will not put a "+" sign to show a positive number. Now that we have the sign figured out, we can concentrate on finding the proper numerical answer.
Taking the absolute value of both numbers, we multiply:

$$\begin{array}{rrr} & 4 & \\ & 3 & 8 \\ \times & 2 & 5 \\ \hline 1 & 9 & 0 \end{array} \Rightarrow \begin{array}{rrrr} & & 1 & \\ & & 3 & 8 \\ \times & & 2 & 5 \\ \hline & 1 & 9 & 0 \\ & 7 & 6 & 0 \\ \hline & 9 & 5 & 0 \end{array}$$

Therefore, $(-38)(-25) = \mathbf{950}$.

Example 7. *Multiply* $34 \cdot 78$.

Solution 7. *You should recognize this as old-fashioned multiplication, and similar to the new addition rules, you do not need to use the new rules when they are not required. All we need to do is use the tower method to multiply the two numbers:*

$$\begin{array}{rrr} & 3 & \\ & 3 & 4 \\ \times & 7 & 8 \\ \hline 2 & 7 & 2 \end{array} \Rightarrow \begin{array}{rrrr} & & 2 & \\ & & 3 & 4 \\ \times & & 7 & 8 \\ \hline & 2 & 7 & 2 \\ 2 & 3 & 8 & 0 \\ \hline 2 & 6 & 5 & 2 \end{array}$$

Thus, $34 \cdot 78 = \mathbf{2,652}$

Example 8. *Divide* $-273 \div 13$.

Solution 8. *Following the new rules:*

1. *Taking the absolute values of both numbers gives* 273 *and* 13*. To divide, we will use long division.*

Scratch work:

$$\begin{array}{r|l} & 21 \\ \cline{2-2} 13 & 273 \\ & \underline{26} \\ & 13 \\ & \underline{13} \\ & 0 \end{array}$$

2. *Since we have one negative number and one positive number in the division, the answer should be negative.*

Therefore, $-273 \div 13 = -\mathbf{21}$.

Example 9. *Divide* $1715 \div (-35)$.

Solution 9. *We will find the sign of the answer first again in this example.*

1. *Since we have one positive number and one negative number in the division, the answer should be negative. So we now know:*
   $1715 \div (-35) = -\_\_\_$.
2. *Taking the absolute values of both numbers gives* 1715 *and* 35*. To divide, we will use long division.*
   Scratch work:

$$\begin{array}{r|l} & 49 \\ \cline{2-2} 35 & 1715 \\ & \underline{140} \\ & 315 \\ & \underline{315} \\ & 0 \end{array}$$

Therefore, $1715 \div (-35) = -\mathbf{49}$.

After you do several problems, you will probably decide whether you prefer to find the sign first or last, and stick with the one method.

Important Note on Finding the Opposite

We have already seen and discussed the operation, *take the opposite of*, but now we may add the following fact.

> • **The operation of *take the opposite of* is always equivalent to multiplying by** -1.

Refer back to example 13 on page 29. We can now redo this example using this new concept, and our integer multiplication rules.

Example 10. *Simplify.*

1. $-(13)$
2. $-(-6)$
3. $-(-87)$
4. $-(32)$
5. -0

Solution 10. *The solutions are:*

1. $-(13) = -1 \cdot 13 = \mathbf{-13}$.
2. $-(-6) = -1 \cdot (-6) = \mathbf{6}$.
3. $-(-87) = -1 \cdot (-87) = \mathbf{87}$.
4. $-(32) = -1 \cdot 32 = \mathbf{-32}$.
5. $-0 = -1 \cdot 0 = \mathbf{0}$.

SECTION 1.4 EXERCISES

(Answers are found on page 371.)

Multiply as indicated.

1. $(-4)(7)$
2. $(-8)(-6)$
3. $(-10)(22)$
4. $(6)(-11)$
5. $(0)(-18)$
6. $(-9)(-2)$
7. $(-17)(-24)$
8. $(15)(-6)$
9. $(-1)(-5)$
10. $(-1)(8)$
11. $(14)(6)$
12. $(-5)(-4)$
13. $(-16)(0)$
14. $(-8)(-9)$
15. $(2)(-34)$
16. $(6)(-7)$
17. $(-8)(-11)$
18. $(-35)(62)$
19. $(21)(-33)$
20. $(-6)(1)$

Divide as indicated.

21. $-25 \div 5$
22. $-32 \div (-8)$
23. $18 \div (-6)$
24. $180 \div (-1)$
25. $0 \div (-2)$
26. $-9 \div (-3)$
27. $-6 \div (-6)$
28. $-22 \div 0$
29. $14 \div (-2)$
30. $-14 \div (-7)$
31. $-345 \div 15$
32. $228 \div (-12)$
33. $340 \div 0$
34. $-144 \div (-16)$
35. $924 \div (-22)$
36. $0 \div (-28)$
37. $-306 \div (-17)$
38. $81 \div (-9)$
39. $0 \div 0$
40. $-350 \div (-25)$

1.5 Grouping Symbols, Exponents and Order of Operations

Road Rules

Three cars approach a four-way stop from three different directions at the same time, who goes first? As you approach a traffic light, you notice that the power is out and the light is not working, what is the proper procedure? When, if ever, is it legal to make a left on red? The answers to these questions and many others are supposed to be known by motorists to help them navigate the roadways and avoid accidents. Mathematics has a similar set of rules which must be followed to avoid incorrect solutions, and it is called the order of operations.

Example 1. *Robin and John wish to simplify the following expression.*

$$3 + 5 \cdot 7$$

Robin decides to do the operations in the order they appear, from left to right, so he finds:

$$\begin{aligned} 3 + 5 \cdot 7 &= 8 \cdot 7 \\ &= 56. \end{aligned}$$

John, meanwhile, likes multiplication better than addition, so he chooses to multiply first:

$$\begin{aligned} 3 + 5 \cdot 7 &= 3 + 35 \\ &= 38. \end{aligned}$$

Obviously both answers cannot be correct, so who is right?

> **Solution 1.** *Many people's intuition tells them that Robin is correct, but this is NOT the case. John did the problem correctly.*

The above example certainly is not obvious, and it can only be done correctly once you know the order of operations. A mnemonic which may be useful when memorizing the order of operations is **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally. The first letters of the words of the mnemonic represent:

Order of Operations

When more than one operation is present in an algebraic expression, the proper order for doing the operations is:

1. **P**arentheses - or any grouping symbol
2. **E**xponents
3. **M**ultiplication and **D**ivision, left to right
4. **A**ddition and **S**ubtraction, left to right

We will look at each of these steps separately, and then we will do many examples combining everything.

Parentheses and other grouping symbols

The first step really should be labeled grouping symbols, but it seems that no one has come up with a cute mnemonic beginning with GS instead of P. There are four separate grouping symbols which you may encounter: (parentheses), [brackets], {braces} and |absolute value|.

Parentheses are the most basic, and possibly the most common, grouping symbol. They simply highlight where you should start. For example, in the expression:

$$5-(4-3)$$

the parentheses indicate that you should subtract the three from the four first to get:

$$5-(1)$$

If there are multiple operations inside the parentheses, you need to apply the order of operations again. For example,

$$5-(2\cdot 2-3)=5-(4-3)=5-(1)$$

as inside the parentheses there is both multiplication and division, and order of operations states that multiplication comes first. We will see more examples of this later.

Once you are down to a single number, as above, you no longer need to consider the parentheses as grouping symbols (as nothing is being grouped like it was before). Sometimes, particularly if the number inside is positive, you may drop the parentheses in these instances:

$$5 - (1) = 5 - 1 = 4$$

However, if there is a number directly in front of the opening parenthesis (or immediately after the closing parenthesis), recall that the parentheses then show multiplication. Therefore:

$$3(4 - 2) = 3(2) = 6$$

If you have trouble remembering this, you may want to put in a multiplication sign as soon as you begin the problem:

$$3(4 - 2) = 3 \cdot (4 - 2) = 3 \cdot (2) = 3 \cdot 2 = 6$$

Also, if you are down to a single negative number in the parentheses, and there is an operation in front of the parentheses, you should not drop the parentheses. For example:

$$5 - (3 - 4) = 5 - (-1) = 6$$

Here we would leave the parentheses to avoid having the ugly "$--$" notation as we mentioned in the previous sections.

In any event, once there is only a single number inside the parentheses, you may move on to the next step of the order of operations.

There may be times when there are two (or more) disjoint sets of parentheses. For example:

$$2 - (3 + 5) - (4 + 2)$$

In this case you simplify whichever one you wish first, or even both at the same time, applying the order of operations inside each one separately. So:

$$2 - (3 + 5) - (4 + 2) = 2 - (8) - (6) = 2 - 8 - 6,$$

etc (we will finish examples like this after we introduce all the basic concepts).

There may also be times when there are two (or more) nested sets of parentheses. For example:

$$2 - (3 + (2 - 4)),$$

or

$$5 + (3 - (2 + (7 - 8)))$$

To simplify, start with the *innermost* grouping symbol, and work your way out. Thus, in the first example above, you would first subtract the four from the two, then add the result to the three, etc.

In these cases, it may be hard to see which opening parenthesis goes with which closing parenthesis, and this is where brackets and braces come in. Both brackets and braces behave exactly like parentheses, and they are only used to make notation easier to read. Therefore, the above two expressions could be written:

$$2 - [3 + (2 - 4)],$$

and

$$5 + \{3 - [2 + (7 - 8)]\}$$

Although you may see this elsewhere, we will save the braces for set notation, and we will just alternate parentheses and brackets. Thus, we will write the second expression:

$$5 + (3 - [2 + (7 - 8)])$$

Once an expression is down to a single set of these grouping symbols, you may revert back to parentheses, and we will do so in our examples. For example,

$$2 - [3 + (2 - 4)] = 2 - [3 + (-2)] = 2 - (1),$$

etc (again, see later for finished examples).

The absolute value bars behave exactly like parentheses, with one big difference. Recall that taking the absolute value of a number is an operation itself, so once you have simplified what is between the absolute value bars to a single number, you MUST IMMEDIATELY take the absolute value of the number. You may replace the absolute value bars with parentheses if you need be, for example:

$$4|3 - 5| - 6 = 4| - 2| - 6 = 4(2) - 6$$

Here we switched to parentheses after taking the absolute value to keep showing multiplication with the four. Just remember that while it is fine to leave parentheses in the problem once you are down to a single number inside (to show multiplication or because you are adding or subtracting a negative, etc.), you must never move past the first step of the order of operations with an absolute value sign still around - you must always take the absolute value of the number first.

Exponents

We have already discussed how multiplication is really just repeated addition, $4 \cdot 3 = 3 + 3 + 3 + 3$, (see page 11). There is also an operation which represents repeated multiplication, and it involves exponents.

Definition: For any positive integer, n, define:

$$a^n = a \cdot a \cdot a \cdot \ldots \cdot a,$$

where there are n factors of the number a. The number a is called the *base*, and the number n is called the *exponent* or *power*.

Example 2. *For each of the following evaluate, and list what is the base and what is the power.*

1. 2^5
2. 1^4
3. 4^3
4. 5^2

Solution 2. *Using the definition, we get:*

1. $2^5 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = \mathbf{32}$.
   The base is 2*; the exponent is* 5.
   This is read as "two to the fifth power".
2. $1^4 = 1 \cdot 1 \cdot 1 \cdot 1 = \mathbf{1}$.
   The base is 1*; the exponent is* 4.
   This is read as "one to the fourth power".
3. $4^3 = 4 \cdot 4 \cdot 4 = \mathbf{64}$.
   The base is 4*; the exponent is* 3.
   This is read as "four to the third power",
   OR read as "four cubed".
4. $5^2 = 5 \cdot 5 = \mathbf{25}$.
   The base is 5*; the exponent is* 2.
   This is read as "five to the second power",
   OR read as "five squared".

Notice that when you read a number with an exponent, the base is read as usual (called a *cardinal* number), while the exponent is read as though it was the place order of someone finishing a race (this is called an *ordinal* number). Two special cases are whenever the exponent is a two or three. If the exponent is a two, the "to the second power" may be replaced with "squared". This comes from the fact that the area of a square is found by taking the length of one of its sides to the second power. If the exponent is a three, the "to the third power" may be replaced with "cubed". This comes from the fact that the volume of a cube is found by taking the length of one of its sides to the third power. Both of the words can be used as commands as well. For example, if you are told to "square five", this will mean "take five to the second power". Similarly, "cube six" means "take six to the third power".

Raising numbers to powers is fairly easy (at least when the power is a positive integer like we are doing here), but you should do enough practice problems to make sure you understand the idea. It is amazing the number of people who eventually treat powers just like factors. For example, they will see 3^3 and write nine as the solution. We would not have introduced new notation to mean the same thing as regular multiplication! If we wanted you to multiply three times three, what is wrong with writing $3 \cdot 3$? You NEED to learn that the power indicates the number of times the base is written as a factor, so:

$$3^3 = 3 \cdot 3 \cdot 3 = 27$$

There are two slight tricks to be wary of. First, some people have trouble interpreting the exponent definition on page 66 when $n = 1$. For example, what is 6^1? The definition states that there should be *one* factor of six, and this means you do not need any multiplication signs. Thus,:

$$6^1 = 6$$

Now we can see that any number to the first power is itself. This rule may also be used in reverse; if you need a number to have an exponent and it does not have an obvious one, you may write the number as itself to the first power. For example,

$$8 = 8^1$$

The second trick is a bit, well, trickier. How would you evaluate -3^2? It may seem to be $(-3)(-3) = 9$, but this is NOT the case. To understand why not, we need to make an important note.

• **Important Note:** For the base of an exponent to be negative, the base MUST be written in parentheses with the power on the right parenthesis.

Therefore, if we really wanted to square negative three, we would write:

$$(-3)^2 = (-3)(-3) = 9$$

So how do we evaluate -3^2? Recall from page 29, that the "$-$" sign may be read as "the opposite of" instead of "negative". The question then becomes, where does taking the opposite fall in our order of operations? Recall from page 60 that taking the opposite is equivalent to multiplying by negative one. Therefore, the order of operations states that *exponents* take precedence over *multiplication*, so:

$$-3^2 = -1 \cdot 3^2 = -1 \cdot (3 \cdot 3) = -1 \cdot (9) = -9$$

Example 3. *Evaluate each of the following.*

1. -5^3
2. $(-2)^4$
3. -7^1
4. $(-9)^1$

Solution 3. *Applying our exponent definition and our important note gives:*

1. $-5^3 = -1 \cdot 5^3 = -1 \cdot (5 \cdot 5 \cdot 5) = -1 \cdot 125 = \mathbf{-125}$.
   Note that if you did this problem wrong and evaluated $(-5)^3$, *you would still have found the correct answer. This will sometimes be the case, but certainly not always (see the* -3^2 *example above, and the next problem in this example), so you want to learn to do it properly.*
2. $(-2)^4 = (-2)(-2)(-2)(-2) = (4)(-2)(-2) = (-8)(-2) = \mathbf{16}$.
3. $-7^1 = -1 \cdot 7^1 = -1 \cdot 7 = \mathbf{-7}$.

4. $(-9)^1 = (-9) = \mathbf{-9}$.

The following definition will allow you to raise numbers to the zeroth power.

Definition: Define:

$$a^0 = 1, \text{ for any } a \neq 0, \text{ and}$$
$$0^0 = \text{undefined.}$$

You will learn how this definition was derived in 10022 (or later in 10006), but we want to present this now as it is an easy definition to learn, and it introduces something else which is undefined.

Example 4. *Evaluate each of the following.*

1. 8^0
2. 13^0
3. 23^0
4. -7^0
5. $(-42)^0$
6. 0^0

Solution 4. *Applying our new definition:*

1. $8^0 = \mathbf{1}$.
2. $13^0 = \mathbf{1}$.
3. $23^0 = \mathbf{1}$.
4. $-7^0 = -1 \cdot 7^0 = -1 \cdot 1 = \mathbf{-1}$.
   This one didn't fool you, did it? This is the same trick as before - if we want the base to be a negative number, it must be in parentheses.
5. $(-42)^0 = \mathbf{1}$

6. $0^0 =$***undefined***.

Multiplication and Division, from left to right

We have already discussed multiplying and dividing integers, so there are only two points to be made. First, even though the M comes before the D in the mnemonic, multiplication and division have equal precedence and should be done from left to right.

Example 5. *Simplify* $12 \div 6 \cdot 2$.

> **Solution 5.** *The only operations are multiplication and division, which carry equal weight on the order of operations, so we will do them from left to right.*
> $12 \div 6 \cdot 2 = 2 \cdot 2 = \mathbf{4}$.

Had we done the multiplication first in the last example, we would have found a different (incorrect) solution as $12 \div 12 = 1$.

Second, the rule of doing multiplication and division from left to right is only important if the operations follow one another, as in the last example. If there is a plus or minus sign (or several plus and/or minus signs) between them, you may do them simultaneously if you wish.

Example 6. *Simplify* $5 \div 5 + 3 \cdot 2$.

> **Solution 6.** *First, we will solve by doing one step at a time, and following the proper order of operations.*

$$\begin{aligned} 5 \div 5 + 3 \cdot 2 &= 1 + 3 \cdot 2 \\ &= 1 + 6 \\ &= \mathbf{7}. \end{aligned}$$

In the original problem, there was a division, an addition, and a multiplication. Following the order of operations, we did the multiplication from left to right, and then did the addition. Because addition (along with subtraction) is the very last step in the order of operations, nothing to the left of the plus sign could interact with anything to its right until all multiplication and division was done. So, in this case, the example could have been solved:

Example 7. *Simplify* $5 \div 5 + 3 \cdot 2$.

Solution 7. *Using the newly learned shortcut,*

$$\begin{aligned} 5 \div 5 + 3 \cdot 2 &= 1 + 6 \\ &= \mathbf{7}. \end{aligned}$$

If you choose to use this shortcut, be sure that no two multiplications and/or divisions follow one another. If you choose to not use the shortcut, you will still get the correct solution (provided you do everything properly, of course!); it will just take an extra step or two.

Addition and Subtraction, from left to right

When you get to this last step, there should be no more absolute value symbols, exponents, multiplications, nor divisions. There may be parentheses still, but there should be no operations inside of them, nor should they be showing multiplication. All that is left is to add or subtract from left to right. The first comment we had about multiplication or division applies here as well - even though A is listed before the S in the mnemonic, addition and subtraction have equal precedence.

Putting it all together

OK, it is time to do some examples. You may wish to refer back to the order of operations table (page 63), and the rules for adding, subtracting, multiplying and dividing integers.

Example 8. *Simplify* $2(3 \cdot 5 - 5^2)^2 - |6|$.

Solution 8. *Starting with step one of the order of operations, we look for any grouping symbols, and we see two sets: one parentheses and one absolute value. They are disjoint so we will work on them simultaneously. Inside the parentheses, there is a multiplication, a subtraction and a power. Following the order of operations, we will do the power first, and then the multiplication, and then the subtraction. As for the absolute value, there is only a single number inside, so we just need to take the absolute value.*

$$\begin{aligned} 2(3 \cdot 5 - 5^2)^2 - |6| &= 2(3 \cdot 5 - 25)^2 - 6 \\ &= 2(15 - 25)^2 - 6 \\ &= 2(-10)^2 - 6. \end{aligned}$$

By the way, if you needed to do scratch work to figure out 5^2 *or* $3 \cdot 5$*, you could have done this to the side. At this point, all that is left inside the parentheses is a single number, so we may move on. There is a multiplication (remember that a number immediately in front of an opening parenthesis means multiply), a power, and a subtraction left. Following the order of operations, we will do the power, then multiply, then subtract. Since the negative sign is inside the parentheses, we will square negative ten. Thus,*

$$\begin{aligned} 2(3 \cdot 5 - 5^2)^2 - |6| &= 2(-10)^2 - 6 \\ &= 2(100) - 6 \\ &= 200 - 6 \\ &= \mathbf{194}. \end{aligned}$$

Example 9. *Simplify* $3 + 2[-6 - (2 - 6)] - 4^2$.

Solution 9. *Looking for grouping symbols first, we see that there are two sets, and they are nested. Recall that in this case we will start on the innermost group, so we will start by subtracting the six from the two, before simplifying the brackets.*

$$\begin{aligned} 3 + 2[-6 - (2 - 6)] - 4^2 &= 3 + 2[-6 - (-4)] - 4^2 \\ &= 3 + 2[-6 + (+4)] - 4^2 \\ &= 3 + 2(-2) - 4^2. \end{aligned}$$

Be careful! The most common mistake in a problem like this is students tend to add the three plus two very early on. You must remember that because the two is immediately before an opening grouping symbol (in this case a bracket), multiplication is implied, and multiplication gains precedence over addition. If you are having trouble remembering this, you may wish to insert a times sign (dot) right away. In this case the above work would have looked like:

$$\begin{aligned} 3+2\cdot[-6-(2-6)]-4^2 &= 3+2\cdot[-6-(-4)]-4^2 \\ &= 3+2\cdot[-6+(+4)]-4^2 \\ &= 3+2\cdot(-2)-4^2. \end{aligned}$$

We wish to comment on three points. First, notice that there was no implied multiplication on the original parentheses, as there was a minus sign in front of them, not a number. Second, once we had simplified inside the brackets down to a single number, we switched backed to parentheses. This is not always done, but it is fairly common and we prefer it. Third, recall that you can not always drop grouping symbols once you are down to a single number. In the last line of above for example, had the inside of the brackets simplified to a positive two, we could have written it as $3+2\cdot 2-4^2$*. But, since we actually have multiplication of a negative number, and we do not write "*$\cdot-$*", we have left the parentheses in. Since the inside of the parentheses is just a single number though, we may consider the grouping symbol stage done, and move on to the next step of order of operations. Well, remaining we have an addition, a multiplication, a subtraction, and a power. Following the order of operations, we will evaluate the power first; then do the multiplication; then add and subtract from left to right.*

$$\begin{aligned} 3+2\cdot[-6-(2-6)]-4^2 &= 3+2\cdot(-2)-4^2 \\ &= 3+2\cdot(-2)-16 \\ &= 3+(-4)-16 \\ &= -1-16 \\ &= -1+(-16) \\ &= \mathbf{-17}. \end{aligned}$$

For the following examples, we will still list every step, but now you are encouraged to supply the proper reasoning.

Example 10. *Simplify* $72 \div 3 - 3(4 - 6 \cdot 3)$.

Solution 10. *Simplifying gives:*

$$\begin{aligned} 72 \div 3 - 3(4 - 6 \cdot 3) &= 72 \div 3 - 3 \cdot (4 - 18) \\ &= 72 \div 3 - 3 \cdot (-14) \\ &= 24 - (-42) \\ &= 24 + 42 \\ &= \mathbf{66}. \end{aligned}$$

Example 11. *Simplify* $-21 \div 3 \cdot 5 - 4^2$.

Solution 11. *Simplifying gives:*

$$\begin{aligned} -21 \div 3 \cdot 5 - 4^2 &= -21 \div 3 \cdot 5 - 16 \\ &= -7 \cdot 5 - 16 \\ &= -35 - 16 \\ &= -35 + (-16) \\ &= -\mathbf{51}. \end{aligned}$$

Example 12. *Simplify* $-6 - 3(-4) \div (3 - 3^2)$.

Solution 12. *Simplifying gives:*

$$\begin{aligned} -6 - 3 \cdot (-4) \div (3 - 3^2) &= -6 - 3 \cdot (-4) \div (3 - 9) \\ &= -6 - 3 \cdot (-4) \div (-6) \\ &= -6 - (-12) \div (-6) \\ &= -6 - 2 \\ &= -6 + (-2) \\ &= -\mathbf{8}. \end{aligned}$$

Example 13. *Simplify* $-2|7-11|-2^3$.

Solution 13. *Simplifying gives:*

$$\begin{aligned} -2\cdot|7-11|-2^3 &= -2\cdot|-4|-2^3 \\ &= -2\cdot 4-2^3 \\ &= -2\cdot 4-8 \\ &= -8-8 \\ &= -8+(-8) \\ &= \mathbf{-16}. \end{aligned}$$

Example 14. *Simplify* $8-2[5^0+(7-10)]$.

Solution 14. *Simplifying gives:*

$$\begin{aligned} 8-2\cdot[5^0+(7-10)] &= 8-2\cdot[5^0+(-3)] \\ &= 8-2\cdot[1+(-3)] \\ &= 8-2\cdot(-2) \\ &= 8-(-4) \\ &= 8+(+4) \\ &= \mathbf{12}. \end{aligned}$$

Example 15. *Simplify* $9^0-(2\cdot 4-4\cdot 3)\div 4$.

Solution 15. *Simplifying gives:*

$$\begin{aligned} 9^0-(2\cdot 4-4\cdot 3)\div 4 &= 9^0-(8-12)\div 4 \\ &= 9^0-(-4)\div 4 \\ &= 1-(-4)\div 4 \\ &= 1-(-1) \\ &= 1+(+1) \\ &= \mathbf{2}. \end{aligned}$$

SECTION 1.5 EXERCISES

(Answers are found on page 372.)

Evaluate each of the following.

1. 5^2
2. 3^4
3. 2^6
4. 1^8
5. -3^0
6. -1^4
7. 17^0
8. 9^1
9. $(-2)^2$
10. 4^3
11. $(-3)^3$
12. $(-1)^6$
13. 0^7
14. $(-12)^0$
15. 7^2
16. 11^1
17. 0^0
18. -8^2
19. 0^1
20. $(-3)^4$

Simplify.

21. $3 - 6 \div 3$
22. $-4 + 8 \cdot 3$
23. $5 - 7 - 2 + 4$
24. $10 \div 2 \cdot (-5)$
25. $4^2 - 2^2$
26. $3 + 5[2 + (1 - 3^2)]$
27. $9^0 - 2 \cdot 8^0$
28. $(5 - 6) - (2^0 - 3^2)$
29. $4^1 - 2(3 - 5^0)^2$
30. $2(6 - 10^0) + 3(5 - 10^1)$
31. $3 - 2[4 + (6 - 4^2)]$
32. $2 - [3^2 + 2(4^1 - 2^2)]$
33. $2|5 - 8| - 10 \div (6^1 - 2^2)$
34. $|3 + (4 - 8)| - |1^0 - 3^2|$
35. $5 \cdot 2 - 2 \div (-1)$
36. $3 + 2 \cdot 6 \div 3 - 8^0$
37. $-3^2 - 2 \cdot 5^0 + (4^2 - 5^2)$
38. $-2^0 + 7^0$
39. $2(3 - 9) - (2^1 - 4^2)$
40. $(4 - 2^2) \cdot 15 \div 3$

1.6 Primes, GCF, & LCM

Prime Numbers

In this section, we will present some tools you may find useful later. For this entire section, we will use only positive integer numbers. We will start with some definitions.

Definition: If a positive integer, N, may be written as a product of two positive integers A and B, i.e. $N = A \cdot B$, then the numbers A and B are called factors of N. We will also say that A and B divide N.

Example 1. *Find all the factors of the number* 6*.*

Solution 1. *Well,*
1 *is a factor of* 6 *as* $1 \cdot 6 = 6$*,*
2 *is a factor of* 6 *as* $2 \cdot 3 = 6$*,*
3 *is a factor of* 6 *as* $2 \cdot 3 = 6$*,*
6 *is a factor of* 6 *as* $1 \cdot 6 = 6$*.*
Therefore, the factors of 6 *are* **{ 1,2,3,6 }.**

By looking at the number of factors that a positive integer has, we can separate the positive integers greater than one into two categories.

Definition: A positive integer greater than 1 whose only factors are 1 and itself is called prime.

Definition: A positive integer greater than 1 which has a positive integer factor other than 1 or itself is called composite.

Important Note:

The number 1 is considered to be NEITHER prime NOR composite.

Example 2. *Determine whether each of the first ten positive integers are prime or composite.*

Solution 2. *Examining each number for factors, we find:*

1	*is*	*neither*	,	*as stated above.*
2	*is*	*prime*	,	*as its only factors are* 1 *and* 2.
3	*is*	*prime*	,	*as its only factors are* 1 *and* 3.
4	*is*	*composite*	,	*as* 2 *is a factor of* 4.
5	*is*	*prime*	,	*as its only factors are* 1 *and* 5.
6	*is*	*composite*	,	*as* 2 *is a factor of* 6.
7	*is*	*prime*	,	*as its only factors are* 1 *and* 7.
8	*is*	*composite*	,	*as* 2 *is a factor of* 8.
9	*is*	*composite*	,	*as* 3 *is a factor of* 9.
10	*is*	*composite*	,	*as* 2 *is a factor of* 10.

Notice that finding a single factor different from 1 or the number itself is enough to make it composite. So, for example, for the number 10 above, we did not have to list all of its factors, just one which wasn't 1 or 10.

An interesting fact about prime numbers is that any positive integer greater than 1 may be written as a product of only primes. For example, $6 = 2 \cdot 3$ and $8 = 2 \cdot 2 \cdot 2$. When you factor a number into the product of only primes, this is called the prime factorization of the number. This concept is the basis for something called the Fundamental Theorem of Arithmetic:

Fundamental Theorem of Arithmetic: Any positive integer greater than 1 may be written as a product of prime numbers, and the factors are unique except, possibly, for ordering.

To find the prime factorization of a number, we will use a factor tree. To see how this is done, look at the following example:

Example 3. *Find the prime factorization of* 24.

Solution 3. *To start, figure out two numbers which multiply together to give you* 24*. We will use* $24 = 4 \cdot 6$ *(at least at first). Write the original number* 24*, and then below it, show two arrows (considered "branches" - hence the title, factor tree) breaking it into its two factors.*

$$\begin{array}{ccc} & 24 & \\ \swarrow & & \searrow \\ 4 & & 6 \end{array}$$

Now we look at the end of our branches. For any factor which is prime, the branch will end here. For any factor which is composite, continue splitting as before until all the branches end on a prime number. So, to continue factoring 24*:*

$$\begin{array}{ccccccc} & & & 24 & & & \\ & & \swarrow & & \searrow & & \\ & 4 & & & & 6 & \\ \swarrow & & \searrow & & \swarrow & & \searrow \\ 2 & & 2 & & 2 & & 3 \end{array}$$

Now all the branches end on a prime number, so the prime factorization of 24 *is:* $24 = \mathbf{2 \cdot 2 \cdot 2 \cdot 3}$*, OR* $24 = \mathbf{2^3 \cdot 3}$*.*

Some comments on the prime factorization. First, note that if we would have written $24 = 6 \cdot 4$ and continued from there, the prime factorization would have been $24 = 2 \cdot 3 \cdot 2 \cdot 2$. We have already discussed how multiplication is commutative, like addition, when multiplying two numbers, but we will also soon learn (section 1.13) that when we are multiplying a bunch of numbers, we may rearrange them in any order we wish. This is what the Fundamental Theorem was referring to when it stated "the factors are unique except, possibly, for ordering." Second comment, even though different arrangements of the factors may all be correct, it is standard to list the factors from smallest to largest as we did in the last example. For a small number of factors, you (or your instructor) may prefer to list every factor as shown in the first solution above, or you may prefer to use exponents to save time and space. Finally, the Fundamental Theorem states that this factorization is unique, but what if you thought of $24 = 3 \cdot 8$ originally, instead of $24 = 4 \cdot 6$? Let us see with another example.

Example 4. *Find the prime factorization of* 24*, starting with* $24 = 3 \cdot 8$*.*

Solution 4. *Redoing our factor tree with this start gives:*

$$\begin{array}{ccccc}
 & 24 & & & \\
 & \swarrow \; \searrow & & & \\
3 & & 8 & & \\
 & & \swarrow \; \searrow & & \\
 & 2 & & 4 & \\
 & & & \swarrow \; \searrow & \\
 & & 2 & & 2
\end{array}$$

Therefore, $24 = \mathbf{2 \cdot 2 \cdot 2 \cdot 3}$*, OR* $24 = \mathbf{2^3 \cdot 3}$*.*

Even though we started factoring differently, the prime factorization ended up the same. You may try doing the prime factorization of 24 again using $24 = 2 \cdot 12$, but you will again get the same end result. One comment on this last example, be careful when some of the branches end at different levels. It is very easy to miss a factor, especially if your branches crowd together. One solution to this is that you may wish to simply "extend" the branch ends with single arrows so that all the factors end up on the same line. Doing this to the last example would give:

Example 5. *Find the prime factorization of* 24*, starting with* $24 = 3 \cdot 8$*.*

Solution 5. *Redoing our factor tree with this start gives:*

$$\begin{array}{ccccc}
 & 24 & & & \\
 & \swarrow \; \searrow & & & \\
3 & & 8 & & \\
\downarrow & & \swarrow \; \searrow & & \\
3 & 2 & & 4 & \\
\downarrow & \downarrow & & \swarrow \; \searrow & \\
3 & 2 & 2 & & 2
\end{array}$$

Therefore, $24 = \mathbf{2 \cdot 2 \cdot 2 \cdot 3}$*, OR* $24 = \mathbf{2^3 \cdot 3}$*.*

Here are some more examples of finding the prime factorization.

Example 6. *Find the prime factorization of* 90.

Solution 6. *We will start with* $90 = 9 \cdot 10$.

$$
\begin{array}{cccccc}
 & & 90 & & & \\
 & & \swarrow \searrow & & & \\
 & 9 & & & 10 & \\
 & \swarrow \searrow & & & \swarrow \searrow & \\
3 & & 3 & 2 & & 5
\end{array}
$$

So the prime factorization of 90 *is:* $90 = \mathbf{2 \cdot 3 \cdot 3 \cdot 5}$, *OR* $90 = \mathbf{2 \cdot 3^2 \cdot 5}$.

Example 7. *Find the prime factorization of* 20.

Solution 7. *We will start with* $20 = 4 \cdot 5$.

$$
\begin{array}{cccc}
 & & 20 & \\
 & & \swarrow \searrow & \\
 & 4 & & 5 \\
 & \swarrow \searrow & & \downarrow \\
2 & & 2 & 5
\end{array}
$$

So the prime factorization of 20 *is:* $20 = \mathbf{2 \cdot 2 \cdot 5}$, *OR* $20 = \mathbf{2^2 \cdot 5}$.

Example 8. *Find the prime factorization of* 110.

Solution 8. *We will start with* $110 = 10 \cdot 11$.

$$
\begin{array}{cccc}
 & & 110 & \\
 & & \swarrow \searrow & \\
 & 10 & & 11 \\
 & \swarrow \searrow & & \downarrow \\
2 & & 5 & 11
\end{array}
$$

So the prime factorization of 110 *is:* $110 = \mathbf{2 \cdot 5 \cdot 11}$.

Example 9. *Find the prime factorization of* 7*.*

> **Solution 9.** 7 *is a prime number, so its only factors are* 1 *and* 7*. In this case, the factor tree would look like:*
>
> $$\begin{array}{c} 7 \\ \downarrow \\ 7 \end{array}$$
>
> *So the prime factorization of* 7 *is* $7 = \mathbf{7}$*.*

In the last example we see that the prime factorization of prime numbers is trivial (and do not require even as much work as we showed).

Useful Note:

The first ten prime numbers are $2, 3, 5, 7, 11, 13, 17, 19, 23$ and 29.

GCF (Greatest Common Factor)

Definition: The greatest common factor of two numbers, a and b, written GCF(a,b), is the largest number which is a factor of both a and b.

Note that the greatest common factor, which we will in the future simply abbreviate as GCF, does not have to be a prime number. For example:

Example 10. *Find the greatest common factor of* 8 *and* 12*.*

> **Solution 10.** *One way to find the greatest common factor is to list out all of the factors of both numbers, and compare.*
>
> $$\begin{array}{rclllllll} 8 & = & 1 & 2 & & 4 & & 8 & \\ & & \updownarrow & \updownarrow & & \updownarrow & & & \\ 12 & = & 1 & 2 & 3 & 4 & 6 & & 12 \end{array}$$

We see that the factors they both have in common are $1, 2$ *and* 4*, so GCF(*8,12*)* $= \mathbf{4}$.

This straightforward comparison of factors is nice way to find the GCF *provided* the numbers are both small or have few factors. For another example:

Example 11. *Find GCF(*10,27*).*

Solution 11. *Using the same technique as last example, we get:*

$$\begin{array}{rclcccccc}
10 & = & 1 & 2 & & 5 & & 10 & \\
 & & \updownarrow & & & & & & \\
27 & = & 1 & & 3 & & 9 & & 27
\end{array}$$

*So, GCF(*10,27*)* $= \mathbf{1}$.

When the GCF of two numbers is 1, we give this a special name.

Definition: Two numbers whose greatest common factor is 1 are called <u>relatively prime</u>.

Notice that two numbers may be relatively prime even if neither number itself is prime (in the last example, 10 and 27 are both composite).

When you are dealing with larger numbers, though, this method of listing out all the factors may be tiresome. Instead, you may use the prime factorization technique to help you.

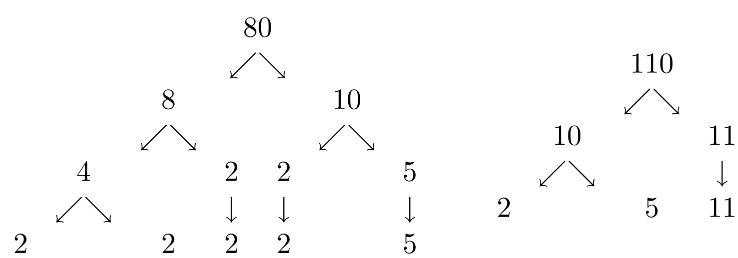
To find the GCF(a,b) using prime factorization.

1. Find the prime factorization of both a and b.
2. Form the prime factorization of GCF(a,b) by listing all the prime factors which are factors of both a and b. There may be repetitions of the same factor!
3. Find GCF(a,b) by multiplying out its prime factorization.

Lets try an example.

Example 12. *Find the GCF(*80,110*).*

Solution 12. *First, we will find the prime factorizations of* 80 *and* 110*.*
Scratch work:



So the prime factorizations are:
$80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^4 \cdot 5$,
$110 = 2 \cdot 5 \cdot 11$.
Comparing the two, we see that the factors they share are one 2 *and one* 5*,so the GCF(*80,110*)* $= 2 \cdot 5 = \mathbf{10}$.

And one more example:

Example 13. *Find the GCF(*60,84*).*

Solution 13. *First, we will find the prime factorizations of* 60 *and* 84.
Scratch work:

$$\begin{tikzcd}[column sep=small]
 & & & 60 \arrow[dll] \arrow[drr] & & & \\
 & 6 \arrow[dl] \arrow[dr] & & & & 10 \arrow[dl] \arrow[dr] & \\
2 & & 3 & & 2 & & 5
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & 84 \arrow[dll] \arrow[drr] & & & \\
 & 4 \arrow[dl] \arrow[dr] & & & & 21 \arrow[dl] \arrow[dr] & \\
2 & & 2 & & 3 & & 7
\end{tikzcd}$$

So the prime factorizations are:
$60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$,
$84 = 2 \cdot 2 \cdot 3 \cdot 7 = 2^2 \cdot 3 \cdot 7$.
Comparing the two, we see that the factors they share are two 2*'s and one* 3*,so the GCF(*60,84*)* $= 2 \cdot 2 \cdot 3 = \mathbf{12}$.

LCM (Least Common Multiple)

A related concept, and one even more important for our purposes, is the least common multiple.

Definition: A multiple of a number, a, is the product of a with an integer.

For example, $4, 8, 0$ and -12 are all multiples of 4, as $4 = 4{\cdot}1$, $8 = 4{\cdot}2$, $0 = 4{\cdot}0$ and $-12 = 4 \cdot (-3)$. Recall that for this section, we will be working with the positive integers only, so for now we are only concerned with positive multiples of a number. Even with this restriction, every positive integer has an infinite amount of positive multiples. We will be concerned with what multiples two numbers have in common.

Definition: An integer which is a multiple of both numbers a and b is called a common multiple of a and b.

Definition: The smallest, positive integer which is a multiple of both a and b is called the least common multiple of a and b.

For example, $12, 24$ and 36 are all common multiples of the numbers 3 and 4, while the number 12 is the least common multiple of 3 and 4. We will abbreviate least common multiple with LCM, and when we wish you to take the LCM of two numbers a and b, we will write this as LCM(a,b). For example, LCM(3,4) $= 12$.

To find the LCM of two numbers, we will present two methods. The first method works best when the two numbers are relatively small. All you do is start with the larger number and find the smallest, positive multiple of it into which the other number divides.

Example 14. *Find the LCM(8,12).*

Solution 14. *The larger of the two numbers is 12, so:*
$12 \cdot 1 = 12$*, does 8 divide into 12? No.*
$12 \cdot 2 = 24$*, does 8 divide into 24? Yes,* ✓.
Therefore, the LCM(8,12) $=$ **24**.

The most multiples you would have to check using this method is up to the product, as the product is obviously always a common multiple. In the last example, this means you would have had to check no further than $12 \cdot 8 = 96$. Because the product of the two numbers is always an easy-to-find common multiple, many people prefer to use it over the LCM when a common multiple is needed, but it is still useful to be able to find the LCM too.

When the numbers are larger, this method can be a bit tedious. In this case, it would be better to use another method which utilizes the prime factorization of the numbers.

<u>To find the LCM(a,b) using prime factorization.</u>

1. Find the prime factorization of both a and b.
2. Form the prime factorization of LCM(a,b) by listing all the prime factors which are factors of either a or b or both.
3. Find LCM(a,b) by multiplying out its prime factorization.

Lets try an example.

Example 15. *Find the LCM(*80,110*).*

Solution 15. *The first step is to find the prime factorization of* 80 *and* 110*. Refer back to example 12 to see the scratch work where we found the prime factorizations to be:* $80 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^4 \cdot 5$,
$110 = 2 \cdot 5 \cdot 11$.
Combining the two, we see that we need four 2*'s, one* 5 *and one* 11*, so:*

$$\begin{aligned} LCM(80{,}110) &= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 11 \\ &= 4 \cdot 2 \cdot 2 \cdot 5 \cdot 11 \\ &= 8 \cdot 2 \cdot 5 \cdot 11 \\ &= 16 \cdot 5 \cdot 11 \\ &= 80 \cdot 11 \\ &= \mathbf{880}. \end{aligned}$$

And a second example.

Example 16. *Find the LCM(*60,84*).*

Solution 16. *First, we will find the prime factorizations of* 60 *and* 84.
Refer back to example 13 *to see the scratch work where we found the prime factorizations to be:* $60 = 2 \cdot 2 \cdot 3 \cdot 5 = 2^2 \cdot 3 \cdot 5$,
$84 = 2 \cdot 2 \cdot 3 \cdot 7 = 2^2 \cdot 3 \cdot 7$.
Combining the two, we see that we need two 2*'s, one* 3*, one* 5 *and one* 7*,so:*

$$\begin{aligned} LCM(60,84) &= 2 \cdot 2 \cdot 3 \cdot 5 \cdot 7 \\ &= 4 \cdot 3 \cdot 5 \cdot 7 \\ &= 12 \cdot 5 \cdot 7 \\ &= 60 \cdot 7 \\ &= \mathbf{420}. \end{aligned}$$

Notice the difference between the GCF and LCM techniques. For the GCF, you only kept prime factors which appeared in the prime factorizations of both a and b. For the LCM, you kept as many factors of any prime number which appeared in either of the prime factorizations of a or b. It is always good to check your work, and the following nifty fact can help you check.

Nifty Fact: For any two positive integers a and b:

$$\text{GCF}(a,b) \cdot \text{LCM}(a,b) = a \cdot b.$$

Example 17. *Use the nifty fact to check on our solutions to examples* 10 *and* 14.

Solution 17. *In those two examples, we found that:*
$GCF(8,12) = 4$ *and*
$LCM(8,12) = 24$*, so:*
$GCF(8,12) \cdot LCM(8,12) = 4 \cdot 24 = 96$.
Meanwhile, the product of 8 *and* 12 *is:*
$8 \cdot 12 = 96$*, so the nifty fact holds,* ✓.

Note that this check is not perfect as it is possible for you to make two mistakes which cancel out. For example, if your GCF has an extra factor of 2, the nifty fact would still check out right provided your LCM has one too few 2's. Also, the nifty fact will only help you check your answers when you find both the GCF and LCM for the same two numbers. Still, in many cases, this quick check can catch a careless mistake.

Example 18. *Use the nifty fact to check on our solutions to examples 12 and 15.*

Solution 18. *In those two examples, we found that:*
GCF($80{,}110$*)* $= 10$ *and*
LCM($80{,}110$*)* $= 880$*, so:*
GCF($80{,}110$*)* $\cdot$ *LCM(*$80{,}110$*)* $= 10 \cdot 880 = 8800$.
Meanwhile, the product of 80 *and* 110 *is:*
$80 \cdot 110 = 8800$*, so the nifty fact holds,* ✓.

Example 19. *Use the nifty fact to check on our solutions to examples 13 and 16.*

Solution 19. *In those two examples, we found that:*
GCF($60{,}84$*)* $= 12$ *and*
LCM($60{,}84$*)* $= 420$*, so:*
GCF($60{,}84$*)* $\cdot$ *LCM(*$60{,}84$*)* $= 12 \cdot 420 = 5040$.
Meanwhile, the product of 60 *and* 84 *is:*
$60 \cdot 84 = 5040$*, so the nifty fact holds,* ✓.

SECTION 1.6 EXERCISES

(Answers are found on page 373.)

Find the prime factorizations of the following numbers.

1. 88
2. 70
3. 65
4. 50
5. 98
6. 100
7. 108
8. 40
9. 135
10. 78

Find the following GCF's.

11. GCF(50,70)
12. GCF(70,98)
13. GCF(40,78)
14. GCF(40,108)
15. GCF(88,98)
16. GCF(50,65)
17. GCF(108,135)
18. GCF(65,70)
19. GCF(100,135)
20. GCF(88,100)

Find the following LCM's. Note that the same numbers are being used to find LCM's as we used to find GCF's in the last group of homework problems. This means that if you do both, you can check your answer with the Nifty Fact from page 88.

21. LCM(50,70)
22. LCM(70,98)
23. LCM(40,78)
24. LCM(40,108)
25. LCM(88,98)
26. LCM(50,65)
27. LCM(108,135)
28. LCM(65,70)
29. LCM(100,135)
30. LCM(88,100)

1.7 Fractions and Mixed Numbers

Fractions: the basics

Definition: A fraction is a ratio of two numbers.

For example, $\frac{3}{5}$, $\frac{22}{10}$, $\frac{\pi}{4}$ and $\frac{-8}{4}$ are all fractions. The fraction bar represents division, so that the fraction $\frac{-8}{4}$ is equal to the integer -2 as $-8 \div 4 = -2$. We may also do the division on the other fractions, like $\frac{22}{10}$, to change the fraction into a decimal (section 1.10) or, when appropriate, into a mixed number (later in this section). The top number of the fraction is called the numerator and the bottom number is called the denominator.

There is a special group of fractions with which we will be concerned, and that is all the fractions where both the numerator and the denominator are integers. This set of numbers has a special name:

rational numbers $= \{\frac{a}{b} \mid a$ and b are both integers and $b \neq 0\}$.

While this is still set notation, we didn't just list all or part of the numbers (along with periods of ellipsis). Rather, the above is set-builder notation. In set-builder notation, you first list an expression; here we have $\frac{a}{b}$. Next, there is always a "|" or "**:**" which means *such that*. And lastly, you describe properties of your original expression. Therefore, the above reads: the rational numbers are all the fractions a over b such that a and b are both integers and b is not equal to zero.

The requirement that the denominator not be zero should come as no surprise since we have mentioned that the fraction bar is equivalent to division. The two division rules given in the introduction (see page 15) translate into the following two facts for fractions:

Fraction Fact 1: For any number a such that $a \neq 0$,

$$\frac{0}{a} = \mathbf{0}.$$

Fraction Fact 2: For any number a,

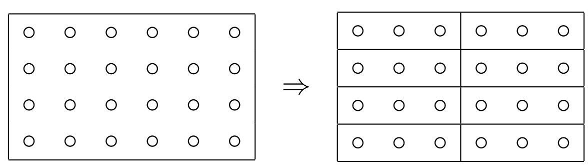
$$\frac{a}{0} = \textbf{undefined}.$$

For all of this section, we will use the word *fraction*, but we will mostly be referring to these specific type of fractions - *rational numbers* - where the numerator and denominator are integers.

Positive fractions have a physical meaning; they can be thought of as part of a whole. The denominator indicates how many equal-size pieces you divide the original object up into, while the numerator indicates how many of the pieces you are interested in.

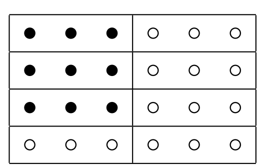
Example 1. *Illustrate* $\frac{3}{8}$.

Solution 1. *We will start with a rectangular cake which we will divide into eight equal-sized pieces.*



Now that the cake is cut into eighths, we will darken the circles in three of the eight pieces to indicate three-eighths ($\frac{3}{8}$).

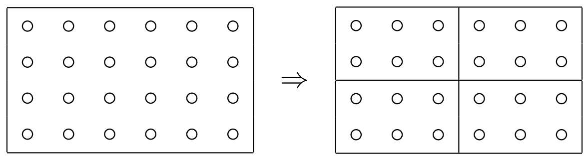
$\frac{3}{8}$ =



Let us do a few more examples.

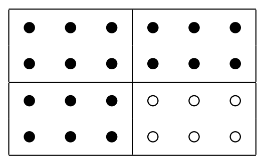
Example 2. *Illustrate* $\frac{3}{4}$.

Solution 2. *We will start with a rectangular cake which we will divide into four equal-sized pieces.*



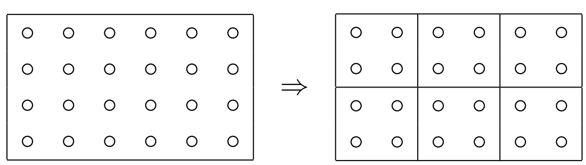
Now that the cake is cut into fourths, we will darken the circles in three of the four pieces to indicate three-fourths ($\frac{3}{4}$).

$$\frac{3}{4} =$$



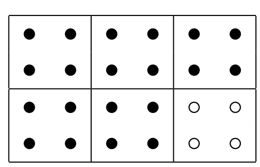
Example 3. *Illustrate* $\dfrac{5}{6}$.

Solution 3. *We will start with a rectangular cake which we will divide into six equal-sized pieces.*



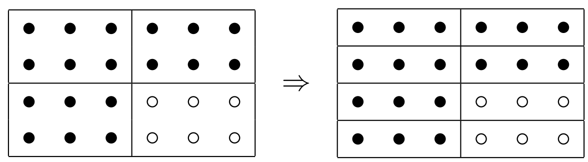
Now that the cake is cut into sixths, we will darken the circles in five of the six pieces to indicate five-sixths ($\frac{5}{6}$).

$$\frac{5}{6} =$$



In all of our examples, we started with a rectangular cake, but the original shape of the object is not important. For the fraction $\frac{3}{8}$, for example, we could have started with a circular pizza, cut into eight equal- sized pieces, and shaded three of the eight to represent three-eighths. All that matters is that you divide the object up into the denominator number of equal-sized pieces, and then indicate the numerator number of these pieces.

Now go back and look at example 2. What would happen if we cut each of the four pieces in half again? Pictorially we get:



The first picture represents $\frac{3}{4}$, while the second picture represents $\frac{6}{8}$, yet they both represent the same amount of the cake! The only difference between the two is the size of the pieces. This leads to an interesting and important fact about fractions:

Interesting and Important Facts:

- Among integers, each integer is unique (no other integer represents the same number as the integer 2, for example).
- Among fractions, for any given fraction, there are an infinite number of other fractions which represent the same amount.

We have already seen that $\frac{3}{4} = \frac{6}{8}$. Recalling that the fraction bar means divide, it is also easy to see that: $2 = \frac{2}{1} = \frac{4}{2} = \frac{10}{5} = \frac{32}{16} = \ldots$. Fractions like these have a special name:

Definition: Different fractions which represent the same number are called equivalent fractions.

We can generate equivalent fractions algebraically by using the following rule:

Fundamental Principle of Fractions:
Let a, b and c be *any* numbers, *except* $b \neq 0$ and $c \neq 0$. Then the following is true:

$$\frac{a}{b} = \frac{a \cdot c}{b \cdot c}$$

This rule is extremely important, and it will be used often when working with fractions. As an example, let us start with the situation where you have a fraction which you wish had a different denominator (but still represented the same amount!). The reason why we may wish to do this will become apparent soon.

Example 4. *Write* $\frac{4}{5}$ *as an equivalent fraction with a denominator of* 15*.*

Solution 4. *Our original fraction has a denominator of* 5 *which we wish to change to a* 15*. The fundamental principle of fractions tells us that we may make equivalent fractions by multiplying numerator and denominator by the same nonzero number. As for which number we wish to multiply by, well to change a* 5 *to a* 15*, we would like to multiply by* 3*. Therefore:*

$$\begin{aligned} \frac{4}{5} &= \frac{4 \cdot 3}{5 \cdot 3} \\ &= \mathbf{\frac{12}{15}} \end{aligned}$$

Example 5. *Write* $\frac{2}{7}$ *as an equivalent fraction with a denominator of* 49*.*

Solution 5. *Applying the fundamental principle of fractions with* $c = 7$ *(as* $7 \cdot 7 = 49$*):*

$$\begin{aligned} \frac{2}{7} &= \frac{2 \cdot 7}{7 \cdot 7} \\ &= \mathbf{\frac{14}{49}} \end{aligned}$$

Example 6. *Write* $\frac{1}{3}$ *as an equivalent fraction with a denominator of* 24.

> **Solution 6.** *This time, to change the* 3 *to a* 24*, we need to multiply by* 8*, so the fundamental principle says:*
>
> $$\begin{aligned} \frac{1}{3} &= \frac{1\cdot 8}{3\cdot 8} \\ &= \mathbf{\frac{8}{24}} \end{aligned}$$

Having all these different fractions representing the same number can be confusing. For example, let us say we do an addition problem where we add two fractions to get a fractional answer (next section). Since there are an infinite number of correct solutions, which do we list in the back? This is not unheard of, and you may encounter situations where there are many possible correct solutions, but the annoying thing here is that all the equivalent fractions represent the same number! In light of this, mathematicians have decided that one form of a fraction is preferable to all of its equivalent forms.

Fractions: simplest form

Definition: A fraction is said to be in simplest form, or in lowest terms, if the greatest common factor of the numerator and denominator is 1.

One method of simplifying a fraction is to imagine that you are using the fundamental principle *backwards*. Instead of starting with a fraction and multiplying numerator and denominator by a common number, if you can rewrite the numerator and denominator as a products with a common factor, you may cancel this common factor. In other words, instead of using the equality to:

$$\frac{a}{b} \rightarrow \frac{a\cdot c}{b\cdot c}$$

you will use the equality to go the other way:

$$\frac{a\cdot c}{b\cdot c} \rightarrow \frac{a}{b}$$

Example 7. *Simplify* $\frac{4}{6}$

Solution 7. *Noticing that the* 4 *and* 6 *have a common factor of* 2*, we will factor both numbers and cancel.*

$$\begin{aligned} \frac{4}{6} &= \frac{2 \cdot 2}{3 \cdot 2} \\ &= \frac{2 \cdot \cancel{2}}{3 \cdot \cancel{2}} \\ &= \mathbf{\frac{2}{3}} \end{aligned}$$

Example 8. *Simplify* $\frac{12}{16}$

Solution 8. *Here, the GCF of* 12 *and* 16 *is* 4*, and this is the would be the best number to factor out. Occasionally, however, especially when the numbers get larger, you may not notice the GCF, but rather just any common factor. That is fine, the problem will still work out, but this means you will have to do several steps to simplify. Here, for example, let us say that we noticed that* 2 *is a common factor of* 12 *and* 16*. Then we get:*

$$\begin{aligned} \frac{12}{16} &= \frac{6 \cdot \cancel{2}}{8 \cdot \cancel{2}} \\ &= \frac{6}{8} \\ &= \frac{3 \cdot \cancel{2}}{4 \cdot \cancel{2}} \\ &= \mathbf{\frac{3}{4}} \end{aligned}$$

So if you do not factor out the GCF right away, you may have several steps where you are factoring out common factors. Just make sure that in your final answer, the numerator and denominator have no common factors other than 1.

Instead of searching for common factors one at a time, a slight modification of the above method is to use the prime factorization. Start off by doing the prime factorization of both the numerator and denominator. Now you will want to cancel any common factors in the numerator and denominator. This may seem to be different from the fundamental principle of fractions, as there may be more than one multiplication sign, but we may modify the fundamental principle in the following way:

> **Fraction Fact 3:** Provided that the only operation in your numerator and denominator is multiplication, you may cancel any factor in the numerator with any matching factor in the denominator.

The provision that only multiplication is present is important! Addition and subtraction kill this ability to cancel. Let us redo our last example using the prime factorization.

Example 9. *Simplify* $\frac{12}{16}$

Solution 9. *First we need to find the prime factorization of* 12 *and* 16.
Scratch work:

$$\begin{array}{ccccccccccccc}
 & 12 & & & & & & & 16 & & & & \\
 & \swarrow\searrow & & & & & & & \swarrow\searrow & & & & \\
3 & & & 4 & & & & 4 & & & 4 & & \\
\downarrow & & & \swarrow\searrow & & & & \swarrow\searrow & & & \swarrow\searrow & & \\
3 & 2 & & & 2 & & 2 & & 2 & 2 & & & 2
\end{array}$$

So, the prime factorizations are:
$12 = 2 \cdot 2 \cdot 3$
$16 = 2 \cdot 2 \cdot 2 \cdot 2$
Replacing the numbers with their prime factorization allows us

to cancel.

$$\begin{aligned}\frac{12}{16} &= \frac{2\cdot 2\cdot 3}{2\cdot 2\cdot 2\cdot 2}\\ &= \frac{\cancel{2}\cdot\cancel{2}\cdot 3}{\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2}\\ &= \frac{3}{2\cdot 2}\\ &= \mathbf{\frac{3}{4}}\end{aligned}$$

This method is nice in that it makes the problem very easy once you have the prime factorizations (provided they are correct). On the other hand, many people do not like to take the time to do the prime factorization, so you will have to decide what works best for you.

If you have ever simplified a fraction before, you may have done so in yet another way, using division instead of canceling factors. To see how this works, we will introduce a modified version of the fundamental principle of fractions:

Fundamental Principle of Fractions, version 2:
Let a, b and c be *any* numbers, *except* $b \neq 0$ and $c \neq 0$. Then the following is true:

$$\frac{a}{b} = \frac{a \div c}{b \div c}$$

Notice that the only difference between this version of the fundamental principle and the other is that the multiplication has been replaced with division. The reason we are allowed to do this will be presented when we introduce reciprocals (page 150). For now, this gives a third method for simplifying fractions. Let us redo our two previous examples using this method.

Example 10. *Simplify* $\dfrac{4}{6}$

Solution 10. *Noticing that the* 4 *and* 6 *have a common factor of* 2*, we will divide numerator and denominator both by it.*

$$\begin{aligned} \frac{4}{6} &= \frac{4 \div 2}{6 \div 2} \\ &= \mathbf{\frac{2}{3}} \end{aligned}$$

Example 11. *Simplify* $\dfrac{12}{16}$

Solution 11. *We will do this problem as before, pretending that our first thought is to divide out a factor of* 2*. Again, this situation of only recognizing a common factor instead of the GCF may happen when you are trying to find factors of large numbers. The bigger the factor you divide out, the faster you will simplify, but provided you do not stop too soon (or give up!), you will still get there in the end.*

$$\begin{aligned} \frac{12}{16} &= \frac{12 \div 2}{16 \div 2} \\ &= \frac{6}{8} \\ &= \frac{6 \div 2}{8 \div 2} \\ &= \mathbf{\frac{3}{4}} \end{aligned}$$

We will do three more examples, one for each of the simplifying methods given.

Example 12. *Simplify* $\dfrac{18}{27}$

Solution 12. *The number* 9 *is a common factor of both* 18 *and* 27 *(the GCF in fact). So:*

$$\begin{aligned}\frac{18}{27} &= \frac{2\cdot \cancel{9}}{3\cdot \cancel{9}}\\ &= \mathbf{\frac{2}{3}}\end{aligned}$$

Example 13. *Simplify* $\dfrac{330}{396}$

Solution 13. *We will simplify this fraction using the prime factorization method. A case like this where the numbers are large, and you would probably have to do several steps using either of the other methods (as the GCF is not obvious, but it is easy to see that* 2*, for example, is a factor), is where this prime factorization method may be easier and just as fast.*
First we need to find the prime factorization of 330 *and* 396*.*
Scratch work:

$$\begin{tikzcd}[column sep=small]
 & & & 330 \arrow[dll] \arrow[drr] & & & \\
 & 10 \arrow[dl] \arrow[dr] & & & & 33 \arrow[dl] \arrow[dr] & \\
2 & & 5 & & 3 & & 11
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & & & 396 \arrow[dll] \arrow[drr] & & & \\
 & 4 \arrow[dl] \arrow[dr] & & & & 99 \arrow[dl] \arrow[dr] & \\
2 \arrow[d] & & 2 \arrow[dl] & & 9 \arrow[dl] \arrow[dr] & & 11 \arrow[d] \\
2 & 2 & & 3 & & 3 & 11
\end{tikzcd}$$

So, the prime factorizations are:
$330 = 2\cdot 3\cdot 5\cdot 11$
$396 = 2\cdot 2\cdot 3\cdot 3\cdot 11$
Replacing the numbers with their prime factorization allows us

to cancel.

$$\begin{aligned}
\frac{330}{396} &= \frac{2\cdot 3\cdot 5\cdot 11}{2\cdot 2\cdot 3\cdot 3\cdot 11} \\
&= \frac{\cancel{2}\cdot \cancel{3}\cdot 5\cdot \cancel{11}}{\cancel{2}\cdot 2\cdot \cancel{3}\cdot 3\cdot \cancel{11}} \\
&= \frac{5}{2\cdot 3} \\
&= \mathbf{\frac{5}{6}}
\end{aligned}$$

Example 14. *Simplify* $\frac{48}{66}$

Solution 14. *The number* 6 *is a factor of both numerator and denominator, so dividing both by it gives:*

$$\begin{aligned}
\frac{48}{66} &= \frac{48\div 6}{66\div 6} \\
&= \mathbf{\frac{8}{11}}
\end{aligned}$$

Mixing in negatives

So far, we have only been dealing with positive fractions. Negative numbers, however, may appear in the numerator and denominator as well, and you need to become adept at handling them. The trick is to remember that the fundamental principle of fractions states that multiplying top and bottom by the same nonzero number makes an equivalent fraction, and one possible nonzero number is negative one. Let us look at a fraction where both numerator and denominator are negative numbers, and see how this helps.

$$\frac{-2}{-7} = \frac{-2\cdot(-1)}{-7\cdot(-1)} = \frac{2}{7} \quad .$$

On the other hand, if the numerator and denominator have opposite signs:

$$\frac{3}{-5} = \frac{3 \cdot (-1)}{-5 \cdot (-1)} = \frac{-3}{5} \quad ,$$

and in either case here, we are dividing two numbers of opposite sign which gives us a negative result. Therefore:

$$\frac{3}{-5} = \frac{-3}{5} = -\frac{3}{5}.$$

Generalizing this, we get:

<u>Fraction Fact 4:</u> For any positive numbers a and b,

$$\frac{-a}{-b} = \frac{a}{b}$$

and

<u>Fraction Fact 5:</u> For any positive numbers a and b,

$$\frac{a}{-b} = \frac{-a}{b} = -\frac{a}{b}$$

When both numerator and denominator have the same sign, we will (obviously?) prefer to have them both positive. When the numerator and denominator have the opposite sign, it is standard to prefer to have the negative sign either in the numerator or out front (depending on the situation). Wanting to move an overall negative sign around in a fraction will occur more often than you might think, so let us do some practice.

Example 15. *Use Fraction Facts 4 and 5 to find one or two equivalent fractions for each of the following, where the equivalent fraction has the same numerical value for its numerator and denominator, but the signs of each have been changed.*

1. $\dfrac{-3}{8}$

2. $\dfrac{-5}{-7}$

3. $\dfrac{2}{9}$

4. $\dfrac{4}{-13}$

5. $-\dfrac{11}{12}$

6. $-\dfrac{-2}{-3}$

7. $-\dfrac{4}{-5}$

Solution 15. *Applying our Fraction Facts yields:*

1. $\dfrac{-3}{8} = \dfrac{\mathbf{3}}{\mathbf{-8}} = -\dfrac{\mathbf{3}}{\mathbf{8}}$

2. $\dfrac{-5}{-7} = \dfrac{\mathbf{5}}{\mathbf{7}}$

3. $\dfrac{2}{9} = \dfrac{\mathbf{-2}}{\mathbf{-9}}$

4. $\dfrac{4}{-13} = \dfrac{\mathbf{-4}}{\mathbf{13}} = -\dfrac{\mathbf{4}}{\mathbf{13}}$

5. $-\frac{11}{12} = \mathbf{\frac{-11}{12}} = \mathbf{\frac{11}{-12}}$

6. *This fraction seems to have way too many "$-$" signs! Since neither fraction fact 4 nor 5 covers this exact situation, we will recall the "$-$" sign all the way in front may be read as "take the opposite of". Thus,*

$$\begin{aligned} -\frac{-2}{-3} &= -\left(\frac{-2}{-3}\right) \\ &= -\left(\frac{2}{3}\right) \\ &= -\mathbf{\frac{2}{3}} \end{aligned}$$

As the opposite of positive two-thirds is negative two-thirds.

7. *Proceeding as we did in the last problem:*

$$\begin{aligned} -\frac{4}{-5} &= -\left(\frac{4}{-5}\right) \\ &= -\left(-\frac{4}{5}\right) \\ &= \mathbf{\frac{4}{5}} \end{aligned}$$

As the opposite of negative four-fifths is positive four-fifths.

We will often want or need to move a negative sign around as part of another problem. When simplifying a fraction, for example, we want there to either be no negative signs or just one overall sign out in front.

Example 16. *Simplify* $\frac{-18}{-22}$

Solution 16. *We will simplify this fraction using the cancelation* *of factor method.*

$$\begin{aligned} \frac{-18}{-22} &= \frac{18}{22} \\ &= \frac{9\cdot \cancel{2}}{11\cdot \cancel{2}} \\ &= \mathbf{\frac{9}{11}} \end{aligned}$$

In the first step we got rid of the negative signs as per Fraction Fact 4, and then simplified as normal.

Example 17. *Simplify* $\frac{15}{-75}$

Solution 17. *We will simplify this fraction using version* 2 *of the fundamental principle of fractions.*

$$\begin{aligned} \frac{15}{-75} &= -\frac{15}{75} \\ &= -\frac{15 \div 15}{75 \div 15} \\ &= -\mathbf{\frac{1}{5}} \end{aligned}$$

Example 18. *Simplify* $\frac{-84}{32}$

Solution 18. *We will simplify this fraction using version* 2 *of the fundamental principle of fractions.*

$$\begin{aligned} \frac{-84}{32} &= -\frac{84}{32} \\ &= -\frac{84 \div 4}{32 \div 4} \\ &= -\mathbf{\frac{21}{8}} \end{aligned}$$

This last example may cause you to pause, but negative twenty-one over eight is the correct answer in lowest terms. If the answer bothers you, this means you are probably familiar with mixed numbers.

Mixed numbers

All fractions may be put into one of two categories.

Definition: A fraction whose numerator is less than its denominator, *in absolute value*, is called a proper fraction.

Definition: A fraction whose numerator is greater than or equal to its denominator, *in absolute value*, is called an improper fraction.

Example 19. *State whether each of the following are proper or improper fractions.*

1. $\frac{8}{9}$
2. $\frac{13}{4}$
3. $\frac{5}{-8}$
4. $\frac{12}{4}$

Solution 19. *According the definitions:*

1. $\frac{8}{9}$ *is a* ***proper fraction****.*
2. $\frac{13}{4}$ *is an* ***improper fraction****.*
3. $\frac{5}{-8}$ *is a* ***proper fraction****.*

   Here we see the importance of taking the absolute value of the numerator and denominator first. Without doing so, we would have said that $5 > -8$*, but in absolute value,* $5 < 8$*.*

4. $\frac{12}{4}$ *is an **improper fraction**.*

True, this fraction is not in lowest terms, and it can even be simplified to an integer, but as written, it is an improper fraction.

You may change an improper fraction into another type of number called a mixed number. A mixed number is a combination of an integer and a fraction.

To change an improper fraction ⟶ mixed number

1. Perform long division, dividing the numerator by the denominator, and taking note of the *quotient* and *remainder*.
2. The quotient is your integer, and next to it make a fraction of the remainder over the old denominator.

Example 20. *Change* $\frac{13}{5}$ *to a mixed number.*

Solution 20. *Following our new rule, we will start by doing long division.*
Scratch work:

$$\begin{array}{rr} & 2 \\ 5 & |\overline{13} \\ & \underline{10} \\ & 3 \end{array}$$

So our quotient is 2 *with a remainder of* 3*. Therefore:*

$$\frac{13}{5} = \mathbf{2\tfrac{3}{5}}$$

Example 21. *Change* $\frac{37}{2}$ *to a mixed number.*

> **Solution 21.** *Following our new rule, we will start by doing long division.*
> *Scratch work:*
>
> $$\begin{array}{r|l} & 18 \\ 2 & \overline{37} \\ & \underline{2} \\ & 17 \\ & \underline{16} \\ & 1 \end{array}$$
>
> *So our quotient is* 18 *with a remainder of* 1*. Therefore:*
>
> $$\frac{37}{2} \quad = \quad \mathbf{18\frac{1}{2}}$$

If there are negative signs in your improper fraction, use the Fraction Facts 4 (page 103) and 5 (page 103) to get either no negative signs, or just one out in front of the fraction. If it is the case where there is one negative sign out in front, change the improper fraction to a mixed number as normal, and keep the negative sign tagging along.

Example 22. *Change* $-\frac{21}{8}$ *to a mixed number.*

> **Solution 22.** *This was the answer to example 18 on page 106. Since the only negative is already out in front, we will change the fraction to a mixed number, and tag along the sign.*
> *Scratch work:*
>
> $$\begin{array}{r|l} & 2 \\ 8 & \overline{21} \\ & \underline{16} \\ & 5 \end{array}$$
>
> *So our quotient is* 2 *with a remainder of* 5*. Therefore:*
>
> $$-\frac{21}{8} \quad = \quad \mathbf{-2\frac{5}{8}}$$

Example 23. *Change* $\dfrac{-51}{10}$ *to a mixed number.*

> **Solution 23.** *Using Fraction Fact* 5*, we may pull the negative sign out front where it will now just tag along. Changing the fraction to a mixed number gives: Scratch work:*
>
> $$\begin{array}{r|l} & 5 \\ 10 & \overline{51} \\ & \underline{50} \\ & 1 \end{array}$$
>
> *So our quotient is* 5 *with a remainder of* 1*. Therefore:*
>
> $$\frac{-51}{10} \quad = \quad \mathbf{-5\tfrac{1}{10}}$$

Example 24. *Change* $\dfrac{-47}{-3}$ *to a mixed number.*

> **Solution 24.** *Using Fraction Fact* 4*, we may get rid of both negative signs. Now, to change the fraction to a mixed number: Scratch work:*
>
> $$\begin{array}{r|l} & 15 \\ 3 & \overline{47} \\ & \underline{3} \\ & 17 \\ & \underline{15} \\ & 2 \end{array}$$
>
> *So our quotient is* 15 *with a remainder of* 2*. Therefore:*
>
> $$\frac{-47}{-3} \quad = \quad \mathbf{15\tfrac{2}{3}}$$

Example 25. *Change* $\dfrac{54}{3}$ *to a mixed number.*

Solution 25. *No negative signs to worry about, so we go ahead and do long division.*
Scratch work:

$$\begin{array}{r|l} & 18 \\ \hline 3 & 54 \\ & \underline{3} \\ & 24 \\ & \underline{24} \\ & 0 \end{array}$$

Oops, we got a remainder of 0*! This means that this improper fraction can be turned into an integer, not a mixed number. Therefore:*

$$\frac{54}{3} \quad = \quad \mathbf{18}$$

It is not always obvious whether an improper fraction will become a mixed number or an integer until you do the long division. To acknowledge this, the directions for similar problems at the end of this section will ask you to convert improper fractions to either a mixed number or an integer.

We also need to know how to change a mixed number into an improper fraction.

To change a mixed number $\longrightarrow$ an improper fraction

1. The improper fraction should have the same sign as the mixed number does.
2. To find the numerator of your fraction, multiply the old denominator by the absolute value of your integer and add this to the old numerator.
3. The denominator of your improper fraction is the same as the denominator of the fractional part of your mixed number.

Example 26. *Change $2\frac{5}{7}$ to an improper fraction.*

Solution 26. *Following our rules, we first note that the fraction will be positive. Then:*

$$\begin{aligned} 2\tfrac{5}{7} &= \frac{7 \cdot 2 + 5}{7} \\ &= \mathbf{\frac{19}{7}} \end{aligned}$$

Example 27. *Change $-3\frac{3}{8}$ to an improper fraction.*

Solution 27. *Following our rules, we first note that the fraction will be negative. Then:*

$$\begin{aligned} -3\tfrac{3}{8} &= -\frac{8 \cdot 3 + 3}{8} \\ &= -\mathbf{\frac{27}{8}} \end{aligned}$$

In this last example, note the importance of multiplying the old denominator by *the absolute value* of the integer. Had you multiplied 8 times -3 and then added 3, you would have gotten a numerator of -21, which is incorrect.

A few more examples.

Example 28. *Change $25\frac{1}{2}$ to an improper fraction.*

Solution 28. *Following our rules, we first note that the fraction will be positive. Then:*

$$\begin{aligned} 25\tfrac{1}{2} &= \frac{2 \cdot 25 + 1}{2} \\ &= \mathbf{\frac{51}{2}} \end{aligned}$$

Example 29. *Change $-13\frac{4}{5}$ to an improper fraction.*

> **Solution 29.** *Following our rules, we first note that the fraction will be negative. Then:*
>
> $$\begin{aligned} -13\tfrac{4}{5} &= -\frac{5 \cdot 13 + 4}{5} \\ &= -\frac{\mathbf{69}}{\mathbf{5}} \end{aligned}$$

Example 30. *Change $11\frac{5}{9}$ to an improper fraction.*

> **Solution 30.** *Following our rules, we first note that the fraction will be positive. Then:*
>
> $$\begin{aligned} 11\tfrac{5}{9} &= \frac{9 \cdot 11 + 5}{9} \\ &= \frac{\mathbf{104}}{\mathbf{9}} \end{aligned}$$

Comparing rational numbers

We may wish to determine which of two rational numbers is the lesser or greater (or even whether they are equal). This turns out to be more difficult than it was with integers, even determining equality! For example, obviously the integer 5 is equal to the integer 5 ($5 = 5$); not as obvious is the fact that the fraction $\frac{3}{7}$ is equal to the fraction $\frac{54}{126}$ ($\frac{3}{7} = \frac{54}{126}$). To learn how to compare fractions, we will look at some obvious cases first, and then show what to do when the answer is not obvious.

> **Obvious case 1:** If you are comparing a positive number to a nonpositive number, the arrow should point towards the nonpositive number (or the mouth should be eating the positive number, if you prefer). If you are comparing a negative number to a nonnegative number, the arrow should point towards the negative number (or the mouth should be eating the nonnegative number, if you prefer).

It sounds a bit more complicated than it is.

Example 31. *Insert a $\leq$ or $>$ between each of the following.*

1. $-\frac{13}{14}$ ___ $\frac{14}{15}$

2. $\frac{25}{8}$ ___ $-\frac{31}{10}$

3. 0 ___ $-\frac{4}{13}$

4. $-\frac{18}{5}$ ___ 0

5. $-\frac{258}{11}$ ___ $\frac{1}{2}$

6. 9 ___ $-\frac{110}{113}$

Solution 31. *Since we are not comparing two numbers of the same sign, we easily get:*

1. $-\frac{13}{14} \leq \frac{14}{15}$

2. $\frac{25}{8} > -\frac{31}{10}$

3. $0 > -\frac{4}{13}$

4. $-\frac{18}{5} \leq 0$

5. $-\frac{258}{11} \leq \frac{1}{2}$

6. $9 \quad > \quad -\dfrac{110}{113}$

The next easiest case, is if the numbers are the same sign, but between different integers.

> **<u>Obvious case 2:</u>** If one or both of the numbers you are comparing is an improper fraction, convert it to a mixed number. Sometimes this will let you estimate the numbers well enough to tell which is the lesser.

Example 32. *Insert a $<$ or $\geq$ between each of the following.*

1. $\dfrac{23}{3} \quad \_\_\_ \quad \dfrac{44}{9}$

2. $\dfrac{25}{4} \quad \_\_\_ \quad 7$

3. $-\dfrac{15}{2} \quad \_\_\_ \quad -\dfrac{16}{3}$

Solution 32. *First changing all the improper fractions to mixed numbers gives:*

1. $\dfrac{23}{3} = 7\dfrac{2}{3}$,

and $\dfrac{44}{9} = 4\dfrac{8}{9}$.

If you needed to do long division scratch work, you could have done this on the side. Now it is easy to see that $7\frac{2}{3} \geq 4\frac{8}{9}$, (as the first number is between 7 and 8 and the second is between 4 and 5). Thus,

$$\mathbf{\frac{23}{3} \quad \geq \quad \frac{44}{9}}$$

2. $\dfrac{25}{4} = 6\dfrac{1}{4}$,

and 7 is an integer.

Now it is easy to see that six-and-a-bit is < seven. Thus,

$$\mathbf{\frac{25}{4} \quad < \quad 7}$$

3. $\dfrac{-15}{2} = -7\dfrac{1}{2}$,

and $\dfrac{-16}{3} = -5\dfrac{1}{3}$.

Now recall that the more negative a number is, the further to the left it is on the number line. So, since seven-and-a-bit is bigger than five-and-a-bit, negative seven-and-a-bit is further to the left on the number line, or $-7\frac{1}{2} < -5\frac{1}{3}$*. Therefore:*

$$-\mathbf{\frac{15}{2}} \quad < \quad -\mathbf{\frac{16}{3}}$$

So what happens if we do not have an obvious case? For example, what if we are comparing two proper fractions of the same sign, like $\frac{5}{11}$ versus $\frac{11}{23}$? Recall that physically $\frac{5}{11}$ means you have divided something (say a rectangular cake, if you prefer) into 11 equal-sized pieces and want 5 of them. $\frac{11}{23}$ means you have divided a similar object into 23 equal-sized pieces and want 11 of them. It is possible that a physical picture could help, say if one of the fractions represents a considerable amount more than the other, but here, both fractions are a bit under half, and it would be hard to judge with the naked eye. But wait, what if instead of dividing the two objects up into a different number of pieces, you divide them both into the same number. Wouldn't this make comparing the two fractions easier? Instead of comparing $\frac{5}{11}$ and $\frac{11}{23}$, we ask you to compare $\frac{5}{11}$ to $\frac{6}{11}$. With the denominators being the same (and thus the size of the pieces each is divided into the same), it is now a simple matter of comparing the numerators.

This is the idea behind comparing fractions. If the two fractions you wish to compare do not have the same denominator, you WILL MAKE them have the same denominator by creating equivalent fractions. First you will need to decide what you would like the denominators to be. A reasonable choice would be the least common multiple of the denominators, but for comparison purposes, the common multiple of their product is more often used.

Example 33. *Insert a $\leq$ or $>$ between the following two fractions.*

$\frac{5}{11}$ ___ $\frac{11}{23}$

Solution 33. *First, we will multiply* $23 \cdot 11$*:*
Scratch work:

$$\begin{array}{cccc} & & 2 & 3 \\ \times & & 1 & 1 \\ \hline & & 2 & 3 \\ & 2 & 3 & 0 \\ \hline & 2 & 5 & 3 \end{array}$$

So a common multiple of the denominators is 253*. We will use the fundamental principle of fractions (page 95) to make equivalent fractions with the appropriate denominator.*

$$\begin{aligned} \frac{5}{11} &= \frac{5 \cdot 23}{11 \cdot 23} \\ &= \frac{115}{253} \end{aligned}$$

while

$$\begin{aligned} \frac{11}{23} &= \frac{11 \cdot 11}{23 \cdot 11} \\ &= \frac{121}{253} \end{aligned}$$

You may have had to do a little bit of scratch work to get the numerators. Now, though, it is easy to compare:

$\frac{115}{253} \leq \frac{121}{253}$

Therefore: $\mathbf{\frac{5}{11} \leq \frac{11}{23}}$

To save a little bit of work, a shortcut was developed, and this shortcut is what you will wish to use when comparing.

> **General case:** To compare two fractions, multiply each denominator by the *other* numerator, and write the product above the corresponding numerator. The same inequality symbol which goes between the two products, goes between the fractions.

Example 34. *Insert a* $\leq$ *or* $>$ *between the following two fractions.*

$\frac{5}{11}$ ___ $\frac{11}{23}$

Solution 34. *Redoing this problem using the shortcut:*

$$\begin{array}{ccc} 115 & & 121 \\ \underline{5} & & \underline{11} \\ 11 & & 23 \end{array}$$

Since $115 \leq 121$*, we get:*

$$\mathbf{\frac{5}{11} \leq \frac{11}{23}}$$

The shortcut works by finding only half of the appropriate equivalent fractions - just the numerators, as you will notice if you compare the numbers 115 and 121 to the true equivalent fractions we got when we did the problem all out before. Some people refer (incorrectly!) to this as cross-multiplying. We will see real cross-multiplying later in Chapter 2.

If you are always careful to look for the obvious cases first, you will only need to use this shortcut on proper fractions. Let us do several more examples.

Example 35. *Insert a* $\leq$ *or* $>$ *between the following two fractions.*

$\frac{3}{5}$ ___ $\frac{9}{16}$

Solution 35. *Using the shortcut:*

$$\begin{array}{ccc} 48 & & 45 \\ \underline{3} & \nwarrow\!\!\nearrow & \underline{9} \\ 5 & \nearrow\!\!\nwarrow & 16 \end{array}$$

Since $48 > 45$*, we get:*

$$\mathbf{\frac{3}{5} > \frac{9}{16}}$$

Example 36. *Insert a* $\leq$ *or* $>$ *between the following two fractions.*

$\frac{28}{5}$ ___ $\frac{37}{7}$

Solution 36. *Since both fractions are the same sign, you may go right to the shortcut if wish, but since the fractions are improper, we will check to see if the answer is obvious if we convert the fractions to mixed numbers.*

$\frac{28}{5} = 5\frac{3}{5}$,

and $\frac{37}{7} = 5\frac{2}{7}$.

Since both mixed numbers have the same integer part, the answer is still not obvious. However, if we figure out which fractional part is larger, this will be enough to tell us which mixed number is larger. This means that by taking the time to change the improper fractions to mixed numbers, we saved ourselves from having to work with the larger numbers. Let us compare the two fractional parts.

$$\begin{array}{ccc} 21 & & 10 \\ \underline{3} & \nwarrow\!\!\nearrow & \underline{2} \\ 5 & \nearrow\!\!\nwarrow & 7 \end{array}$$

Since $21 > 10$*, we get:*

$\frac{3}{5} > \frac{2}{7} \Rightarrow 5\frac{3}{5} > 5\frac{2}{7}$*, so:*

$$\mathbf{\frac{28}{5} > \frac{37}{7}}$$

Example 37. *Insert a $\leq$ or $>$ between the following two fractions.*

$$-\frac{38}{11} \quad \_\_\_\_ \quad -\frac{17}{5}$$

Solution 37. *Both fractions are negative, so that is no help. Again, you could jump right to the shortcut, but be careful about the negative signs. Instead of trying to include them in the product, you should just ignore them and compare the absolute value of the fractions, and then recall that a larger negative value is further to the left on the number line. We, though, our going to convert the two improper fractions to mixed numbers first.*

$-\frac{38}{11} = -3\frac{5}{11}$,

and $-\frac{17}{5} = -3\frac{2}{5}$.

Since both mixed numbers have the same integer part, the answer is still not obvious. Just like before though, we only need to compare the fractional parts. So,

$$\begin{array}{ccc} 25 & & 22 \\ \underline{5} & & \underline{2} \\ 11 & & 5 \end{array}$$

Since $25 > 22$, *we get:*

$$\begin{array}{ccc} \frac{5}{11} & > & \frac{2}{5} \\ & \Downarrow & \\ 3\frac{5}{11} & > & 3\frac{2}{5} \\ & \Downarrow & \\ -3\frac{5}{11} & \leq & -3\frac{2}{5} \end{array}$$

as the larger negative number is further to the left on the number line. Therefore:

$$-\mathbf{\frac{38}{11}} \quad \leq \quad -\mathbf{\frac{17}{5}}$$

Improper Fractions vs. Mixed Numbers

Now having seen both improper fractions and mixed numbers, you may wonder which form is more preferable. Many people prefer mixed numbers, but the truth is they both have good points.

- *In general,* mixed numbers are better when you wish to know how large the number is or wish to approximate it.
- *In general,* improper fractions are better when you wish to perform a mathematical operations (addition, subtraction, etc.).

As you can see, both forms has it uses. Therefore, a note on how we will present solutions is in order.

IMPORTANT NOTATION: For the rest of this ebook:

- *except* for problems where we specifically ask you to do otherwise (like example 4 on page 95), we will expect all final answers involving fractions (including the fractional part of mixed numbers) to be *simplified.*
- *except* for problems where we ask you to change an improper fraction to a mixed number (like example 20 on page 108) or vice versa (like example 26 on page 112), we will NOT care whether you leave your final answer a mixed number or improper fraction. The solutions to exercises and our examples will contain both.

And finally, one more thing to keep in mind about mixed numbers.

Important Fact about Mixed Numbers

Even though we put the fractional part of a mixed number right next to the integral part, there is a *hidden* plus sign there.

Example 38. $6\frac{2}{3} = 6 + \frac{2}{3}$

While this may not seem earth-shattering, it is actually quite significant. Even though it is fairly common to hide a multiplication sign in mathematics (you will see several cases of this in Core Math courses), it is almost unheard of to hide an addition sign. Remembering the hidden plus sign in a mixed number may help you to understand why the various shortcuts we use with mixed numbers work, and we will demonstrate this in the next couple of sections.

One other note, for negative mixed numbers, the fraction is the same sign as the integer.

Example 39.

$$\begin{aligned} -5\tfrac{1}{4} &= -\left(5 + \tfrac{1}{4}\right) \\ &= -5 + \left(-\tfrac{1}{4}\right) \end{aligned}$$

SECTION 1.7 EXERCISES

(Answers are found on page 374.)

Rewrite the following fractions as equivalent fractions with the same numerical values in the numerator and denominator but either no negative signs, or only one negative sign out in front.

1. $\dfrac{-2}{-7}$
2. $\dfrac{-3}{5}$
3. $\dfrac{5}{-9}$
4. $\dfrac{-11}{-13}$
5. $\dfrac{-2}{3}$
6. $\dfrac{-21}{-22}$
7. $\dfrac{-5}{6}$
8. $\dfrac{-3}{-4}$
9. $\dfrac{3}{-11}$
10. $\dfrac{7}{-10}$

Rewrite each fraction as an equivalent fraction with the given denominator.

11. $\frac{2}{5}$ with a denominator of 20
12. $\frac{-1}{4}$ with a denominator of 8
13. $-\frac{3}{7}$ with a denominator of 21
14. $\frac{1}{2}$ with a denominator of 60
15. $-\frac{2}{3}$ with a denominator of 18
16. $\frac{5}{8}$ with a denominator of 32
17. $-\frac{4}{5}$ with a denominator of 45
18. $\frac{7}{8}$ with a denominator of 48
19. $\frac{-1}{6}$ with a denominator of 42
20. $\frac{2}{9}$ with a denominator of 72

Put the following fractions in simplest form.

21. $\frac{24}{36}$
22. $-\frac{3}{15}$
23. $-\frac{10}{25}$
24. $\frac{33}{77}$
25. $-\frac{32}{72}$
26. $\frac{10}{16}$
27. $-\frac{24}{40}$
28. $\frac{13}{39}$
29. $\frac{20}{50}$
30. $-\frac{65}{80}$

Change the following improper fractions to mixed numbers.

31. $\frac{11}{3}$
32. $-\frac{15}{2}$
33. $\frac{57}{4}$
34. $-\frac{35}{8}$
35. $\frac{62}{3}$
36. $-\frac{29}{2}$
37. $\frac{54}{11}$
38. $-\frac{104}{9}$
39. $\frac{82}{3}$
40. $-\frac{73}{5}$

Change the following mixed numbers to improper fractions.

41. $3\frac{2}{7}$
42. $-2\frac{1}{5}$
43. $5\frac{3}{4}$
44. $-1\frac{10}{11}$
45. $11\frac{2}{3}$
46. $-5\frac{7}{8}$
47. $9\frac{3}{10}$
48. $-4\frac{1}{6}$
49. $8\frac{5}{9}$
50. $-2\frac{11}{24}$

Insert a $\leq$ or $>$ between each of the following pairs of numbers to make a true statement.

51. $-\frac{3}{5}$ ___ $\frac{1}{2}$
52. $\frac{4}{9}$ ___ $\frac{11}{24}$
53. $\frac{8}{13}$ ___ $\frac{3}{5}$
54. $-\frac{5}{7}$ ___ $-\frac{11}{15}$
55. $\frac{31}{8}$ ___ $\frac{52}{19}$
56. $-\frac{100}{3}$ ___ $-\frac{65}{2}$
57. $\frac{15}{17}$ ___ $-\frac{110}{133}$
58. $\frac{11}{5}$ ___ $\frac{9}{4}$
59. $-\frac{5}{6}$ ___ $-\frac{4}{5}$
60. 6 ___ $\frac{53}{9}$

1.8 Fraction Addition and Subtraction

Adding or Subtracting Two Fractions

Imagine you have a pizza which is cut into eight equal-sized pieces, and you take two pieces. This means you are going to eat two-eighths ($\frac{2}{8}$) of the pizza. After eating them, you are still hungry, so you take another piece. How much of the pizza have you eaten now? Isn't it fairly obvious that you have eaten three-eighths ($\frac{3}{8} = \frac{2}{8} + \frac{1}{8}$)? This shows that addition (and subtraction) of fractions is easy *provided the two fractions have the same denominator*; i.e. the size of the pieces are the same. This leads to the following rules.

Adding Fractions with the Same Denominator

Let a, b and c be *any* numbers, *except* $b \neq 0$. Then:

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

and

Subtracting Fractions with the Same Denominator

Let a, b and c be *any* numbers, *except* $b \neq 0$. Then:

$$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

Example 1. *Add* $\dfrac{3}{11} + \dfrac{6}{11}$

Solution 1. *Following our rule, we get:*

$$\begin{aligned} \frac{3}{11} + \frac{6}{11} &= \frac{3+6}{11} \\ &= \mathbf{\frac{9}{11}} \end{aligned}$$

Example 2. *Add* $\frac{3}{8} + \frac{1}{8}$

Solution 2. *Following our rule, we get:*

$$\begin{aligned} \frac{3}{8} + \frac{1}{8} &= \frac{3+1}{8} \\ &= \frac{4}{8} \\ &= \frac{4 \div 4}{8 \div 4} \\ &= \mathbf{\frac{1}{2}} \end{aligned}$$

This last example illustrates an important point. Even if both of your original fractions are simplified, you may have to simplify your final answer.

Example 3. *Subtract* $\frac{5}{9} - \frac{1}{9}$

Solution 3. *Following our rule, we get:*

$$\begin{aligned} \frac{5}{9} - \frac{1}{9} &= \frac{5-1}{9} \\ &= \mathbf{\frac{4}{9}} \end{aligned}$$

Example 4. *Subtract* $\dfrac{7}{12} - \dfrac{5}{12}$

Solution 4. *Following our rule, we get:*

$$\begin{aligned} \frac{7}{12} - \frac{5}{12} &= \frac{7-5}{12} \\ &= \frac{2}{12} \\ &= \frac{2 \div 2}{12 \div 2} \\ &= \mathbf{\frac{1}{6}} \end{aligned}$$

So far, so good, but what if we wish to add (or subtract) fractions which have different denominators? Think back to the pizza example. Now say we have two pizzas; one cut into eight equal-sized pieces with pepperoni, and one cut into six equal-sized pieces with sausage. The overall size of the two pizzas is the same, so this means that the pieces of the sausage pizza are larger. Now you take two pieces of the pepperoni pizza and one of the sausage. How much total pizza have you taken? The word "piece" can be ambiguous in the English language, so you could feasibly get away with saying three pieces, but obviously you have more than you would have had, had you taken three of the smaller pieces, and less than you would have had, had you taken three of the larger pieces. Therefore, we know that $\frac{2}{8} + \frac{1}{6} > \frac{3}{8}$ and $\frac{2}{8} + \frac{1}{6} < \frac{3}{6}$. While it is nice to have bounds on our answer, we still want to know what $\frac{2}{8} + \frac{1}{6}$ equals. If you stop to think about it for a moment, you will realize that we have all the information that we need. First, we know that it is easy to add (or subtract) fractions when they have the same denominator. Second, we know that we may *change the denominator* of our fractions *without changing their value*, by making equivalent fractions (fundamental principle of fractions, page 95). This is all we need to add or subtract any two fractions we desire.

Important and Useful Fact

To add or subtract two fractions with different denominators, you will first MAKE THEM HAVE THE SAME DENOMINATOR by forming equivalent fractions, and then add or subtract as before.

Example 5. *Let us solve our pizza problem. Notice that* $\frac{2}{8} = \frac{1}{4}$ *in simplified form, so we are really trying to add:*

$\dfrac{1}{4} + \dfrac{1}{6}$.

Solution 5. *A common multiple of the two denominators is* 12*, so we will start by making equivalent fractions, then adding.*

$$\begin{aligned} \frac{1}{4} + \frac{1}{6} &= \frac{1 \cdot 3}{4 \cdot 3} + \frac{1 \cdot 2}{6 \cdot 2} \\ &= \frac{3}{12} + \frac{2}{12} \\ &= \frac{3+2}{12} \\ &= \mathbf{\frac{5}{12}} \end{aligned}$$

Example 6. *Subtract* $\dfrac{7}{10} - \dfrac{2}{5}$.

Solution 6. *A common multiple of the two denominators is* 10*,* *so we will start by making equivalent fractions, then subtracting.*

$$\begin{aligned}\frac{7}{10}-\frac{2}{5} &= \frac{7}{10}-\frac{2\cdot 2}{5\cdot 2}\\ &= \frac{7}{10}-\frac{4}{10}\\ &= \frac{7-4}{10}\\ &= \mathbf{\frac{3}{10}}\end{aligned}$$

You can see that we will be making a lot of equivalent fractions when adding and subtracting, so finding a common multiple is important. We could use the common multiple of the product, as we did for comparisons, but while this is an easy common multiple to find, there are a couple of problems with it. First, we would prefer to keep the common multiple as small as possible since we will be doing our scratch work by hand instead of using a calculator. Second, currently we are only working with two fractions at a time, but later we will see cases where we need to find a common multiple of the denominators of three or even four fractions at once, and this exacerbates the problem of the product of the denominators being large. Therefore, we will prefer to work with the *least* common multiple of the denominators.

Least Common Denominator (LCD)

Definition: The least common multiple of the denominators of all the fractions involved is called the least common denominator, or LCD.

If necessary, you may wish to go back to the section on finding least common multiples (section 1.6) to recall the two methods for finding the LCD. Looking back at the previous two examples, you will see that we used the LCD for the common denominator both times. Let us do a few more examples of adding and subtracting fractions with different denominators using the LCD.

Example 7. *Add* $2+\dfrac{5}{7}$.

Solution 7. *Any integer can be changed to a fraction easily by putting it over* 1*. Doing so gives:*

$$2 + \frac{5}{7} \quad = \quad \frac{2}{1} + \frac{5}{7}$$

The LCD of the two denominators is 7*, so:*

$$\begin{aligned}
\frac{2}{1} + \frac{5}{7} &= \frac{2 \cdot 7}{1 \cdot 7} + \frac{5}{7} \\
&= \frac{14}{7} + \frac{5}{7} \\
&= \frac{14 + 5}{7} \\
&= \mathbf{\frac{19}{7}} \quad \textit{or} \quad \mathbf{2\frac{5}{7}}
\end{aligned}$$

Compare the original problem to the mixed number version of the final answer. Does it seem trivial? It should! Here is the first example where the important fact (page 121) about mixed numbers having a hidden plus sign shows up. As you can see from this example, if you are adding an integer and a proper fraction (it is important for both to be the same sign), the solution as a mixed number is trivial. As for the solution as an improper fraction, you may solve the long way, as in the last example, or you may just convert the mixed number to an improper fraction using the shortcut we learned last section. See example 26 on page 112 and compare to the last example.

This last example also shows how the shortcut for changing a mixed number to an improper fraction was developed. When you make the integer a fraction, it will always go over 1, and the LCD of 1 and another number is always the other number. Therefore, the denominator of the improper fraction will always be the same as the denominator of the fractional part of the mixed number. As for the numerator, the integer will always need to be multiplied by the denominator of the fractional part (to make an equivalent fraction with the right denominator), and the product will then be added to the numerator of the fractional part. While it is easier to just use the shortcut rather than the long way we did in the last example, what this means is that if you ever forget the shortcut, you can do one problem the long way as above to refresh your memory.

Back to examples.

Example 8. *Add* $\frac{9}{8} + \frac{11}{10}$.

Solution 8. *Here the fractions are improper, but provided we do not change them to mixed numbers, the method is still the same. The LCD of the two denominators is* 40*, so:*

$$\begin{aligned}
\frac{9}{8} + \frac{11}{10} &= \frac{9 \cdot 5}{8 \cdot 5} + \frac{11 \cdot 4}{10 \cdot 4} \\
&= \frac{45}{40} + \frac{44}{40} \\
&= \frac{45 + 44}{40} \\
&= \mathbf{\frac{89}{40}} \quad \textit{or} \quad \mathbf{2\frac{9}{40}}
\end{aligned}$$

Example 9. *Subtract* $\frac{3}{8} - \frac{1}{4}$.

Solution 9. *The LCD is* 8*, so:*

$$\begin{aligned}
\frac{3}{8} - \frac{1}{4} &= \frac{3}{8} - \frac{1 \cdot 2}{4 \cdot 2} \\
&= \frac{3}{8} - \frac{2}{8} \\
&= \frac{3 - 2}{8} \\
&= \mathbf{\frac{1}{8}}
\end{aligned}$$

Example 10. *Subtract* $\frac{11}{6} - \frac{2}{9}$.

Solution 10. *The LCD is* 18, *so:*

$$\begin{aligned} \frac{11}{6} - \frac{2}{9} &= \frac{11 \cdot 3}{6 \cdot 3} - \frac{2 \cdot 2}{9 \cdot 2} \\ &= \frac{33}{18} - \frac{4}{18} \\ &= \frac{33 - 4}{18} \\ &= \mathbf{\frac{29}{18}} \quad \textit{or} \quad \mathbf{1\frac{11}{18}} \end{aligned}$$

Mixed Number Addition

To do addition involving mixed numbers, there are two methods you could use. The first is:

Mixed Number + or −, method 1:

You may ALWAYS convert all mixed numbers into improper fractions, and then add or subtract following the fraction rules.

The first method is probably the easiest. However, if you wish, there is another method, which we will call the mixed-number addition-shortcut, you may use. To see how it works, let us look at an example:

Example 11. *Add* $2\frac{3}{8} + 4\frac{1}{4}$.

Solution 11. *All the shortcuts involving mixed numbers stem from remembering the hidden plus sign. The following work is slightly beyond the scope of your current knowledge, as we will* *use the commutative and associative properties of addition (we will see these in more detail in section 1.13 on page 216), but follow along as best you can.*

$$\begin{aligned}
2\tfrac{3}{8}+4\tfrac{1}{4} &= \left(2+\tfrac{3}{8}\right)+\left(4+\tfrac{1}{4}\right)\\
&= (2+4)+\left(\tfrac{3}{8}+\tfrac{1}{4}\right)\\
&= 6+\left(\tfrac{3}{8}+\tfrac{1\cdot 2}{4\cdot 2}\right)\\
&= 6+\left(\tfrac{3}{8}+\tfrac{2}{8}\right)\\
&= 6+\tfrac{5}{8}\\
&= \mathbf{6\frac{5}{8}} \quad \textit{or} \quad \mathbf{\frac{53}{8}}
\end{aligned}$$

This shows that the mixed-number addition-shortcut is:

Mixed Number +, method 2:

1. Add the two fractions together. If the result is an improper fraction, convert it to a mixed number.
2. Add the two integers together.
3. Add the two above results together.

Example 12. *Add* $5\frac{3}{4}+4\frac{1}{6}$.

Solution 12. *Using our shortcut, first we will add the fractions.*

$$\begin{aligned}\frac{3}{4}+\frac{1}{6} &= \frac{3\cdot 3}{4\cdot 3}+\frac{1\cdot 2}{6\cdot 2}\\ &= \frac{9}{12}+\frac{2}{12}\\ &= \frac{11}{12}\end{aligned}$$

Then add the integers.

$$5+4 = 9$$

And the final answer is the sum of these two results. So,

$5\frac{3}{4}+4\frac{1}{6} = \mathbf{9\frac{11}{12}}$ *or* $\mathbf{\frac{119}{12}}$

Example 13. *Add* $7\frac{1}{2}+2\frac{4}{5}$.

Solution 13. *Using our shortcut, first we will add the fractions.*

$$\begin{aligned}\frac{1}{2}+\frac{4}{5} &= \frac{1\cdot 5}{2\cdot 5}+\frac{4\cdot 2}{5\cdot 2}\\ &= \frac{5}{10}+\frac{8}{10}\\ &= \frac{13}{10}\\ &= 1\tfrac{3}{10}\end{aligned}$$

Then add the integers.

$$7+2 = 9$$

And the final answer is the sum of these two results. So,

$$\begin{aligned}7\tfrac{1}{2}+2\tfrac{4}{5} &= 9+1\tfrac{3}{10}\\ &= \mathbf{10\frac{3}{10}} \quad \text{or} \quad \mathbf{\frac{103}{10}}\end{aligned}$$

This last example shows a situation where the fractional sum must be converted to a mixed number before adding to the integer sum. DO NOT give an answer along the lines of $9\frac{13}{10}$ as this is an abomination of all that is pure and true, much like Frankenstein's monster (as opposed to Frankenstein who was just a mad scientist). Your answer must either be a true fraction in lowest terms (either proper or improper) OR a true mixed number which consists of an integer plus a proper fraction. Notice that the last step where we added an integer to an improper fraction may be thought of as an application of this shortcut method, where the integer is thought of as a mixed number with zero fractional part.

Example 14. *Add* $10\frac{2}{3} + 1\frac{4}{9}$.

Solution 14. *Using our shortcut, first we will add the fractions.*

$$\begin{aligned} \frac{2}{3} + \frac{4}{9} &= \frac{2 \cdot 3}{3 \cdot 3} + \frac{4}{9} \\ &= \frac{6}{9} + \frac{4}{9} \\ &= \frac{10}{9} \\ &= 1\tfrac{1}{9} \end{aligned}$$

Then add the integers.

$$10 + 1 = 11$$

And the final answer is the sum of these two results. So,

$$\begin{aligned} 10\tfrac{2}{3} + 1\tfrac{4}{9} &= 11 + 1\tfrac{1}{9} \\ &= \mathbf{12\frac{1}{9}} \quad \textit{or} \quad \mathbf{\frac{109}{9}} \end{aligned}$$

This having to change an improper fraction to a mixed number if the fractional sum is too large is similar to "carrying" in the tower method of addition. If this was the only trick, most people would probably prefer the shortcut, as it keeps the numbers being added and multiplied smaller.

However, once we mix in negative numbers, we shall see that there is yet another trick to be wary of.

Mixed Number Subtraction

Again, the first method to subtract mixed numbers is to just change everything to fractions. The second method is again a shortcut method, whose rules are:

Mixed Number $-$, method 2:

1. Subtract the two fractions in the order that they appeared. If necessary, borrow 1 from the integer.
2. Subtract the two integers in the order that they appeared.
3. Add the two above results together.

Example 15. *Subtract* $4\frac{3}{4} - 1\frac{1}{2}$.

Solution 15. *Using our shortcut, first we will subtract the fractions.*

$$\begin{aligned} \frac{3}{4} - \frac{1}{2} &= \frac{3}{4} - \frac{1 \cdot 2}{2 \cdot 2} \\ &= \frac{3}{4} - \frac{2}{4} \\ &= \frac{1}{4} \end{aligned}$$

Then subtract the integers.

$$4 - 1 = 3$$

And the final answer is the sum of these two results. So,

$4\frac{3}{4} - 1\frac{1}{2} = \mathbf{3\frac{1}{4}}$ *or* $\mathbf{\frac{13}{4}}$

Example 16. *Subtract* $3\frac{1}{10} - 2\frac{4}{5}$.

Solution 16. *Using our shortcut, first we will subtract the fractions.*

$$\begin{aligned}\frac{1}{10} - \frac{4}{5} &= \frac{1}{10} - \frac{4 \cdot 2}{5 \cdot 2} \\ &= \frac{1}{10} - \frac{8}{10}\end{aligned}$$

Here we notice a problem, as the smaller fraction is first. If we were using the tower method for old-fashioned subtraction, we would look to borrow ten. Here we will borrow 1 *whole from the integer. So, cross out the* 3 *and make it a* 2 *and add this borrowed* 1 *to the first fraction. On your paper, you could just plug in a* 1 *to your previous work; here we will redo the work.*

$$\begin{aligned}1\frac{1}{10} - \frac{4}{5} &= 1\frac{1}{10} - \frac{4 \cdot 2}{5 \cdot 2} \\ &= 1\frac{1}{10} - \frac{8}{10}\end{aligned}$$

Now we will change the mixed number to a fraction and subtract.

$$\begin{aligned}1\frac{1}{10} - \frac{8}{10} &= \frac{11}{10} - \frac{8}{10} \\ &= \frac{3}{10}\end{aligned}$$

Then subtract the integers. Don't forget we borrowed 1 *from the* 3*.*

$$2 - 2 = 0$$

And the final answer is the sum of these two results. So,

$3\frac{1}{10} - 2\frac{4}{5} = \mathbf{\frac{3}{10}}$

Notice that the integers may totally cancel. That is fine, it just means your answer is a proper fraction. You should NOT get a negative number using this method - that is handled differently as we will see when we mix in negative numbers later on. For now, let us do one more example.

Example 17. *Subtract* $5\frac{1}{9} - \frac{2}{3}$.

Solution 17. *Our second number is not a mixed number, but since it is a proper fraction, it is fine. We will just act like it is a mixed number with integral part equal to* 0. *Had it been an improper fraction, we would have had to made both numbers improper fractions or made both mixed numbers. First we will subtract the fractions.*

$$\begin{aligned}\frac{1}{9} - \frac{2}{3} &= \frac{1}{9} - \frac{2 \cdot 3}{3 \cdot 3} \\ &= \frac{1}{9} - \frac{6}{9}\end{aligned}$$

We need to borrow 1 *from the* 5, *to get:*

$$\begin{aligned}1\frac{1}{9} - \frac{6}{9} &= \frac{10}{9} - \frac{6}{9} \\ &= \frac{4}{9}\end{aligned}$$

Then subtract the integers. Don't forget we borrowed 1 *from the* 5.

$$4 - 0 = 4$$

And the final answer is the sum of these two results. So,

$5\frac{1}{9} - \frac{2}{3} = \mathbf{4\frac{4}{9}}$ *or* $\mathbf{\frac{40}{9}}$

Mixing in negatives

So far this section, we have only dealt with nice, positive numbers. What happens to our rules and methods if we mix in some negative numbers?

If you are adding or subtracting fractions:

1. Find the LCD of the fractions.
2. Make equivalent fractions and combine into a single fraction, leaving all signs as is.
3. You may now have to use the rules for addition or subtraction of integers in the numerator.

Bottom line: you may put off worrying about fancy addition or subtraction rules until you have a single fraction, and then it is just a matter of using the integer addition and subtraction rules in the numerator.

Example 18. *Subtract* $-\frac{2}{3} - \left(-\frac{3}{4}\right)$

Solution 18. *The LCD is* 12*, so following these new rules, we get:*

$$\begin{aligned} -\frac{2}{3} - \left(-\frac{3}{4}\right) &= -\frac{2 \cdot 4}{3 \cdot 4} - \left(-\frac{3 \cdot 3}{4 \cdot 3}\right) \\ &= -\frac{8}{12} - \left(-\frac{9}{12}\right) \\ &= \frac{-8 - (-9)}{12} \end{aligned}$$

Note that we moved the negative signs to the numerators. Now we will subtract as we learned to do back in section 1.3.

$$\begin{aligned} -\frac{2}{3} - \left(-\frac{3}{4}\right) &= \frac{-8 - (-9)}{12} \\ &= \frac{-8 + (+9)}{12} \\ &= \mathbf{\frac{1}{12}} \end{aligned}$$

Example 19. *Add* $\dfrac{11}{20} + \left(-\dfrac{4}{5}\right)$

Solution 19. *The LCD is* 20*, so:*

$$\begin{aligned}
\frac{11}{20} + \left(-\frac{4}{5}\right) &= \frac{11}{20} + \left(-\frac{4 \cdot 4}{5 \cdot 4}\right) \\
&= \frac{11}{20} + \left(-\frac{16}{20}\right) \\
&= \frac{11 + (-16)}{20} \\
&= \frac{-5}{20} \\
&= -\frac{5 \div 5}{20 \div 5} \\
&= -\frac{\mathbf{1}}{\mathbf{4}}
\end{aligned}$$

If choose to add or subtract mixed numbers using the shortcuts, however, the order in which you do things changes.

<u>If you are adding or subtracting mixed numbers:</u>

1. You MUST take signs into account first. See section 1.3 or Appendix A.
2. Do scratch work by applying shortcuts.
3. Make your answer have the correct sign.

This is where you may find it to be easier to always convert mixed numbers to fractions. Any of the following examples involving mixed numbers will be done twice - once by changing to improper fractions and once using the shortcuts.

Example 20. *Add* $-3\frac{1}{2} + 1\frac{1}{5}$

Solution 20. *First we will solve this by changing everything to fractions.*

$$\begin{aligned} -3\tfrac{1}{2} + 1\tfrac{1}{5} &= -\frac{7}{2} + \frac{6}{5} \\ &= -\frac{7 \cdot 5}{2 \cdot 5} + \frac{6 \cdot 2}{5 \cdot 2} \\ &= -\frac{35}{10} + \frac{12}{10} \\ &= \frac{-35 + 12}{10} \\ &= -\mathbf{\frac{23}{10}} \quad \textit{or} \quad \mathbf{-2\frac{3}{10}} \end{aligned}$$

Now let us redo the example leaving the mixed numbers as is.

Example 21. *Add* $-3\frac{1}{2} + 1\frac{1}{5}$

Solution 21. *Since we are adding a negative number to a positive one, we need to take the absolute value of both, and subtract larger minus smaller.*
Scratch work:
We want to subtract $3\frac{1}{2} - 1\frac{1}{5}$*, so first we subtract the fractions.*

$$\begin{aligned} \frac{1}{2} - \frac{1}{5} &= \frac{1 \cdot 5}{2 \cdot 5} - \frac{1 \cdot 2}{5 \cdot 2} \\ &= \frac{5}{10} - \frac{2}{10} \\ &= \frac{3}{10} \end{aligned}$$

Next the integers.

$$3 - 1 = 2$$

So the right numerical answer is $2\frac{3}{10}$*, but we need to think about the sign. The larger of the two numbers in absolute value was the* $3\frac{1}{2}$*, and it was originally negative, so the answer should be negative.*
Therefore,

$$-3\tfrac{1}{2} + 1\tfrac{1}{5} \quad = \quad \mathbf{-2\frac{3}{10}} \quad or \quad -\mathbf{\frac{23}{10}}$$

Example 22. *Subtract* $2\frac{5}{12} - 8\frac{2}{3}$

Solution 22. *First we will solve this by changing everything to fractions.*

$$\begin{aligned}
2\tfrac{5}{12} - 8\tfrac{2}{3} &= \frac{29}{12} - \frac{26}{3} \\
&= \frac{29}{12} - \frac{26 \cdot 4}{3 \cdot 4} \\
&= \frac{29}{12} - \frac{104}{12} \\
&= \frac{29 - 104}{12} \\
&= -\frac{75}{12} \\
&= -\frac{75 \div 3}{12 \div 3} \\
&= -\mathbf{\frac{25}{4}} \quad or \quad \mathbf{-6\frac{1}{4}}
\end{aligned}$$

Example 23. *Subtract* $2\frac{5}{12} - 8\frac{2}{3}$

Solution 23. *Leaving the numbers as mixed, we need to note that although both numbers are positive, the smaller number is first. Using the subtraction shortcut 2 (page 53), we need to subtract the other way around, and then just make the answer negative. Scratch work:*
We want to subtract $8\frac{2}{3} - 2\frac{5}{12}$, so first we subtract the fractions.

$$\begin{aligned}
\frac{2}{3} - \frac{5}{12} &= \frac{2 \cdot 4}{3 \cdot 4} - \frac{5}{12} \\
&= \frac{8}{12} - \frac{5}{12} \\
&= \frac{3}{12} \\
&= \frac{3 \div 3}{12 \div 3} \\
&= \frac{1}{4}
\end{aligned}$$

Next the integers.

$$8 - 2 = 6$$

Combining the two and making the answer negative gives:

$$2\tfrac{5}{12} - 8\tfrac{2}{3} = -\mathbf{6\frac{1}{4}} \quad \textit{or} \quad -\mathbf{\frac{25}{4}}$$

Example 24. *Subtract* $-\frac{2}{9} - \frac{3}{5}$

Solution 24. *There are no improper fractions nor mixed numbers here, so we will only solve this problem once. The LCD is*

45, *so:*

$$
\begin{aligned}
-\frac{2}{9}-\frac{3}{5} &= -\frac{2\cdot 5}{9\cdot 5}-\frac{3\cdot 9}{5\cdot 9} \\
&= -\frac{10}{45}-\frac{27}{45} \\
&= \frac{-10-27}{45} \\
&= \frac{-10+(-27)}{45} \\
&= -\frac{\mathbf{37}}{\mathbf{45}}
\end{aligned}
$$

SECTION 1.8 EXERCISES

(Answers are found on page 375.)

Add or subtract. Your final answer may be either a fraction or a mixed number, but in either case, it should be simplified.

1. $\frac{3}{8}+\frac{3}{4}$
2. $\frac{1}{2}+\frac{1}{6}$
3. $\frac{4}{5}-\frac{3}{10}$
4. $\frac{5}{12}-\frac{2}{3}$
5. $2\frac{1}{3}+4\frac{1}{5}$
6. $3\frac{1}{2}-1\frac{3}{4}$
7. $\frac{15}{4}+\left(-\frac{1}{3}\right)$
8. $\frac{5}{3}-4\frac{1}{2}$
9. $-\frac{5}{9}-\frac{2}{3}$
10. $-\frac{1}{4}+\frac{15}{2}$
11. $2\frac{1}{4}-4\frac{1}{3}$
12. $-\frac{8}{9}+\left(-\frac{1}{7}\right)$
13. $\frac{9}{10}-\frac{2}{5}$
14. $\frac{20}{3}+\frac{11}{4}$
15. $4\frac{1}{4}-\frac{5}{6}$
16. $-6\frac{3}{5}+2\frac{1}{2}$

17. $1\frac{1}{3} - 3\frac{1}{6}$

18. $8\frac{3}{4} - 5\frac{1}{4}$

19. $\frac{4}{5} + \frac{6}{7}$

20. $-\frac{1}{4} - \frac{5}{8}$

21. $3\frac{1}{3} - 1\frac{5}{8}$

22. $-\frac{7}{10} - \frac{3}{2}$

23. $\frac{3}{4} - \frac{7}{8}$

24. $2\frac{1}{5} + (-1\frac{1}{2})$

25. $-1\frac{5}{9} - 6\frac{1}{3}$

26. $0 - \frac{3}{5}$

27. $\frac{5}{6} + \frac{1}{7}$

28. $-2\frac{5}{6} - 5\frac{2}{3}$

29. $3 + \frac{4}{7}$

30. $-\frac{13}{16} + \frac{5}{8}$

1.9 Fraction Multiplication and Division

Fraction Multiplication

To multiply two fractions, you will need the following rule:

Multiplying Fractions

Let a, b, c and d be *any* numbers, *except* $b \neq 0$ and $d \neq 0$. Then:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Unlike addition where people would like to simply add numerators and denominators and not worry about an LCD, fraction multiplication is exactly what you would like it to be. All you need do is multiply the two numerators and the two denominators separately, and simplify your final answer if necessary.

Example 1. *Multiply* $\frac{2}{5} \cdot \frac{3}{7}$

Solution 1. *Following our new rule:*

$$\begin{aligned} \frac{2}{5} \cdot \frac{3}{7} &= \frac{2 \cdot 3}{5 \cdot 7} \\ &= \mathbf{\frac{6}{35}} \end{aligned}$$

Example 2. *Multiply* $\frac{5}{8} \cdot \frac{6}{15}$

Solution 2. *Following our new rule:*

$$\begin{aligned}
\frac{5}{8}\cdot\frac{6}{15} &= \frac{5\cdot 6}{8\cdot 15}\\
&= \frac{30}{120}\\
&= \frac{30\div 10}{120\div 10}\\
&= \frac{3\div 3}{12\div 3}\\
&= \mathbf{\frac{1}{4}}
\end{aligned}$$

Seems pretty straightforward, doesn't it? There are two points you want to keep in mind when multiplying fractions.

Multiplying Fractions Note 1

Before multiplying out the numerators and denominators, you may cancel any factor of either numerator with any matching factor of either denominator.

The idea here is that prior to multiplying, you have a partially factored form of your answer. Although it probably will not be the prime factorization (it was in the first example, though), it is still a factored form, and it may be easier to recognize common factors. Also note that while any cancelation will usually be from one fraction's numerator to the other fraction's denominator, the cancelation may be from the same fraction's numerator and denominator. This is rare as it means that one of the original fractions was not in lowest terms, but it could happen (see the last example). Redoing the last example with this in mind gives:

Example 3. *Multiply* $\frac{5}{8}\cdot\frac{6}{15}$

Solution 3. *Following our new rule:*

$$\begin{aligned}
\frac{5}{8}\cdot\frac{6}{15} &= \frac{\cancel{5}\,1}{\cancel{8}\,4}\cdot\frac{\cancel{6}\,3}{\cancel{15}\,3} \\
&= \frac{1}{4}\cdot\frac{\cancel{3}\,1}{\cancel{3}\,1} \\
&= \mathbf{\frac{1}{4}}
\end{aligned}$$

In the first step, we canceled the common factor of 5 *between the* 5 *and* 15*, and the common factor of* 2 *between the* 6 *and the* 8*. In the second step, we noticed that the second fraction's numerator and denominator had a common factor of* 3*, so we canceled that. Finally, we multiplied the reduced numerators and denominators.*

While not a necessary process, note 1 will make the multiplication and simplification easier.

Multiplying Fractions Note 2

Do NOT go LCD crazy. While LCD's are very important in dealing with fractions, they are NOT necessary for fraction multiplication and fraction division.

While you may still get the problem right if you make equivalent fractions (they are equivalent, after all), you will be doing unnecessary work which will only complicate the problem. It is better to learn that no LCD is needed for fraction multiplication and division.

Mixing in Negatives

Recall that when you multiplied two integers, you determined the numerical value of the answer, and the sign of the answer separately. The same is true for fractions. When there are negatives involved, you will do your scratch work ignoring the signs and multiplying by the fraction rule we have already learned, and then at the end (or at the beginning, your choice), you will calculate the sign of the answer following the same rules as before (see Appendix A if you need to review). Some examples:

Example 4. *Multiply* $-\frac{5}{6} \cdot \frac{3}{5}$

Solution 4. *First we will find the numerical value of the answer, so ignore the negative sign.*
Scratch work:

$$\begin{aligned} \frac{5}{6} \cdot \frac{3}{5} &= \frac{\cancel{5}\ 1}{\cancel{6}\ 2} \cdot \frac{\cancel{3}\ 1}{\cancel{5}\ 1} \\ &= \frac{1}{2} \end{aligned}$$

As for sign, a negative times a positive makes a negative, so:

$-\frac{5}{6} \cdot \frac{3}{5} = -\mathbf{\frac{1}{2}}$

Example 5. *Multiply* $-\frac{3}{10} \cdot \left(-\frac{2}{7}\right)$

Solution 5. *First we will find the numerical value of the answer, so ignore the negative signs.*
Scratch work:

$$\begin{aligned} \frac{3}{10} \cdot \frac{2}{7} &= \frac{3}{\cancel{10}\ 5} \cdot \frac{\cancel{2}\ 1}{7} \\ &= \frac{3}{35} \end{aligned}$$

As for sign, a negative times a negative makes a positive, so:

$-\frac{3}{10} \cdot \left(-\frac{2}{7}\right) = \mathbf{\frac{3}{35}}$

Example 6. *Multiply* $-5 \cdot \left(-\frac{1}{5}\right)$

Solution 6. *For multiplying or dividing, you do not want to mix integers with fractions, so let us change the integer to a fraction by putting it over* 1*. Again, we will start with scratch work which ignores the signs. Scratch work:*

$$\begin{aligned} \frac{5}{1} \cdot \frac{1}{5} &= \frac{\cancel{5}\,1}{1} \cdot \frac{1}{\cancel{5}\,1} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

As for sign, a negative times a negative makes a positive, so:

$-5 \cdot \left(-\frac{1}{5}\right) = \mathbf{1}$

Reciprocals

Definition: Two numbers which multiply to give a product of positive one are called reciprocals.

Given a fraction, to find its reciprocal, just flip the fraction (stays the same sign). Given an integer or a mixed number, make it a fraction, and then do the same.

Example 7. *Find the reciprocal of the following numbers.*

1. $-\frac{4}{11}$

2. $\frac{13}{2}$

3. 5

4. -1

5. $-2\frac{1}{3}$

Solution 7. *We have:*

1. *The reciprocal of* $-\frac{4}{11}$ *is* $-\mathbf{\frac{11}{4}}$
2. *The reciprocal of* $\frac{13}{2}$ *is* $\mathbf{\frac{2}{13}}$
3. *The reciprocal of* $5\left(=\frac{5}{1}\right)$ *is* $\mathbf{\frac{1}{5}}$
4. *The reciprocal of* $-1\left(=-\frac{1}{1}\right)$ *is* $\left(-\frac{1}{1}=\right)\mathbf{-1}$
5. *The reciprocal of* $-2\frac{1}{3}\left(=-\frac{7}{3}\right)$ *is* $-\mathbf{\frac{3}{7}}$

Two interesting facts about reciprocals:

Interesting Reciprocal Fact 1

Positive and negative one are each their own reciprocals. These are the only two numbers for which this is true.

Interesting Reciprocal Fact 2

Zero has no reciprocal. This is the only number for which this is true.

Reciprocals are very useful. For one thing, we will use them to define fraction division. Let us look at an example.

Example 8. *Divide* $\frac{3}{4} \div \frac{2}{7}$

> **Solution 8.** *Recall that the fraction bar shows division, so we could rewrite this problem as simplifying the following fraction:*
>
> $$\frac{3}{4} \div \frac{2}{7} \quad = \quad \frac{\frac{3}{4}}{\frac{2}{7}}$$
>
> *This ugly looking fraction is called a complex fraction, and you will learn more about them later. For now, just recall the fundamental principle of fractions which says that we may multiply numerator and denominator by the same number to make an equivalent fraction. We will choose to multiply by the reciprocal of the denominator.*
>
> $$\begin{aligned} \frac{3}{4} \div \frac{2}{7} &= \frac{\frac{3}{4} \cdot \frac{7}{2}}{\frac{2}{7} \cdot \frac{7}{2}} \\ &= \frac{\frac{3}{4} \cdot \frac{7}{2}}{1} \\ &= \frac{3}{4} \cdot \frac{7}{2} \\ &= \mathbf{\frac{21}{8}} \end{aligned}$$
>
> *In the above, we used the facts that the product of two reciprocals is* 1*, and any number divided by* 1 *is itself.*

This concept of division by a fraction turning into multiplication by its reciprocal generalizes regardless what fraction we are dividing by, and this leads to the following shortcut for fraction division.

Dividing Fractions

Let a, b, c and d be *any* numbers, *except* $b \neq 0$, $c \neq 0$ and $d \neq 0$. Then:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

Similar to being able to switch subtraction to addition (or vice versa) by changing the number following the operation to its opposite, this means that you may change division to multiplication (or vice versa) by changing the number following the operation to its reciprocal (provided zero is not involved). Whereas, changing subtraction to addition (or vice versa) is commonly done, changing division to multiplication is only really useful when fractions are involved. For example, when we get to decimal division, you will have no desire to change the division to multiplication - it is easier to just divide. Besides the rule for fraction division above, don't forget that we have also seen the second version of the fundamental principle of fractions. The reason we sometimes prefer to change the multiplication sign in the original version to a division sign is that we would prefer to talk about division by an integer over multiplication by a fraction. For example, if you have a fraction where the numerator and denominator have a common factor of two, it is nicer to think of dividing numerator and denominator by 2 than multiplying by $\frac{1}{2}$. In any case, now that we have seen reciprocals and this rule for switching between multiplication and division, it is easy to see that the two versions of the fundamental principle are the same. Before we do some examples, one important hint:

Fraction Division Hint

Do NOT cancel factors while you still have a division sign. You MUST change to multiplication first.

Ok, now for some examples.

Example 9. *Divide* $\frac{4}{5} \div \left(-\frac{3}{10}\right)$

Solution 9. *We will worry about the sign of our answer at the end. For the scratch work, we need to first change the division to multiplication.*
Scratch work:

$$\begin{aligned}\frac{4}{5} \div \frac{3}{10} &= \frac{4}{5} \cdot \frac{10}{3} \\ &= \frac{4}{\cancel{5}\,1} \cdot \frac{\cancel{10}\,2}{3} \\ &= \frac{8}{3}\end{aligned}$$

As for sign, a positive divided by a negative makes a negative, so:

$\frac{4}{5} \div \left(-\frac{3}{10}\right) = -\mathbf{\frac{8}{3}}$

Example 10. *Divide* $-\frac{4}{7} \div (-8)$

Solution 10. *We will worry about the sign of our answer at the end. For the scratch work, we need to first change the division to multiplication.*
Scratch work:

$$\begin{aligned}\frac{4}{7} \div \frac{8}{1} &= \frac{4}{7} \cdot \frac{1}{8} \\ &= \frac{\cancel{4}\,1}{7} \cdot \frac{1}{\cancel{8}\,2} \\ &= \frac{1}{14}\end{aligned}$$

As for sign, a negative divided by a negative makes a positive, so, $-\frac{4}{7} \div (-8) = \mathbf{\frac{1}{14}}$

Multiplying and Dividing Mixed Numbers

To multiply or divide mixed numbers, you only need to remember one rule.

Mixed Number Multiplication and Division

You MUST ALWAYS convert all mixed numbers into improper fractions first to multiply or divide.

The "shortcuts" for mixed numbers are not worth doing in the case of multiplication and division (see Appendix B on page 365).

Example 11. *Multiply* $-2\frac{1}{2} \cdot 1\frac{1}{5}$

Solution 11. *Changing to improper fractions, and ignoring the signs for the moment:*

$$\begin{aligned} 2\frac{1}{2} \cdot 1\frac{1}{5} &= \frac{5}{2} \cdot \frac{6}{5} \\ &= \frac{\cancel{5}\,1}{\cancel{2}\,1} \cdot \frac{\cancel{6}\,3}{\cancel{5}\,1} \\ &= 3 \end{aligned}$$

A negative number multiplied by a positive number makes a negative number, so:
$-2\frac{1}{2} \cdot 1\frac{1}{5} = \mathbf{-3}$

Example 12. *Divide* $2\frac{1}{4} \div 3\frac{1}{3}$

Solution 12. *Changing to improper fractions:*

$$\begin{aligned} 2\frac{1}{4} \div 3\frac{1}{3} &= \frac{9}{4} \div \frac{10}{3} \\ &= \frac{9}{4} \cdot \frac{3}{10} \\ &= \mathbf{\frac{27}{40}} \end{aligned}$$

This example illustrates why you must wait until you switch to multiplication before canceling. Once you switch here, you can see that the 3 *and* 9 *in the numerators have no factor greater than* 1 *in common with the* 4 *and* 10 *in the denominators.*

Example 13. *Divide* $-3\frac{1}{5} \div (-4)$

Solution 13. *Changing to improper fractions, and ignoring the signs for the moment:*

$$\begin{aligned} 3\frac{1}{5} \div 4 &= \frac{16}{5} \div \frac{4}{1} \\ &= \frac{16}{5} \cdot \frac{1}{4} \\ &= \frac{\cancel{16}\ 4}{5} \cdot \frac{1}{\cancel{4}\ 1} \\ &= \frac{4}{5} \end{aligned}$$

A negative number divided by a negative number makes a positive number, so:
$-3\frac{1}{5} \div (-4) = \mathbf{\frac{4}{5}}$

Order of Operations

Now that we can add, subtract, multiply and divide fractions, lets do some order of operations problems involving them. Most of the problems will be the same types as before, with some fractions mixed in with the integers, but we will also add problems involving large fractions. For example,

Example 14. *Simplify* $\dfrac{10(-1)-(-2)(-3)}{2[-8\div(-2-2)]}$

To do these type of problems, you may either recall that the fraction bar means divide and rewrite the problems as [numerator]÷[denominator], or, perhaps more simply, do one step of order of operations on the numerator and denominator separately until there is a single number in the numerator, and a single number in the denominator. At this point, treat as a fraction and simplify if possible. So, to solve the last example:

Solution 14. *We will apply order of operations on the numerator and denominator separately.*

$$\begin{aligned}\frac{10(-1)-(-2)(-3)}{2[-8\div(-2-2)]} &= \frac{-10-6}{2[-8\div(-4)]}\\ &= \frac{-16}{2(2)}\end{aligned}$$

At this point, the numerator is just a single number, so it will just tag along until we finish simplifying the denominator.

$$\begin{aligned}\frac{10(-1)-(-2)(-3)}{2[-8\div(-2-2)]} &= \frac{-16}{2(2)}\\ &= \frac{-16}{4}\\ &= \mathbf{-4}\end{aligned}$$

Some more order of operations examples.

Example 15. *Simplify* $\dfrac{8+3\cdot 6}{4-2^4}$

Solution 15. *We will apply order of operations on the numerator and denominator separately.*

$$\begin{aligned}\frac{8+3\cdot 6}{4-2^4} &= \frac{8+18}{4-16} \\ &= \frac{26}{-12} \\ &= -\frac{26\div 2}{12\div 2} \\ &= -\mathbf{\frac{13}{6}} \quad \textit{or} \quad -\mathbf{2\frac{1}{6}}\end{aligned}$$

Example 16. *Simplify* $\frac{1}{2}[2-5(7^1-6^0)]$

Solution 16. *Starting with the innermost grouping symbols:*

$$\begin{aligned}\tfrac{1}{2}[2-5(7^1-6^0)] &= \tfrac{1}{2}[2-5(7-1)] \\ &= \tfrac{1}{2}[2-5(6)] \\ &= \tfrac{1}{2}(2-30) \\ &= \tfrac{1}{2}(-28) \\ &= \frac{1}{2}\cdot\frac{-28}{1} \\ &= \mathbf{-14}\end{aligned}$$

Example 17. *Simplify* $\dfrac{5^2 - 3^3}{2 \cdot 8 - 4^2}$

Solution 17. *We will apply order of operations on the numerator and denominator separately.*

$$\begin{aligned}\frac{5^2 - 3^3}{2 \cdot 8 - 4^2} &= \frac{25 - 27}{2 \cdot 8 - 16} \\ &= \frac{-2}{16 - 16} \\ &= \frac{-2}{0} \\ &= \textbf{\textit{undefined}}\end{aligned}$$

Don't forget that division by zero, or a zero in the denominator of a fraction, is undefined.

SECTION 1.9 EXERCISES

(Answers are found on page 376.)

Find the reciprocal of the following numbers.

1. $\dfrac{3}{5}$
2. $-\dfrac{7}{11}$
3. -9
4. $2\dfrac{1}{5}$
5. $-3\dfrac{1}{9}$
6. $\dfrac{22}{35}$
7. 0
8. 84
9. $-\dfrac{12}{17}$
10. $6\dfrac{1}{7}$

Multiply or divide. Your final answer may be either a fraction or a mixed number, but in either case, it should be simplified.

11. $\frac{3}{20} \cdot \frac{5}{27}$

12. $-\frac{6}{25} \cdot \left(-\frac{5}{9}\right)$

13. $2\frac{1}{4} \div \left(-5\frac{1}{3}\right)$

14. $3\frac{1}{5} \cdot \frac{7}{32}$

15. $6 \cdot \left(-\frac{1}{4}\right)$

16. $\frac{125}{3} \div \left(-\frac{10}{3}\right)$

17. $-2\frac{3}{4} \div \left(-1\frac{5}{6}\right)$

18. $\frac{5}{8} \cdot 1\frac{3}{5}$

19. $-\frac{10}{49} \cdot \frac{7}{8}$

20. $\frac{3}{4} \div (-8)$

21. $2\frac{1}{8} \div 1\frac{1}{6}$

22. $1\frac{3}{8} \cdot 2\frac{2}{7}$

23. $-\frac{10}{21} \cdot \left(-\frac{7}{15}\right)$

24. $0 \div \frac{3}{13}$

25. $\frac{3}{13} \div 0$

26. $-\frac{17}{18} \cdot \frac{6}{11}$

27. $-\frac{5}{6} \div \left(-\frac{3}{10}\right)$

28. $-\frac{7}{9} \cdot 1\frac{2}{7}$

29. $\frac{8}{27} \cdot \frac{2}{3}$

30. $\frac{8}{27} \div \frac{2}{3}$

31. $-5 \div \left(-\frac{1}{2}\right)$

32. $-2\frac{2}{3} \cdot \frac{6}{11}$

33. $\frac{15}{17} \div (-5)$

34. $\frac{9}{14} \div \frac{1}{7}$

35. $7 \div \left(-\frac{1}{3}\right)$

36. $-\frac{4}{5} \cdot \left(-\frac{6}{7}\right)$

37. $\frac{8}{15} \cdot \frac{5}{6}$

38. $-\frac{1}{2} \div 2$

39. $-3\frac{3}{5} \cdot 1\frac{1}{9}$

40. $\frac{5}{7} \div \frac{5}{7}$

Simplify.

41. $\dfrac{3-2\cdot 5}{7(3^1-5^0)}$

42. $\dfrac{6\div 2\cdot 3}{5-4(2-3)}$

43. $\dfrac{4^2+3^3}{5^0-8^0}$

44. $\dfrac{|2-8|-(3-5)}{2^3-1^4}$

45. $\dfrac{|5-2|-|2-5|}{6\cdot 3-4\cdot 4}$

46. $\dfrac{1}{2}\cdot\dfrac{2}{3}\div\dfrac{3}{4}-\dfrac{2}{3}$

47. $1\dfrac{1}{2}+\dfrac{2}{5}\cdot\dfrac{15}{14}$

48. $1-\dfrac{3}{8}\cdot\dfrac{12}{5}+\dfrac{3}{5}$

49. $-\dfrac{5}{12}+3\div 12$

50. $-\dfrac{4}{15}\cdot\dfrac{3}{7}\div\dfrac{3}{2}\div\left(-\dfrac{8}{5}\right)$

1.10 Decimals and Percents

Decimals - the basics

We use a base-10 numeration system. This means that the number 142 represents 2 ones, 4 tens, and 1 hundred, or:

$142 = 1 \cdot 100 + 4 \cdot 10 + 2 \cdot 1$

This is called expanded form of the number. Notice that each digit further to the left represents ten times more than the previous digit. Conversely, this means that each digit further to the right is one-tenth as much as the previous digit ($10 = \frac{1}{10} \cdot 100, \quad 1 = \frac{1}{10} \cdot 10$, etc.). If we wished to extend this concept, we could put a digit to the right of the one's digit, and this would represent *tenths* (as $\frac{1}{10} = \frac{1}{10} \cdot 1$). The only problem would be recognizing where the one's digit would be. For example, if we put a 5 to the right of our 142 from before, we would get 1425, and the 5 would be thought to be the one's digit. This is where the decimal point comes in.

Important Fact about the Decimal Point

The number immediately to the LEFT of the decimal point is the one's digit.

Thus, the number 142.50799 in expanded form would be:

$142.50799 = 1 \cdot 100 + 4 \cdot 10 + 2 \cdot 1 + 5 \cdot \frac{1}{10} + 0 \cdot \frac{1}{100} + 7 \cdot \frac{1}{1{,}000} + 9 \cdot \frac{1}{10{,}000} + 9 \cdot \frac{1}{100{,}000}$

To write or read such numbers, we need to be a bit more careful than we are with integers. For example, officially the number 142 should be written (or read) as "*one hundred forty-two*". Sometimes, though, the sloppier "*one hundred and forty-two*" is used, which is ok since "and" often means "plus" in mathematics. Thus, "*one hundred and forty-two*" is $100 + 42 = 142$. If you are dealing with a decimal, on the other hand, you cannot afford to be so casual, as the word "and" is reserved for the decimal point.

To Write (or Read) a Decimal Number

1. Write (or say) the number to the left of the decimal point as normal - the word "and" may NOT be used.
2. Write (or say) "and" for the decimal point.
3. Write (or say) the number to the right of the decimal point as normal - the word "and" may NOT be used - and tack-on the place value of the last (right-most) digit.

Example 1. *Write, in words, the number* 142.50799*.*

Solution 1. *Following our rules, we write:*
142.50799 *is one hundred forty-two and fifty thousand seven hundred ninety-nine hundred-thousandths.*

You may need to count across to find the place value of the last digit; just note that the place values going to the right are similar to the integral ones going to the left with the "s"'s replaced with "ths"'s (and no oneths!). So, instead of:

tens, hundreds, thousands, ten-thousands, hundred-thousands, millions, . . .

the decimal place values are:

tenths, hundredths, thousandths, ten-thousandths, hundred-thousandths, millionths, . . .

Example 2. *Write, in words, the number* -32.001*.*

Solution 2. *Following our rules, we write:*
-32.001 *is negative thirty-two and one thousandth.*

Note, since the decimal portion was just the number 1, we dropped the "s" on "thousandths". There is an informal way to write and read decimals as well. Instead of using "and" for the decimal point, some people will just say "point". When you do this, all the numbers after the decimal point (and sometimes even the ones in front) are usually read individually. Therefore, the last example could have been written *negative thirty-two point zero zero one* or *negative three two point zero zero one*. In all of our examples and in all of our exercises, we will use the more formal language.

Example 3. *Write, in words, the number* 500.20.

Solution 3. *Following our rules, we write:*
500.20 *is five hundred and twenty hundredths.*

Necessary or Unnecessary?

In mathematics, sometimes we will have unnecessary zeros. To be able to tell if a zero is necessary or unnecesary, use the following rule.

In *mathematics*, a zero in a number is UNNECESSARY if:

- it is to the left of the decimal point, and there are no nonzero numbers to its left.
- it is to the right of the decimal point, and there are no nonzero numbers to its right.

The first case is less common, although you may see room numbers identified as 02 or 002. Since there are no nonzero numbers located to the left of the zeros, they are unnecessary and may be removed or added as desired. Thus,

$$002 = 02 = 2.$$

The most common use of an unnecessary zero to the left of the decimal point occurs when the integral part of a number is zero. In that case, while it is fine to just start with the decimal point, it is more common to put a zero first. For example, $.5 = 0.5$. We will tend to put in the zero when there is no integer.

Unnecessary zeros to the right of the decimal point are much more useful, and we will use them frequently when dealing with decimals. The trouble is there can be a bit of a controversy about whether the zeros are actually necessary or not. For example, let us say you are going on a vacation, and you need to weigh a suitcase to be sure that it is not too heavy. In your home, you put the suitcase on a digital scale, and it reads 28 pounds. At the airport, the airline weighs your suitcase, and they mark it at 28.0 pounds. Is there any difference between the two numbers? In this case, yes, the zero is significant. The discrepancy occurs because in mathematics, we are referring to exact numbers. When we write 28, we mean the number 28. In the sciences and the real world, most numbers are *measurements of a physical quantity up to a certain degree of accuracy.* So, when you weighed your suitcase with your home scale, even assuming it is accurate, it told you that your suitcase 28 pounds *to the nearest pound.* The actual weight could be anywhere between 27.5 pounds to just under 28.5 pounds. When the airline weighed the suitcase and marked it as 28.0 pounds, they are claiming that it weighs 28.0 pounds to the nearest tenth of a pound. Now the actual weight is known to be between 27.95 pounds to just under 28.05 pounds. For this course, we will usually be using exact numbers, but keep this point in mind for other courses.

Example 4. *Write the following numbers with only necessary zeros.*

1. 0407.501
2. 100.0300
3. 0.1008

4. 007

Solution 4. *Removing all unnecessary zeros:*

1. $0407.501 = \mathbf{407.501}$
2. $100.0300 = \mathbf{100.03}$
3. $0.1008 = .\mathbf{1008}$
4. $007 = \mathbf{7}$
   Provide we are talking about the number, of course. If you are a British secret agent, the double-o is a license to kill and is necessary.

Changing Fractions $\longrightarrow$ Decimals

To convert a fraction all you need to do is remember that the fraction bar means divide, similar to how you change an improper fraction to a mixed number.

Rules for changing fractions $\longrightarrow$ decimals

Perform long division, dividing numerator by denominator. Do not stop dividing until you get either a remainder of zero, or have fallen into a repetitive pattern which will continue forever. Note, you may have to add unnecessary zeros to the dividend (the number you are dividing into). The place-values of your quotient are right above the equivalent place-values of your dividend, so the decimal points will be in-line as well.

Example 5. *Change the fraction* $\dfrac{7}{2}$ *to a decimal.*

Solution 5. *We will start by doing long division.*
Scratch work:

$$\begin{array}{rl} & 3.5 \\ 2 & |\overline{7.0} \\ & \underline{6} \\ & 10 \\ & \underline{10} \\ & 0 \end{array}$$

Therefore, $\frac{7}{2} = \mathbf{3.5}$

Example 6. *Change the fraction* $\frac{5}{8}$ *to a decimal.*

Solution 6. *We will start by doing long division.*
Scratch work:

$$\begin{array}{rl} & 0.625 \\ 8 & |\overline{5.000} \\ & \underline{48} \\ & 20 \\ & \underline{16} \\ & 40 \\ & \underline{40} \\ & 0 \end{array}$$

Therefore, $\frac{5}{8} = \mathbf{0.625}$

Both of the last two examples had decimals which eventually stopped. These have a special name.

Definition: A decimal with a finite number of digits is called a <u>terminating decimal</u>.

It is possible for the decimal to go on forever, but fall into a pattern.

Definition: A number whose decimal representation goes on forever, but whose digits eventually fall into a repetitive pattern is called a repeating decimal.

Example 7. *Change the fraction* $-\dfrac{2}{11}$ *to a decimal.*

> **Solution 7.** *The negative will make no difference in our scratch work; it will just tag along at the end. We will start by doing long division.*
> *Scratch work:*
>
> $$\begin{array}{r|l} & 0.1818\ldots \\ 11 & \overline{2.0000} \\ & \underline{11} \\ & \;\;90 \\ & \;\;\underline{88} \\ & \;\;\;\;20 \\ & \;\;\;\;\underline{11} \\ & \;\;\;\;\;\;90 \\ & \;\;\;\;\;\;\underline{88} \\ & \;\;\;\;\;\;\;\;2 \\ & \;\;\;\;\;\;\;\;\vdots \end{array}$$
>
> *At this point we can tell that we are stuck in a loop.*
>
> *Therefore,* $-\dfrac{2}{11} = \mathbf{-0.1818}\ldots$

A nicer way to write repeating decimals is to put a bar over the part of the decimal which repeats. For example,

$-0.1818\ldots = -0.\overline{18}$
$0.3333\ldots = 0.\overline{3}$
$-0.51234234234\ldots = -0.51\overline{234}$

Example 8. *Change the fraction* $\dfrac{5}{12}$ *to a decimal.*

Solution 8. *We will start by doing long division.*
Scratch work:

$$
\begin{array}{r@{\,}l}
 & 0.4166\ldots \\
12 & \overline{|5.0000} \\
 & \underline{48} \\
 & \;\;20 \\
 & \;\;\underline{12} \\
 & \;\;\;\;80 \\
 & \;\;\;\;\underline{72} \\
 & \;\;\;\;\;\;80 \\
 & \;\;\;\;\;\;\underline{72} \\
 & \;\;\;\;\;\;\;\;8 \\
 & \;\;\;\;\;\;\;\;\vdots
\end{array}
$$

At this point we can tell that we are stuck in a loop.

Therefore, $\dfrac{5}{12} = \mathbf{0.41\overline{6}}$

There are other types of decimals which neither terminate nor repeat. The most famous such number is pi, π, whose decimal representation some (crazy??) people memorize up to millions of digits. This won't arise in any of the fractions which we ask you to change to a decimal, because of the following fact.

Important Fact 1 about Rational Numbers

The decimal representation of any rational number either terminates or repeats.

Since we will only be asking you to convert rational numbers (integers divided by integers) to decimals, you may be confident that decimal will either eventually terminate or repeat. It is interesting to note that you can tell which is going to happen before you divide.

Interesting Note about Rational Numbers

If you have a rational number written as a fraction in lowest terms, look at the prime factorization of the denominator. If there are only factors of two and/or five, its decimal representation will terminate. If there are any other prime numbers in the factorization, its decimal representation will repeat.

The fraction being in lowest terms is important since obviously you could introduce other prime numbers into a denominator by making an equivalent fraction. Looking back at examples 5 and 6, we see that the prime factorization of the denominators in these cases ($2 = 2$ and $8 = 2 \cdot 2 \cdot 2$, respectively) only had factors of 2 or 5 (only 2 in these cases), so they should become terminating decimals. The prime factorization of the denominators from examples 7 and 8 ($11 = 11$ and $12 = 2 \cdot 2 \cdot 3$, respectively), on the other hand, contained other prime numbers, so they should become repeating decimals.

It should be noted that the reverse of Important Fact 1 is also true.

Important Fact 2 about Rational Numbers

Every number whose decimal representation terminates or repeats is a rational number.

Changing Decimals $\longrightarrow$ Fractions

In this course, we will only ask you to change terminating decimals to fractions. Changing repeating decimals to fractions is a bit harder, and is covered in other courses (blatant plug for Math 14001: Basic Math Concepts I and Math 14002: Basic Math Concepts II).

Rules for changing terminating decimals $\longrightarrow$ fractions

1. Place entire number with the decimal point removed in the numerator.
2. In the denominator, place the power of ten which matched the last place-value of the original decimal.
3. Simplify, if desired.

Because many of these examples will involve very large numbers, we will not ask you to simplify your final answer (although, since the denominator is a power of 10, you would only need to check if a 2 or 5 divides into the numerator, possibly repeatedly).

Example 9. *Change* 62.14607 *to a fraction. You do NOT need to simplify.*

Solution 9. *First, we will remove the decimal point and place the number into the numerator, inserting commas to make it easier to read.*

$$62.14607 \quad = \quad \frac{6,214,607}{}$$

The last digit (the 7*) was originally in the hundred-thousandths spot, so* 100,000 *goes into the denominator. Therefore,*

$$62.14607 \quad = \quad \mathbf{\frac{6,214,607}{100,000}}$$

Note, you may change decimals with a nonzero integral part directly to a mixed number if you leave the integer alone and just do the above on the decimal part only.

Example 10. *Change* 62.14607 *to a mixed number. You do NOT need to simplify.*

Solution 10. *Working only with the decimal part, we get:*

$$0.14607 = \frac{14,607}{}$$

The last digit (the 7*) was originally in the hundred-thousandths spot, so* 100, 000 *goes into the denominator. Therefore,*

$$62.14607 = \mathbf{62\tfrac{14,607}{100,000}}$$

Example 11. *Change* -3.886 *to a fraction. You do NOT need to simplify.*

Solution 11. *First, we will remove the decimal point and place the number into the numerator, inserting commas to make it easier to read.*

$$-3.886 = \frac{-3,886}{}$$

The last digit was originally in the thousandths spot, so 1, 000 *goes into the denominator. Therefore,*

$$-3.886 = -\frac{\mathbf{3,886}}{\mathbf{1,000}}$$

Example 12. *Change* -3.886 *to a mixed number. You do NOT need to simplify.*

Solution 12. *Working only with the decimal part, we get:*

$$0.886 = \frac{886}{}$$

The last digit was originally in the thousandths spot, so 1, 000 *goes into the denominator. Therefore,*

$$-3.886 = \mathbf{-3\tfrac{886}{1,000}}$$

Example 13. *Change* 0.900416 *to a fraction. You do NOT need to simplify.*

Solution 13. *First, we will remove the decimal point and place the number into the numerator, inserting commas to make it easier to read.*

$$0.900416 \quad = \quad \frac{900,416}{}$$

The last digit was originally in the millionths spot, so $1,000,000$ *goes into the denominator. Therefore,*

$$0.900416 \quad = \quad \mathbf{\frac{900,416}{1,000,000}}$$

Rounding Decimals

You may be asked to round a decimal to a certain place-value. If so, there are only two digits which are important: the digit at the place-value in question, and the digit to its immediate right.

To Round a Decimal to a desired Place-Value:

- If the desired place-value is to the right of the decimal point AND the number to the immediate right is a $0, 1, 2, 3$, or 4, just drop all the digits to the right of the desired place.
- If the desired place-value is to the right of the decimal point AND the number to the immediate right is a $5, 6, 7, 8$, or 9, drop all the digits to the right of the desired place AND add one to the desired digit. If this digit is a nine, you may have to carry.
- If the desired place-value is to the left of the decimal point AND the number to the immediate right is a $0, 1, 2, 3$, or 4, change any digits between desired place-value and the decimal point to zeros and drop any digits after the decimal point.
- If the desired place-value is to the left of the decimal point AND the number to the immediate right is a $5, 6, 7, 8$, or 9, change any digits between desired place-value and the decimal point to zeros and drop any digits after the decimal point, AND add one to the desired digit. If this digit is a nine, you may have to carry.

Example 14. *Round the number* 137.08996 *to the nearest:*

1. *ten-thousandth.*
2. *thousandth.*
3. *hundredth.*
4. *tenth.*
5. *hundred.*

Solution 14. *To start, we will always underline the digit in question, and then follow our rules.*

1. *nearest ten-thousandth,* $137.089\underline{9}6$
The digit to the right of the nine is a six, so we will drop the 6 *and add one to the nine. This makes a zero (ten), carry the one. The next number is also a nine so plus one again makes a zero, carry the one. The next number is an eight, so plus one makes nine.*

   Therefore, 137.08996 *rounded to the nearest ten-thousandth is* **137.0900**.

   Note, since we were asked to round to the nearest ten-thousandth, it is good form to leave the unnecessary zeros to show that this is the case.

2. *nearest thousandth,* $137.08\underline{9}96$
The digit to the right of the nine is a nine, so we drop the 96 *and add one to the nine. This makes a zero, carry the one. The next number is an eight, plus one makes a nine.*

   Therefore, 137.08996 *rounded to the nearest thousandth is* **137.090**.

3. *hundredth,* $137.0\underline{8}996$
The digit to the right of the eight is a nine, so we drop the 996 *and add one to the eight to make it a nine.*

   Therefore, 137.08996 *rounded to the nearest hundredth is* **137.09**.

4. *tenth,* $137.\underline{0}8996$
The digit to the right of the zero is an eight so we drop the 8996 *and add one to the zero to get a one.*

   Therefore, 137.08996 *rounded to the nearest tenth is* **137.1**.

5. *hundred,* $\underline{1}37.08996$
Here is a case where we are rounding to a place-value which is left of the decimal point. So, after seeing that the number to the right of the one is a three, we only drop the .08996, *and change the* 37 *to zeros. As for the one, since the number to its right was a* 3, *we leave it a one.*

   Therefore, 137.08996 *rounded to the nearest hundred is* **100**.

Example 15. *Round the number* $-2.39\overline{15}$ *to the nearest:*

1. *ten-thousandth.*
2. *thousandth.*
3. *hundredth.*
4. *tenth.*

Solution 15. *With repeating decimals, write out enough of the decimal to see the two important positions.*

1. *ten-thousandth,* $-2.391\underline{5}1\ldots$
   The digit to the right is a one, so we drop it and the rest of the decimal and leave the 5 *alone.*

   Therefore, $-2.39\overline{15}$ *rounded to the nearest ten-thousandth is* $\mathbf{-2.3915}$.
2. *thousandth,* $-2.39\underline{1}5\ldots$
   The digit to the right is a five, so we drop it and the rest of the decimal and add one to the 1.

   Therefore, $-2.39\overline{15}$ *rounded to the nearest thousandth is* $\mathbf{-2.392}$.
3. *hundredth,* $-2.3\underline{9}1\ldots$
   The digit to the right is a one, so we drop it and the rest of the decimal and leave the 9 *alone.*

   Therefore, $-2.39\overline{15}$ *rounded to the nearest hundredth is* $\mathbf{-2.39}$.
4. *tenth,* $-2.\underline{3}9\ldots$
   The digit to the right is a nine, so we drop it and the rest of the decimal and add one to the 3.

   Therefore, $-2.39\overline{15}$ *rounded to the nearest tenth is* $\mathbf{-2.4}$.

Comparisons and Orderings

To compare two decimals, use the following guidelines:

To Compare Two Decimals:

1. First compare the integral parts, including positive versus negative. If there is a difference here, this will determine which is the lesser.
2. If the integral parts are the same, start comparing place-value to place-value, starting with tenths and moving to the right, until there is a difference. Do not forget that you may add unnecessary zeros to a terminating decimal if it helps.

Example 16. *Insert a $<$ or $\geq$ between the following two numbers:*
$18.54 \quad \underline{\qquad} \quad 18.5389$

Solution 16. *First, we compare the two integral parts, but here they are both* 18. *Next, we need to start comparing the decimal places - to do so we will write one number above the other, and start by comparing the tenths.*

$$\begin{array}{ccccccc} & & & \downarrow & & & \\ 1 & 8 & . & 5 & 4 & 0 & 0 \\ 1 & 8 & . & 5 & 3 & 8 & 9 \end{array}$$

The tenths are the same, move on to hundredths.

$$\begin{array}{ccccccc} & & & & \downarrow & & \\ 1 & 8 & . & 5 & 4 & 0 & 0 \\ 1 & 8 & . & 5 & 3 & 8 & 9 \end{array}$$

$4 \geq 3$, *so* $\mathbf{18.54 \geq 18.5389}$

This method is easy enough to expand, so that instead of comparing two decimals at at time, you could be comparing several. This is called ordering the decimals.

Example 17. *Write the following decimals from least to greatest:*
$-134.545, -143.5, -134.\overline{54}, -134.54\overline{45}$

Solution 17. *Before we begin ordering the numbers, a couple of comments. First, we could just rewrite the numbers with commas between them, but to emphasize how we are ordering them, we will write our solution as:*
$\_\_\_ < \_\_\_ < \_\_\_ < \_\_\_.$
Second, be careful with the negative signs. Since all the numbers are the same sign, there is no obvious least or greatest, but don't forget that a large negative number is further to the left on the number line than a small negative number. We will order the numbers as if they were all positive, and then flip their order when we make them negative.
Looking at the numbers: $134.545, 143.5, 134.\overline{54}, 134.54\overline{45}$*, notice that one has a different integral part than the others. Since* $143 > 134$*, we know that the* 143 *number is the largest:*
$\_\_\_ < \_\_\_ < \_\_\_ < 143.5.$
To order the other three numbers, we will write the three above and below each other and compare the decimal places, step-by-step. We will write out plenty of decimal places for all three to allow us to compare - note that this will mean adding a couple of unnecessary zeros to the terminating decimal.

$$\begin{array}{ccccccccc} & & & & \downarrow & & & & \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 0 & 0 \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 4 & 5 & \dots \\ 1 & 3 & 4 & . & 5 & 4 & 4 & 5 & 4 & \dots \end{array}$$

The tenths are the same, move on to hundredths.

$$\begin{array}{ccccccccc} & & & & & \downarrow & & & \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 0 & 0 \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 4 & 5 & \dots \\ 1 & 3 & 4 & . & 5 & 4 & 4 & 5 & 4 & \dots \end{array}$$

The hundredths are the same, move on to the thousandths.

$$\begin{array}{ccccccccc} & & & & & & \downarrow & & \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 0 & 0 \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 4 & 5 & \dots \\ 1 & 3 & 4 & . & 5 & 4 & 4 & 5 & 4 & \dots \end{array}$$

$4 < 5$, so the bottom number is the smallest of the three. Updating our ordering gives:
$134.54\overline{45} <$ ___ $<$ ___ < 143.5.
As for the last two numbers, move on to the ten-thousandths.

$$\begin{array}{cccccccccc} & & & & & & & \downarrow & & \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 0 & 0 & \\ 1 & 3 & 4 & . & 5 & 4 & 5 & 4 & 5 & \ldots \end{array}$$

$0 < 4$, so the first number is the smallest. Therefore:
$134.54\overline{45} < 134.545 < 134.\overline{54} < 143.5$.
Finally, changing all the numbers back to negatives switches the ordering, so:
$\mathbf{-143.5 < -134.\overline{54} < -134.545 < -134.54\overline{45}}$.

One last comment, if you are asked to compare a decimal with a fraction, you must either change the decimal to a fraction and compare fractions, OR change the fraction to a decimal and compare the decimals (this last is probably the easiest).

Example 18. *Insert a $\leq$ or $>$ between the following two numbers.*

0.572 ___ $\dfrac{4}{7}$

Solution 18. *We will start by changing the fraction to a decimal.*
Scratch work:

$$\begin{array}{r|l} & 0.5714\ldots \\ 7 & \overline{4.0000} \\ & \underline{35} \\ & \;\;50 \\ & \;\;\underline{49} \\ & \;\;\;\;10 \\ & \;\;\;\;\;\;\underline{7} \\ & \;\;\;\;\;\;30 \\ & \;\;\;\;\;\;\underline{28} \\ & \;\;\;\;\;\;\;\;2 \\ & \;\;\;\;\;\;\;\;\vdots \end{array}$$

We haven't started repeating yet, but the other decimal only has three decimal places, so four is the most we will need for our fraction. Comparing the two, notice that the change occurs in the thousandths place:

$$\begin{array}{cccccccc} & & & & \downarrow & & \\ 0 & . & 5 & 7 & 2 & 0 & \\ 0 & . & 5 & 7 & 1 & 4 & \ldots \end{array}$$

Since $2 > 1$, $\quad 0.572 > 0.5714\ldots$, *so:*

$$\mathbf{0.572 > \frac{4}{7}}$$

<u>Percents</u>

Percentages are another way of describing parts of a whole. When dealing with one, you will almost always wish to change the percent into a fraction or decimal, and this is what we will look at here. In the next section we will look at some applications.

<u>To Change a Percent $\longrightarrow$ Fraction</u>

1. Drop the % sign.
2. Put number over 100.
3. Simplify.

This transformation is obvious if you know a little Latin. The word *percent* is roughly Latin for $\div 100$.

Example 19. *Change the following percentages to fractions. Simplify your answer if possible.*

1. 17%

2. 82%

3. 145%
4. 3.9%

Solution 19. *We have:*

1. $17\% = \mathbf{\frac{17}{100}}$

2. $$\begin{aligned} 82\% &= \frac{82}{100} \\ &= \frac{82 \div 2}{100 \div 2} \\ &= \mathbf{\frac{41}{50}} \end{aligned}$$

3. $$\begin{aligned} 145\% &= \frac{145}{100} \\ &= \frac{145 \div 5}{100 \div 5} \\ &= \mathbf{\frac{29}{20}} \end{aligned}$$

4. $3.9\% = \mathbf{\frac{3.9}{100}}$

   This last answer may be simplified - we will see how in the next section.

Changing a percent to a decimal is even easier.

To Change a Percent $\longrightarrow$ Decimal

1. Drop the % sign.
2. Move the decimal point two places to the left.

Example 20. *Change the following percentages to decimals.*

1. 17%
2. 82%
3. 145%
4. 3.9%

Solution 20. *We have:*

1. $17\% = \mathbf{0.17}$
2. $82\% = \mathbf{0.82}$
3. $145\% = \mathbf{1.45}$
4. $3.9\% = \mathbf{0.039}$

Note that if there aren't enough digits to move the decimal point past, you may need to add in unnecessary zeros ($3.9 = 03.9$, for example). Also, this last method is easily reversed.

To Change a Decimal $\longrightarrow$ Percent

1. Move the decimal point two places to the right.
2. Add a % sign.

Example 21. *Change the following decimals to percents.*

1. 2.37
2. 0.9
3. 1.5407
4. $5.\overline{6}$

Solution 21. *We have:*

1. $2.37 = \mathbf{237}\%$
2. $0.9 = \mathbf{90}\%$
3. $1.5407 = \mathbf{154.07}\%$
4. $5.\overline{6} = \mathbf{566.\overline{6}}\%$

Changing fractions to percents is the hardest transformation involving percents, and we will show two methods. The first is the general method which will always work, while the second is a faster method which only works well if the denominator of the fraction is a FACTOR of 100.

To Change a Fraction $\longrightarrow$ Percent, General Method

1. Change the fraction to a decimal using long division.
2. Change the decimal to a percent as explained before.

To Change a Fraction $\longrightarrow$ Percent, Shortcut

1. Make an equivalent fraction with a denominator of 100.
2. Take the new numerator and add a % sign.

Example 22. *Change the following fractions to percents.*

1. $\frac{1}{8}$

2. $\frac{5}{6}$

3. $\frac{3}{10}$

4. $\frac{14}{25}$

Solution 22. *We have:*

1. $\frac{1}{8}$, *8 is not a factor of 100, so we will use the general method.*

   Scratch work:

$$\begin{array}{r|l} & \overline{0.125} \\ 8 & 1.000 \\ & \underline{8} \\ & 20 \\ & \underline{16} \\ & \;\;40 \\ & \;\;\underline{40} \\ & \;\;\;0 \end{array}$$

   So, $\frac{1}{8} = 0.125 = \mathbf{12.5}\%$

2. $\frac{5}{6}$, *6 is not a factor of 100, so we will use the general method.*

   Scratch work:

$$
\begin{array}{r|l}
 & \;\;0.833\ldots \\
6 & \overline{5.000} \\
 & \underline{48} \\
 & \;\;20 \\
 & \;\;\underline{18} \\
 & \;\;\;\;20 \\
 & \;\;\;\;\underline{18} \\
 & \;\;\;\;\;\;2 \\
 & \;\;\;\;\;\;\vdots
\end{array}
$$

So, $\frac{5}{6} = 0.8\overline{3} = \mathbf{83.\overline{3}}\%$

3. $\frac{3}{10}$, 10 *is a factor of* 100, *so we will use the shortcut.*

$$
\begin{aligned}
\frac{3}{10} &= \frac{3 \cdot 10}{10 \cdot 10} \\
&= \frac{30}{100} \\
&= \mathbf{30}\%
\end{aligned}
$$

4. $\frac{14}{25}$, 25 *is a factor of* 100, *so we will use the shortcut.*

$$
\begin{aligned}
\frac{14}{25} &= \frac{14 \cdot 4}{25 \cdot 4} \\
&= \frac{56}{100} \\
&= \mathbf{56}\%
\end{aligned}
$$

SECTION 1.10 EXERCISES

(Answers are found on page 378.)

Change the following fractions to decimals.

1. $\dfrac{1}{2}$
2. $\dfrac{7}{9}$
3. $\dfrac{3}{8}$
4. $\dfrac{9}{100}$
5. $\dfrac{8}{15}$
6. $\dfrac{10}{11}$
7. $\dfrac{7}{25}$
8. $\dfrac{13}{16}$
9. $\dfrac{19}{4}$
10. $\dfrac{5}{22}$

Change the following decimals to fractions or mixed numbers. For these problems, you do NOT have to simplify your answer.

11. 3.7630073
12. 14.2178
13. -9.00035
14. 5.124992
15. 156.2
16. -55.88
17. 0.000007
18. 11.011
19. -18.7002
20. 0.0306

Round the following decimals to the nearest hundredth.

21. 22.0849
22. -73.095128
23. $1,350.0072$
24. -3.1415
25. 56.999
26. -18.55532

Insert a $<$ or $\geq$ between each of the following pairs of numbers to make a true statement.

27. $1.5\overline{6}$ ___ 1.56
28. $-23.\overline{52}$ ___ $-23.5\overline{2}$
29. 8.67 ___ -8.67
30. $4.\overline{3}$ ___ $4.\overline{30}$
31. -2.9192 ___ $-2.\overline{91}$
32. $\frac{3}{5}$ ___ 0.6
33. $\frac{4}{9}$ ___ $0.\overline{45}$
34. $\frac{9}{14}$ ___ 0.643
35. $\frac{16}{5}$ ___ 3.1
36. $-\frac{1}{6}$ ___ -0.167

Order the following decimals from least to greatest.

37. $32.5\overline{43}$, 32.54343, $32.5\overline{4}$, 32.5
38. -18.71, $-18.7\overline{1}$, -18.7101, $-18.\overline{7}$
39. $32.\overline{32}$, $23.\overline{23}$, $32.\overline{23}$, $23.\overline{32}$
40. $-0.\overline{01}$, -0.0101, $-0.0\overline{1}$, -0.1

Rewrite the following percents as fractions. Simplify your answer if possible.

41. 24%
42. 33%
43. 5%
44. 130%
45. 19%
46. 75%
47. 80%
48. 1%
49. 0%
50. 200%

Rewrite the following percents as decimals.

51. 24%
52. 33%
53. 5%
54. 130%
55. 19%
56. 75%
57. 80%
58. 1%

59. 0%

60. 200%

Rewrite the following decimals as percents.

61. 0.52

62. 0.031

63. 0.18

64. 2.5

65. 9.99

66. 0.005

67. 0.7

68. 0.44

69. 0.225

70. 1.0

Rewrite the following fractions as percents.

71. $\frac{1}{3}$

72. $\frac{1}{2}$

73. $\frac{1}{4}$

74. $\frac{1}{5}$

75. $\frac{1}{6}$

76. $\frac{3}{4}$

77. $\frac{5}{9}$

78. $\frac{2}{25}$

79. $\frac{7}{10}$

80. $\frac{17}{50}$

1.11 Decimal Operations

For all of this section, we will be dealing exclusively with terminating decimals.

Addition and Subtraction

To add or subtract two decimals without a calculator, we will use a vertical tower method similar to how we do old-fashioned addition and subtraction without a calculator (see page 8). Just as before, you must make sure that the place-values of the two numbers are aligned. With whole numbers this meant that the two numbers would be right-aligned, while this need not be the case for decimals. Instead, just remember to align the decimal points before adding or subtracting. Note, if you are adding two decimals, or subtracting two decimals where the decimal after the minus sign has fewer decimal places, you may add unnecessary zeros to make both decimals have the same number of decimal places if you prefer. If you are subtracting two decimals where the decimal before the minus sign has fewer decimal places, you MUST add in unnecessary zeros to have something to take-away from.

Example 1. *Add* $13.8277 + 559.02$

Solution 1. *We will align the numbers first. We are going to add in zeros, but this is optional here.*
Scratch work:

$$\begin{array}{ccccccccc} & & 1 & & & & & & \\ & & 1 & 3 & . & 8 & 2 & 7 & 7 \\ + & 5 & 5 & 9 & . & 0 & 2 & 0 & 0 \\ \hline & 5 & 7 & 2 & . & 8 & 4 & 7 & 7 \end{array}$$

So: $13.8277 + 559.02 = \mathbf{572.8477}$.

Example 2. *Add* $463.987 + 295.3966$

Solution 2. *We will align the numbers first. We are going to add in a zero, but this is optional here.*
Scratch work:

$$\begin{array}{rcccccccc} & 1 & & 1 & & 1 & 1 & & \\ & 4 & 6 & 3 & . & 9 & 8 & 7 & 0 \\ + & 2 & 9 & 5 & . & 3 & 9 & 6 & 6 \\ \hline & 7 & 5 & 9 & . & 3 & 8 & 3 & 6 \end{array}$$

So: $463.987 + 295.3966 = \mathbf{759.3836}$

Example 3. *Subtract* $427.25 - 93.812$

Solution 3. *Here, we need to add a zero to the top number.*
Scratch work:

$$\begin{array}{rccccccc} & 3 & 12 & 6 & & 12 & 4 & 10 \\ & \cancel{4} & \cancel{2} & \cancel{7} & . & \cancel{2} & \cancel{5} & \cancel{0} \\ - & & 9 & 3 & . & 8 & 1 & 2 \\ \hline & 3 & 3 & 3 & . & 4 & 3 & 8 \end{array}$$

So: $427.25 - 93.812 = \mathbf{333.438}$

Example 4. *Subtract* $190.854 - 89.91$

Solution 4. *Here, we will add a zero to the bottom number, but this is only a matter of preference.*
Scratch work:

$$\begin{array}{rccccccc} & & & 9 & & & & \\ & & 8 & \cancel{10} & & 18 & & \\ & 1 & \cancel{9} & \cancel{0} & . & \cancel{8} & 5 & 4 \\ - & & 8 & 9 & . & 9 & 1 & 0 \\ \hline & 1 & 0 & 0 & . & 9 & 4 & 4 \end{array}$$

So: $190.854 - 89.91 = \mathbf{100.944}$

Mixing in Negatives

So far, we have looked at adding and subtracting positive decimals, but what if a negative number is involved?

If you adding or subtracting decimals:

1. You MUST take signs into account first. See Appendix A.
2. Do scratch work as shown in previous examples.
3. Make your answer have the correct sign.

Example 5. *Add* $-4.823 + 17.5$

Solution 5. *When adding a negative number and a positive number, we take the absolute value of both numbers, and subtract larger minus smaller.*
Scratch work:

$$\begin{array}{cccccccc} & & & & 14 & 9 & \\ & & 6 & & \cancel{4} & \cancel{1}0 & 10 \\ & 1 & \cancel{7} & . & \cancel{5} & \cancel{0} & \cancel{0} \\ - & & 4 & . & 8 & 2 & 3 \\ \hline & 1 & 2 & . & 6 & 7 & 7 \end{array}$$

The larger number in absolute value is the 17.5, *and it was originally positive, so the result should be positive.*
So: $-4.823 + 17.5 = \mathbf{12.677}$

Example 6. *Subtract* $-2.97 - 3.414$

Solution 6. *This is obviously not straight-forward subtraction, so let us change the subtraction to addition.*
$-2.97 - 3.414 = -2.97 + (-3.414)$
When adding a negative number and a negative number, we take the absolute value of both numbers, and add.
Scratch work:

$$\begin{array}{rcccccc} & 1 & & & & \\ & 2 & . & 9 & 7 & 0 \\ + & 3 & . & 4 & 1 & 4 \\ \hline & 6 & . & 3 & 8 & 4 \end{array}$$

When adding two numbers of the same sign, the sum has the common sign. Therefore, here, the answer is negative.
So: $-2.97 - 3.414 = -\mathbf{6.384}$

Example 7. *Subtract* $5.18 - 13.091$

Solution 7. *We are subtracting two positive numbers, but the smaller is first, so we will use subtraction shortcut* 2 *(page* 53*), although you could just change the subtraction to addition and follow the addition rules if you prefer. Subtraction shortcut* 2 *says to subtract the numbers the other way around, and make the answer negative.*
Scratch work:

$$\begin{array}{rcccccc} & & 12 & & & & \\ & 0 & \cancel{2} & & 10 & & \\ & \cancel{1} & \cancel{3} & . & \cancel{0} & 9 & 1 \\ - & & 5 & . & 1 & 8 & 0 \\ \hline & & 7 & . & 9 & 1 & 1 \end{array}$$

So: $5.18 - 13.091 = -\mathbf{7.911}$

Multiplication

To multiply two positive decimals, we will use a vertical tower method just like we used for old-fashioned multiplication of large numbers without a calculator.

To multiply decimals:

1. Write the two factors as if they were positive integers being multiplied using the old-fashioned tower method (page 13). You do NOT need to align place-values, and you do NOT wish to write-in any unnecessary zeros (except, possibly, a zero for zero integral part).
2. Multiply as before, ignoring the decimal points.
3. The final product should have the same number of decimal places as the two factors *together*.

Example 8. *Multiply* $3.27 \cdot 4.7$

Solution 8. *Scratch work:*

$$\begin{array}{rrrr} & 1 & 4 & \\ & 3. & 2 & 7 \\ \times & & 4. & 7 \\ \hline 2 & 2 & 8 & 9 \end{array} \Rightarrow \begin{array}{rrrrr} & & 1 & 2 & \\ & & 3 & 2 & 7 \\ \times & & & 4 & 7 \\ \hline & 2 & 2 & 8 & 9 \\ 1 & 3 & 0 & 8 & 0 \\ \hline 1 & 5 & 3 & 6 & 9 \end{array}$$

So, had we been multiplying $327 \cdot 47$*, the answer would have been*

$15,369$ *(see example 11 on page 13). Instead, we now look back at the factors and see that one factor had two decimal places, while the other had one decimal place for a total of three decimal places. Making our product have three decimal places gives:*
$3.27 \cdot 4.7 = \mathbf{15.369}$

Note that in the above example, we didn't fill in the decimal place in the product in the scratch work to emphasize that it was done last. In future examples, we will fill it in, just remember that it is the last thing you do in the scratch work.

Example 9. *Multiply* $0.91 \cdot 2.08$

> **Solution 9.** *Don't forget that multiplication is commutative, so we will choose to put the* 0.91 *number on bottom. Also, the zero on the* 0.91 *is unnecessary so you do not need to write it if you wish. We will leave it on, but we will not bother to multiply by it when we do the scratch work. Scratch work:*
>
> $$\begin{array}{cccc} & 2. & 0 & 8 \\ \times & 0. & 9 & 1 \\ \hline & 2 & 0 & 8 \end{array} \Rightarrow \begin{array}{cccccc} & & & & 7 & \\ & & & 2. & 0 & 8 \\ \times & & & 0. & 9 & 1 \\ \hline & & & 2 & 0 & 8 \\ & 1 & 8 & 7 & 2 & 0 \\ \hline & 1. & 8 & 9 & 2 & 8 \end{array}$$
>
> *So,* $2.08 \cdot 0.91 = \mathbf{1.8928}$

Mixing in Negatives

Recall that the sign rules for multiplication are independent of the scratch work, regardless of the type of numbers involved. Therefore, mixing in negatives only make the problems slightly harder.

Example 10. *Multiply* $-4.35 \cdot 72$

> **Solution 10.** *Ignoring the negative sign for the moment, we get:*

Scratch work:

$$\begin{array}{rrrr} & & 1 & \\ & 4. & 3 & 5 \\ \times & & 7 & 2 \\ \hline & 8 & 7 & 0 \end{array} \Rightarrow \begin{array}{rrrrr} & & 2 & 3 & \\ & & 4. & 3 & 5 \\ \times & & & 7 & 2 \\ \hline & & 8 & 7 & 0 \\ 3 & 0 & 4 & 5 & 0 \\ \hline 3 & 1 & 3. & 2 & 0 \end{array}$$

And a negative number times a positive number makes a negative number, so, $-4.35 \cdot 72 = -\mathbf{313.2}$

Notice in the last problem that when we did our multiplying, the rightmost digit of the product was a zero. While this zero was unnecessary and we did not include it when we wrote the final answer, *it was necessary when we were determining where to put the decimal point in the product.* In this case, the product should have two decimal places, so we moved it past the 0 and the 2. Only after the decimal point is in place does the zero become unnecessary, and thus optional.

Example 11. *Multiply* $-18.4 \cdot -0.15$

Solution 11. *Ignoring the negative signs for the moment, we get: Scratch work:*

$$\begin{array}{rrrr} & 4 & 2 & \\ & 1 & 8. & 4 \\ \times & 0. & 1 & 5 \\ \hline & 9 & 2 & 0 \end{array} \Rightarrow \begin{array}{rrrrr} & & 1 & 8. & 4 \\ \times & & 0. & 1 & 5 \\ \hline & & 9 & 2 & 0 \\ & 1 & 8 & 4 & 0 \\ \hline & 2. & 7 & 6 & 0 \end{array}$$

And a negative number times a negative number makes a positive number, so, $-18.4 \cdot -0.15 = \mathbf{2.76}$

Example 12. *Multiply* $51.54 \cdot -10$

Solution 12. *Ignoring the negative sign for the moment, we get:*
Scratch work:

$$\begin{array}{rrrrr} & 5 & 1. & 5 & 4 \\ \times & & & 1 & 0 \\ \hline & 0 & 0 & 0 & 0 \end{array} \Rightarrow \begin{array}{rrrrr} & 5 & 1. & 5 & 4 \\ \times & & & 1 & 0 \\ \hline & 0 & 0 & 0 & 0 \\ 5 & 1 & 5 & 4 & 0 \\ \hline 5 & 1 & 5. & 4 & 0 \end{array}$$

And a positive number times a negative number makes a negative

number, so, $51.54 \cdot -10 = \mathbf{-515.4}$

Because we use a base-ten numeration system, multiplying by powers of 10 is actually easier than we made it look in the last example. Back in the days when you were doing old-fashioned multiplication, you may have heard that when you multiplied a number by a power of 10, you just added the same number of zeros to the number as there were zeros in the power of 10. This rule may be generalized for any decimal number, not just integers.

To multiply by a power of ten:

When multiplying a number by a power of ten, simply move the decimal point in the number to the right by the same number of places as there are zeros in the power of ten. You will still need to figure out the sign of the product as usual.

Example 13. *Multiply* $-32.609 \cdot 100$

Solution 13. *Ignoring the negative sign for a moment, we will multiply* $32.609 \cdot 100$*. Because one of the factors is* 100*, the multiplication is trivial; we just need to move the decimal point*

in 32.609 *two places to the right (as there are two zeros in* 100*).*
Thus, the numerical answer is 3,260.9*.*
As for sign, a negative times a positive makes a negative, so:
$-32.609 \cdot 100 = \mathbf{-3,260.9}$

Example 14. *Multiply* $-87.93 \cdot -10,000$

Solution 14. *Ignoring the negative signs for a moment, we need to move the decimal point in* 87.93 *four places to the right. If you need to, you may add unnecessary zeros to a number, in this case, you may make* $87.93 = 87.9300$ *so that there are enough digits to move past (some people prefer to just do this in their heads). Thus, the numerical answer is* 879,300*.*
As for sign, a negative times a negative makes a positive, so:
$-87.93 \cdot -10,000 = \mathbf{879,300}$

This property of multiplication by powers of ten is useful for reducing fractions as well.

Simplifying fractions which contain terminating decimals

Last section we saw the fraction: $\frac{3.9}{100}$. A fraction, such as this, whose numerator or denominator (or both) are terminating fractions are not considered to be in lowest terms. To simplify such a fraction:

Simplifying fractions involving terminating decimals

1. Make an equivalent fraction by multiplying numerator and denominator by a power of ten large enough to make both integers.

2. Simplify.

Example 15. *Simplify* $\frac{3.9}{100}$

> **Solution 15.** *The numerator has one decimal place; the denominator has none. Therefore, the most number of places we need the decimal to move is one, so we will multiply numerator and denominator by* 10*, and then simplify.*

$$\begin{aligned}\frac{3.9}{100} &= \frac{3.9 \cdot 10}{100 \cdot 10} \\ &= \mathbf{\frac{39}{1000}}\end{aligned}$$

The last solution was in lowest terms, but if you are having trouble seeing it, you may wish to do the prime factorization of both the numerator and denominator. In the last case, you would find that:
$39 = 3 \cdot 13$, and
$1000 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$.
Now it is easier to see that there are no common factors greater than 1.

Example 16. *Simplify* $\frac{0.28}{24.5}$

> **Solution 16.** *The numerator has two decimal places; the denominator has one. Therefore, the most number of places we need the decimal to move is two, so we will multiply numerator and denominator by* 100*, and then simplify using the prime factorizations.*

$$\begin{aligned}\frac{0.28}{24.5} &= \frac{0.28 \cdot 100}{24.5 \cdot 100} \\ &= \frac{28}{2450} \\ &= \frac{2 \cdot 2 \cdot 7}{2 \cdot 5 \cdot 5 \cdot 7 \cdot 7} \\ &= \frac{2}{5 \cdot 5 \cdot 7} \\ &= \mathbf{\frac{2}{175}}\end{aligned}$$

Division

To do decimal division, we will look at a couple of different situations. The easiest situation is when the dividend and divisor are both positive, and the divisor is an integer.

> **Dividing a positive decimal by a positive integer.**
>
> 1. Perform long division, same as you do for whole numbers (see page 17).
> 2. The decimal point in the quotient is directly above the decimal point in the dividend (i.e. the place-values of the dividend and quotient are in line).

Example 17. *Divide* $6.9 \div 3$

Solution 17. *Performing long division we get:*
Scratch work:

$$\begin{array}{r r}
 & 2.3 \\
3 & \big|\overline{6.9} \\
 & \underline{6} \\
 & 09 \\
 & \underline{9} \\
 & 0
\end{array}$$

So, $6.9 \div 3 = \mathbf{2.3}$

Example 18. *Divide* $7.36 \div 5$

Solution 18. *Performing long division we get:*
Scratch work:

$$\begin{array}{r|l} & 1.472 \\ \cline{2-2} 5 & 7.360 \\ & \underline{5} \\ & 23 \\ & \underline{20} \\ & 36 \\ & \underline{35} \\ & 10 \\ & \underline{10} \\ & 0 \end{array}$$

So, $7.36 \div 5 = \mathbf{1.472}$

Note that we had to add a zero to the dividend when we were dividing. Just as you did for changing fractions to decimals, you should keep dividing until either you get a zero remainder, or the division falls into a repetitive pattern.

So, what do you do when the divisor has decimal places too? Let us look at an example.

Example 19. *Divide* $1.04 \div 0.2$

Solution 19. *If we change the division problem to a fraction, we will have a fraction containing decimals which we just learned how to simplify. Applying the same idea here gives:*

$$\begin{aligned} \frac{1.04}{0.2} &= \frac{1.04 \cdot 10}{0.2 \cdot 10} \\ &= \frac{10.4}{2} \end{aligned}$$

Had we really been trying to simplify the fraction, we would have multiplied numerator and denominator by 100 *instead of* 10, *but we were really only concerned with removing the decimal from the denominator. After all, this shows that:*

$$1.04 \div 0.2 = 10.4 \div 2$$

since they are equivalent fractions, and now the divisor is an integer so we can divide as before.
Scratch work:

$$\begin{array}{r|l} & 5.2 \\ 2 & \overline{10.4} \\ & \underline{10} \\ & 04 \\ & \underline{4} \\ & 0 \end{array}$$

So, $1.04 \div 0.2 = \mathbf{5.2}$

This method may be summarized by:

Dividing a positive, terminating decimal by a positive, terminating decimal

1. Move the decimal point in the divisor to the right as many places as necessary to make the divisor an integer.
2. Move the decimal point in the dividend to the right the same number of places as you moved the decimal point in step 1, to keep the quotient the same (i.e. you are making equivalent fractions). Note, you may have to add zeros.
3. Divide, using the long division rule for dividing a positive decimal by a positive integer.

Example 20. *Divide* $6.3 \div 0.28$

Solution 20. *First, we will move the decimal points:*

$$0.28 \;\; |\overline{6.3} \quad \Rightarrow \quad 28 \;\; |\overline{630}$$

Performing the long division we get:
Scratch work:

$$\begin{array}{rl} & 22.5 \\ 28 & |\overline{630.0} \\ & \underline{56} \\ & 70 \\ & \underline{56} \\ & 140 \\ & \underline{140} \\ & 0 \end{array}$$

So, $6.3 \div 0.28 = \mathbf{22.5}$

Mixing in Negatives

Division behaves like multiplication in that the sign rules have nothing to do with your scratch work. Therefore, do scratch work using the absolute value of all numbers, and then make the sign of your answer correct.

Example 21. *Divide* $-19.872 \div 0.54$

Solution 21. *Ignoring the negative sign for the moment, we will move the decimal points:*

$$0.54 \quad |\overline{19.872} \quad \Rightarrow \quad 54 \quad |\overline{1987.2}$$

Performing the long division we get:
Scratch work:

$$\begin{array}{rl} & 36.8 \\ 54 & |\overline{1987.2} \\ & \underline{162} \\ & 367 \\ & \underline{324} \\ & 432 \\ & \underline{432} \\ & 0 \end{array}$$

Now, to get the proper sign, recall that a negative divided by a positive makes a negative.
So, $-19.872 \div 0.54 = \mathbf{-36.8}$

Example 22. *Divide* $-2.83 \div -0.3$

Solution 22. *Ignoring the negative signs for the moment, we will move the decimal points:*

$$0.3 \quad |\overline{2.83} \quad \Rightarrow \quad 3 \quad |\overline{28.3}$$

Performing the long division we get:
Scratch work:

$$\begin{array}{rl} & 9.433\ldots \\ 3 & |\overline{28.300\ldots} \\ & \underline{27} \\ & 13 \\ & \underline{12} \\ & 10 \\ & \underline{9} \\ & 10 \\ & \underline{9} \\ & 1 \\ & \vdots \end{array}$$

This quotient is now seen to be a repeating decimal. To get the proper sign, recall that a negative divided by a negative makes a positive.
So, $-2.83 \div -0.3 = \mathbf{9.4\overline{3}}$

Percent applications

If asked to find a certain percent of a number, you need to know:

To take a percentage of a number

- Change the percent to a decimal (or fraction).
- The word "of" in this context (and often in math) means "multiply".

Example 23. *What is* 8% *of* 250*?*

Solution 23. *Changing the percent to a decimal and multiplying gives:*

$$8\% \textit{ of } 250 \quad = \quad 0.08 \cdot 250$$

So we need to decimal multiplication: Scratch work:

$$\begin{array}{rrrrr} & & 4 & & \\ & & 2 & 5 & 0 \\ \times & & 0. & 0 & 8 \\ \hline & 2 & 0. & 0 & 0 \end{array}$$

Therefore, 8% *of* 250 = **20**

Example 24. *What is* 12% *of* 88*?*

Solution 24. *Changing the percent to a decimal and multiplying gives:*

$$12\% \textit{ of } 88 \quad = \quad 0.12 \cdot 88$$

So we need to decimal multiplication: Scratch work:

$$\begin{array}{rrrr} & & 1 & \\ & & 8 & 8 \\ \times & 0. & 1 & 2 \\ \hline & 1 & 7 & 6 \end{array} \Rightarrow \begin{array}{rrrrr} & & & 8 & 8 \\ \times & & 0. & 1 & 2 \\ \hline & & 1 & 7 & 6 \\ & & 8 & 8 & 0 \\ \hline & 1 & 0. & 5 & 6 \end{array}$$

Therefore, 12% *of* 88 = **10.56**

SECTION 1.11 EXERCISES

(Answers are found on page 380.)

Add or subtract.

1. $52.389 + 712.9333$
2. $34.6 + 2.8944$
3. $17.34 - 12.7$
4. $821.077 - 49.75$
5. $-28.567 + 14.39$
6. $7.355 - 11.68$
7. $-29.05 - 243.782$
8. $227.34 + (-18.854)$
9. $100.76 - 87.3$
10. $15.435 - (-254.36)$
11. $-43.709 - (-300.56)$
12. $194.538 + 3.88$
13. $-34.209 + 4.6993$
14. $56.3 - 129.44$
15. $-2.89 - 7.33$
16. $14.23 - (-323.18)$
17. $56.02 + (-34.185)$
18. $0.24 - 156$
19. $-19.4 + 6.75$
20. $566.2 - (-18.493)$

Multiply.

21. $43.05 \cdot 2.9$
22. $18.5 \cdot 0.043$
23. $-33.6 \cdot 7.2$
24. $-14.8 \cdot (-0.25)$
25. $0.672 \cdot (-3.1)$
26. $1.02 \cdot 9.6$
27. $-31.7 \cdot (-8.3)$
28. $24.1 \cdot (-0.034)$
29. $-6.2 \cdot 4.6$
30. $-0.33 \cdot 0.28$
31. $17.372 \cdot 100$
32. $0.000472 \cdot (-10,000)$
33. $-425.2834 \cdot 10$
34. $-2.40023 \cdot (-1,000)$
35. $-33.961 \cdot 10$
36. $207.0034 \cdot 100,000$
37. $2.019 \cdot (-100)$
38. $-0.000156 \cdot (-10,000)$
39. $320.1 \cdot 1,000$
40. $-0.02005 \cdot 10,000$

Divide. Write your answer as a decimal.

41. $163 \div 5$
42. $32.788 \div 14$
43. $3.542 \div (-2.3)$
44. $-60.12 \div 0.3$
45. $1.6328 \div 5.2$
46. $-53 \div -4$
47. $17.71 \div (-5.5)$
48. $-6.817 \div 3.4$
49. $-7.505 \div (-2.5)$
50. $31.08 \div (-0.6)$

Find the following amounts.

51. 5% of 26
52. 7% of 48
53. 2% of 135
54. 15% of 28
55. 75% of 92
56. 16% of 111
57. 82% of 90
58. 33% of 78
59. 50% of 35
60. 100% of 19

1.12 Introduction to Radicals

We have seen how addition and subtraction can undo each other. For example, $5 + 7 = 12$ and $12 - 7 = 5$. More generally, if we tell you that we are taking a number and adding 7 to it, and then we tell you the result, you could tell what the original number was by subtracting 7. There is a similar relationship between multiplication and division (provided you are not multiplying or dividing by zero). For example, we tell you that we took a number, multiplied it by 3, and got a result of 24. Can you tell what number we started with? Certainly, all you have to do is divide our result by 3 to undo my multiplication (thus, we started with 8). This property of being able to undo an operation is important in mathematics. In this section, we wish to define the operation which undoes "raising a number to a power".

Definition: The number a is said to be an <u>*nth* root</u> of the number b if $a^n = b$.

Example 1. *The following are all examples of roots:*

1. 4 *is a second root (or, more commonly, square root) of* 16 *as* $4 \cdot 4 = 16$.
2. 2 *is a third root (or, more commonly, cube root) of* 8 *as* $2 \cdot 2 \cdot 2 = 8$.
3. 3 *is a fourth root of* 81 *as* $3 \cdot 3 \cdot 3 \cdot 3 = 81$.
4. 2 *is a fifth root of* 32 *as* $2^5 = 32$.

Notice that just as we commonly use "squared" instead of "to the second power", we even more commonly say "square root" instead of "second root". The same is also true for saying "cube root" instead of "third root".

We need a notation for this concept of "taking a root of a number", and we will use a radical. We would like to use the notation:

$$\sqrt[index]{radicand}$$

to mean:

$$\sqrt[n]{b} = a \quad \Leftrightarrow \quad a^n = b$$

but there is a problem. In the last example, we mentioned that 4 is a square root of 16 as $4^2 = 16$, but -4 is *also* a square root of 16 since $(-4)^2 = (-4)(-4) = 16$. This is bothersome as we do not wish the radical to be ambiguous. Also, while this is a problem with square roots and fourth roots and so on, it is not true for roots with odd indices. For example, 2 is a cube root of 8 as $2^3 = 8$, AND it is THE ONLY cube root of 8 (as $(-2)^3 = (-2)(-2)(-2) = -8$ not 8). Therefore we need to define the radical differently when the index is even versus when the index is odd.

Radicals with even indices

First, note that a square root is the most common root you will encounter. As a result, it was decided that instead of writing the index of 2 on the radical, it will be understood that if there is no obvious index, the radical will represent a *square root*. For example, we will take $\sqrt{25}$ to mean "square root of 25" (we just need to clear up the ambiguity of which square root).

When the index is even, the radical's behavior depends entirely on the sign of the radicand. Let us look at the three possible cases.

1. The radicand is positive.

   - In this case, there will be two roots, one positive and one negative. We will define the radical to mean the principal square root which is the positive square root. The other square root will be represented by putting a "$-$" in front of the radical (recall this means to "take the opposite of" or, equivalently, "multiply by negative one" which when applied to the principal square root will give a negative number).

     Example 2. *Evaluate the following.*

     (a) $\sqrt{25}$
     (b) $-\sqrt{25}$
     (c) $\sqrt{4}$
     (d) $-\sqrt{4}$

     Solution 2. *We get:*

     (a) $\sqrt{25} = \mathbf{5}$

     The principal square root of 25 *is* 5*.*

     (b) $-\sqrt{25} = -\mathbf{5}$

     The non principal square root of 25 *is* -5*,* *OR*

The opposite of the principal square root of 25 *is* -5.

(c) $\sqrt{4} = \mathbf{2}$

The principal square root of 4 *is* 4.

(d) $-\sqrt{4} = -\mathbf{2}$

The non principal square root of 4 *is* -2,

OR

The opposite of the principal square root of 4 *is* -2.

2. The radicand is zero.
   - There is no ambiguity here as $\sqrt[n]{0} = 0$. Recall zero is neither positive nor negative.

3. The radicand is negative.
   - This is troublesome as
     - a positive number to an even power is positive.
     - zero to an even power is zero.
     - a negative number to an even power is positive.

   Therefore, it seems to be *impossible* to find an even root of a negative number. In fact, mathematicians have defined such roots, but the answers are not real numbers, so we will not be concerned with the solutions in this course. Instead,

Radical Fact 1:

Whenever there is a negative radicand for an even indexed radical, you may immediately say that this represents **no real number**.

Example 3. *Evaluate the following.*

(a) $\sqrt{-25}$
(b) $\sqrt{-13}$
(c) $\sqrt{-1}$
(d) $-\sqrt{-8}$

Solution 3. *As we just learned:*

(a) $\sqrt{-25} =$ ***no real number***
(b) $\sqrt{-13} =$ ***no real number***
(c) $\sqrt{-1} =$ ***no real number***
(d) $\sqrt{-8} =$ ***no real number***

That is pretty easy, isn't it? Some people write **no solution** or **undefined** in these instances, but since there are actually solutions, just ones dealing with numbers which aren't real numbers, we prefer the more accurate **no real number**.

Radicals with odd indices

Even though you may be more familiar with square roots, radicals with odd indices are actually much nicer to work with as *regardless* of what the radicand is (positive, zero, negative), radicals with odd indices are always:

1. unique.
2. real numbers.

No longer do we need to define a principal root - there is always only one root.

Example 4. *Evaluate the following:*

1. $\sqrt[3]{27}$
2. $\sqrt[3]{-27}$
3. $\sqrt[5]{32}$
4. $\sqrt[5]{-32}$

Solution 4. *Using our definition of roots, we get:*

1. $\sqrt[3]{27} = \mathbf{3}$

   The cube root of 27 *is* 3 *as* $3^3 = 27$.

2. $\sqrt[3]{-27} = -\mathbf{3}$

   The cube root of -27 *is* -3 *as* $(-3)^3 = -27$.

3. $\sqrt[5]{32} = \mathbf{2}$

   The fifth root of 32 *is* 2 *as* $2^5 = 32$.

4. $\sqrt[5]{-32} = -\mathbf{2}$

   The fifth root of -32 *is* -2 *as* $(-2)^5 = -32$.

If you don't like working with the negative number under the radical, there is an optional rule you may use.

Radical Fact 2:

Whenever there is a negative radicand for an odd indexed radical, you may *pull it out in front of the radical.*

Example 5. *Evaluate* $\sqrt[3]{-27}$

Solution 5. *We did this problem in the last example, but this time we will use our new rule.*

$$\sqrt[3]{-27} = -\sqrt[3]{27} = -\mathbf{3}$$

Tables for Perfect Squares and Perfect Cubes

You may find it useful to be able to recognize perfect squares and/or perfect cubes. The following tables, or tables for higher powers, may be reproduced easily.

First Ten Positive Perfect Squares

$1 \cdot 1 = \mathbf{1}$
$2 \cdot 2 = \mathbf{4}$
$3 \cdot 3 = \mathbf{9}$
$4 \cdot 4 = \mathbf{16}$
$5 \cdot 5 = \mathbf{25}$
$6 \cdot 6 = \mathbf{36}$
$7 \cdot 7 = \mathbf{49}$
$8 \cdot 8 = \mathbf{64}$
$9 \cdot 9 = \mathbf{81}$
$10 \cdot 10 = \mathbf{100}$

The bold-faced numbers are the perfect squares, so one of the factors to the left would be its principal square root.

First Six Positive Perfect Cubes

$1 \cdot 1 \cdot 1 = \mathbf{1}$
$2 \cdot 2 \cdot 2 = \mathbf{8}$
$3 \cdot 3 \cdot 3 = \mathbf{27}$
$4 \cdot 4 \cdot 4 = \mathbf{64}$
$5 \cdot 5 \cdot 5 = \mathbf{125}$
$6 \cdot 6 \cdot 6 = \mathbf{216}$

The bold-faced numbers are the perfect cubes, so one of the factors to the left would be its cube root.

Large radicals

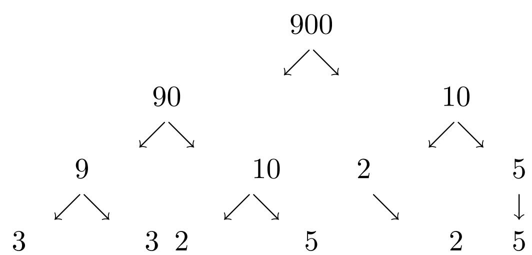
While tables are nice when simplifying smaller radicals, they are not as practical for larger radicands. To simplify these, we recommend you find the prime factorization of the radicand, and then use the following fact:

Radical Fact 3:

When simplifying nth roots, every n like factors under the radical may be pulled outside the radical as a single factor. Any such factor comes out *multiplying.*

Example 6. *Simplify* $\sqrt{900}$

Solution 6. *First, let us find the prime factorization of* 900.



So, $900 = 2\cdot 2\cdot 3\cdot 3\cdot 5\cdot 5$. *Writing these factors under the radical, grouping together like pairs of factors (since this is a square root so* $n = 2$*), and using our new fact gives:*

$$\begin{aligned}\sqrt{900} &= \sqrt{(2\cdot 2)\cdot(3\cdot 3)\cdot(5\cdot 5)}\\ &= 2\cdot 3\cdot 5\\ &= \mathbf{30}\end{aligned}$$

So for square roots, you need to group together *two* of the same factors; we did this with parentheses. Then radical fact 3 says each grouping pulls out of the radical as a single factor, so a pair of two's under the square root comes out as a single two, etc. When all the factors under the radical pair up, you may drop the radical once you have pulled out the single factors. You may wonder what happens when not all the factors pair up - these cases will be looked at in 10022 (or later in the semester in 10006). For now, all of our examples and problems will have numbers whose factors pull out perfectly.

Also, note that radical fact 3 does not require the factors to be prime numbers. Had you noticed that $900 = 30 \cdot 30$, you could have simplified by:

$$\sqrt{900} = \sqrt{(30 \cdot 30)} = \mathbf{30}$$

In general though, you will have an easier time just doing the prime factorization.

Example 7. *Simplify* $\sqrt[3]{42,875}$

Solution 7. *First, let us find the prime factorization of* $42,875$.

$$\begin{array}{ccccccc}
 & 42,875 & & & & & \\
 & \swarrow\ \searrow & & & & & \\
5 & & 8,575 & & & & \\
\downarrow & & \swarrow\ \searrow & & & & \\
5 & 5 & & 1,715 & & & \\
\downarrow & \downarrow & & \swarrow\ \searrow & & & \\
5 & 5 & 5 & & 343 & & \\
\downarrow & \downarrow & \downarrow & & \swarrow\ \searrow & & \\
5 & 5 & 5 & 7 & & 49 & \\
\downarrow & \downarrow & \downarrow & \downarrow & & \swarrow\ \searrow & \\
5 & 5 & 5 & 7 & 7 & & 7
\end{array}$$

So, $42,875 = 5 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 7$. *This time, though, we are dealing with a cube root, so we must group three like-factors together.*

$$\begin{aligned}
\sqrt[3]{42,875} &= \sqrt[3]{(5 \cdot 5 \cdot 5) \cdot (7 \cdot 7 \cdot 7)} \\
&= 5 \cdot 7 \\
&= \mathbf{35}
\end{aligned}$$

So the type of root determines how many like-factors must be grouped together, and then one factor of each group pulls outside the radical (multiplying any other factors which come out).

One last note must be made.

Important Note on Radicals

For any number but 0 and 1, there is a difference in squaring the number and cubing the number. Similarly, for any number but 0 and 1, there is a difference in taking a square root and taking a cube root. Therefore, do NOT become lazy and just drop the index of the radical!

Many students like to simply quit writing in the index, but this IS NOT correct, regardless of what is meant.

SECTION 1.12 EXERCISES

(Answers are found on page 381.)

Simplify the following radicals.

1. $\sqrt{49}$
2. $\sqrt{-49}$
3. $-\sqrt{49}$
4. $\sqrt[3]{125}$
5. $\sqrt[3]{-125}$
6. $-\sqrt[3]{125}$
7. $\sqrt{64}$
8. $\sqrt[3]{64}$
9. $\sqrt{-81}$
10. $\sqrt[3]{-8}$
11. $\sqrt[4]{-16}$
12. $\sqrt[5]{-32}$
13. $-\sqrt{36}$
14. $-\sqrt[3]{216}$
15. $\sqrt{484}$
16. $\sqrt[3]{4,096}$
17. $\sqrt{576}$
18. $\sqrt[3]{-19,683}$
19. $\sqrt{-10,000}$
20. $-\sqrt{1764}$

1.13 Properties of Real Numbers

Many of the following properties of real numbers have already been introduced, but we want to summarize all of them in one section.

First, is the *commutative property.* For an operation to be commutative, it shouldn't matter which number goes before the operation, and which goes after the operation. Both addition and multiplication are commutative.

> Commutative Property of Addition:
> For any real numbers a and b,
>
> $$a + b = b + a$$

> Commutative Property of Multiplication:
> For any real numbers a and b,
>
> $$a \cdot b = b \cdot a$$

We have already used the commutative property of addition. Notice that subtraction is not commutative ($5 - 3 \neq 3 - 5$, for example), BUT recall that we can change subtraction to addition. Therefore,

$$5 - 3 = 5 + (-3) = -3 + 5$$

This shows that as long as you keep the "$-$" sign if front of the same numbers (possibly changing it from a negative sign to a minus sign or vice versa), you may use the commutative property of addition in conjunction with subtraction.

Similarly, division is not commutative ($4 \div 2 \neq 2 \div 4$, for example), BUT we may change division to multiplication and use the commutative property of multiplication. For example,

$$4 \div 2 = 4 \cdot \frac{1}{2} = \frac{1}{2} \cdot 4$$

Next is the *associative property.* For an operation to be associative, you need to be doing the same operation twice, and it shouldn't matter which of the two you do first. Again, addition and multiplication are both associative.

Associative Property of Addition:
For any real numbers a, b and c,

$$(a + b) + c = a + (b + c)$$

Associative Property of Multiplication:
For any real numbers a, b and c,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

As before, neither subtraction nor division is associative, but we may change subtraction to addition or division to multiplication if necessary.

While both properties are important, they are most often used in conjunction. For example, since addition is both commutative and associative, whenever we have a situation where we have several additions going on (and nothing else!), we may rearrange the numbers and add in any order we wish. We used this useful property when we developed the shortcut for mixed number addition (see example 11 on page 132).

A property which we haven't seen yet is the *distributive property*. Technically, it is called the Distributive Property of Multiplication over Addition or the Distributive Property of Multiplication over Subtraction, but we will generally just use the abbreviated title of distributive property.

Distributive Property:
For any real numbers a, b and c,

$$a(b + c) = a \cdot b + a \cdot c$$

$$(b + c)a = b \cdot a + c \cdot a$$

$$a(b - c) = a \cdot b - a \cdot c$$

$$(b - c)a = b \cdot a - c \cdot a$$

Example 1. *Use the distributive property to simplify* $3(4+5)$.

Solution 1. *Using the distributive property:*

$$\begin{aligned} 3(4+5) &= 3 \cdot 4 + 3 \cdot 5 \\ &= 12 + 15 \\ &= \mathbf{27} \end{aligned}$$

You may check the last result using the normal order of operations on the original problem. This begs the question, why don't we just use the order of operations instead of this new property? Well, soon we will be dealing with expressions like $(x+2)$ which cannot be added, and to get rid of the parentheses, we will need to use the distributive property.

Example 2. *State the property used:*

1. $5+4=4+5$
2. $(2+3)+7=2+(3+7)$
3. $(2 \cdot 3) \cdot 7 = (3 \cdot 2) \cdot 7$
4. $(2+3) \cdot 7 = 7 \cdot (2+3)$
5. $2(7+3) = 2 \cdot 7 + 2 \cdot 3$

Solution 2. *You should have said:*

1. $5+4=4+5$ *is the commutative property of addition since the* 4 *and* 5 *changed places.*
2. $(2+3)+7=2+(3+7)$ *is the associative property of addition since* 2 *and* 3 *are grouped together on the left side and* 3 *and* 7 *are grouped on the right side.*
3. $(2 \cdot 3) \cdot 7 = (3 \cdot 2) \cdot 7$ *is the commutative property of multiplication since the grouping did not change, but the* 2 *and* 3 *changed places.*

4. $(2+3)\cdot 7 = 7\cdot(2+3)$ *is commutative property of multiplication since the group* $(2+3)$ *changes places with* 7 *around multiplication.*
5. $2(7+3) = 2\cdot 7 + 2\cdot 3$ *is the distributive property since the* 2 *is distributed across addition.*

The distributive property may also be used when there is no obvious number in front of the opening parenthesis. In this case, you may act as though there is a $1\cdot$ in front.

Example 3. *Remove the parentheses by using the distributive property. You don't need to simplify.*

1. $(5+2)-3$
2. $4+(9-15)$

Solution 3. *We get:*

1.
$$\begin{aligned}(5+2)-3 &= 1\cdot(5+2)-3\\ &= 1\cdot 5+1\cdot 2-3\\ &= \mathbf{5+2-3}\end{aligned}$$

2.
$$\begin{aligned}4+(9-15) &= 4+1\cdot(9-15)\\ &= 4+1\cdot 9-1\cdot 15\\ &= \mathbf{4+9-15}\end{aligned}$$

After telling you not to simplify, you may wonder why we did several steps, but there was a reason for this. In the very beginning, you may need to write out every step, as we did. You want to practice this technique, though, until you can go from the the original problem to the final answer in one step. You should be able to imagine the "$1\cdot$" and multiply, without having to write it all down. Once you get rid of the parentheses, of course you will add or subtract where possible.

The distributive property is also used when there is a minus sign or negative sign in front of the opening parenthesis, but now you will be distributing a -1.

Example 4. *Remove the parentheses by using the distributive property. You don't need to simplify.*

1. $-(4+6)-3$
2. $13-(2+1)$

Solution 4. *We get:*

1.
$$\begin{aligned} -(4+6)-3 &= -1\cdot(4+6)-3 \\ &= -1\cdot 4+(-1)\cdot 6-3 \\ &= \mathbf{-4-6-3} \end{aligned}$$

2.
$$\begin{aligned} 13-(2-1) &= 13+(-1)\cdot(2-1) \\ &= 13+(-1)\cdot 2-(-1)\cdot 1 \\ &= \mathbf{13-2+1} \end{aligned}$$

Again, even though we showed several steps, you want to get good enough to go from the original to the end in one step. This is sometimes called distributing the minus sign (or negative sign).

Now that we have seen the commutative property of addition, the associative property of addition, and the distributive property, we can illustrate why the shortcut for mixed number subtraction works.

Example 5. *Refer back to example 15 on page 136. Using our new knowledge of the above properties, show why the subtraction shortcut works.*

Solution 5. *We wanted to subtract* $4\frac{3}{4}-1\frac{1}{2}$.

$$\begin{aligned} 4\tfrac{3}{4}-1\tfrac{1}{2} &= \left(4+\tfrac{3}{4}\right)-\left(1+\tfrac{1}{2}\right) \\ &= 4+\tfrac{3}{4}-1-\tfrac{1}{2} \\ &= (4-1)+\left(\tfrac{3}{4}-\tfrac{1}{2}\right) \end{aligned}$$

which is our shortcut. The rest of the problem would follow as before.

Zero plays an important role in addition, as it has the unique property of not changing another number when being added to it.

Additive Identity: For any real number a, $$a + 0 = 0 + a = a$$

It is also important to know that no matter what number you start with, there is always another number you may add to it to get zero.

Additive Inverse: For any real number a, there is a number $(-a)$, such that: $$a + (-a) = (-a) + a = 0$$

Note that the additive inverse of a number is just its *opposite*.

Zero also plays an important role in multiplication, but it is not the multiplicative identity as multiplying by zero generally does change the number you are multiplying. Rather, the number 1 has the property of not changing a number it is multiplying.

Multiplicative Identity: For any real number a, $$a \cdot 1 = 1 \cdot a = a$$

If you start with any number but 0, there is another number which you may multiply by to go back to 1.

<u>Multiplicative Inverse:</u>
For any real number a *except* $a \neq 0$,

$$a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$$

A number's multiplicative inverse is its *reciprocal.*

Example 6. *State (a) the additive inverse, and (b) the multiplicative inverse, for the following numbers.*

1. 5
2. $\frac{3}{7}$
3. -3
4. 0

Solution 6. *We have:*

1. 5
   (a) *The additive inverse of* 5 *is* $-\mathbf{5}$.
   (b) *The multiplicative inverse of* 5 *is* $\mathbf{\frac{1}{5}}$.
2. $\frac{3}{7}$
   (a) *The additive inverse of* $\frac{3}{7}$ *is* $-\mathbf{\frac{3}{7}}$.
   (b) *The multiplicative inverse of* $\frac{3}{7}$ *is* $\mathbf{\frac{7}{3}}$.
3. -3
   (a) *The additive inverse of* -3 *is* $\mathbf{3}$.
   (b) *The multiplicative inverse of* -3 *is* $-\mathbf{\frac{1}{3}}$.
4. 0
   (a) *The additive inverse of* 0 *is* $\mathbf{0}$.
   (b) 0 *does NOT have a multiplicative inverse, so* ***none***.

Example 7. *State the property used:*

1. $5 \cdot 1 = 5$
2. $0 + (3 + 2) = (3 + 2)$
3. $\frac{1}{3} \cdot 3 = 1$
4. $-\frac{4}{5} + \frac{4}{5} = 0$

Solution 7. *We have:*

1. $5 \cdot 1 = 5$ *is multiplicative identity.*
2. $0 + (3 + 2) = (3 + 2)$ *is additive identity.*
3. $\frac{1}{3} \cdot 3 = 1$ *is multiplicative inverse.*
4. $-\frac{4}{5} + \frac{4}{5} = 0$ *is additive inverse.*

Applications

These properties of real numbers may be used to simplify addition and multiplication problems. Let us look at several examples.

Example 8. *At a fast food restaurant, your order costs:*

$$\$1.29 + \$0.99 + \$0.99 + \$0.99$$

How much do you need to pay?

Solution 8. *Many people would either use a calculator (or just believe the cashier) or would estimate the solution, but it is easy to calculate the exact value if we apply the real number properties that we just learned. To avoid the ugly-looking numbers we are starting with, let us add a penny to each one, so now we wish to add:*

$$\$1.30 + \$1.00 + \$1.00 + \$1.00 = \$4.30$$

At the moment, this is an estimate, but to go back to the exact value, recall that we added in four pennies, so all we need to do is subtract away four cents:

$$\$4.30 - \$0.04 = \$4.26$$

So, your bill will be \$4.26 *(provided there is no tax - say that you get the food to go and don't order a pop or sweet tea). You really should have been able to do this addition in your head.*

The idea of adding in four cents provided we later take away four cents is the additive identity (as we are actually adding zero) and additive inverse (as our zero is four plus ones and four minus ones) properties. We then used the commutative and associative properties of addition to rearrange the numbers nicely. Written out:

$$\begin{aligned}
1.29 + .99 + .99 + .99 &= 1.29 + .99 + .99 + .99 + 4(.01 - .01) \\
&= (1.29 + .01) + 3(.99 + .01) - .04 \\
&= 1.30 + 1 + 1 + 1 - .04 \\
&= 4.30 - .04 \\
&= 4.26
\end{aligned}$$

Example 9. *Add* $98 + 156$*.*

Solution 9. *To add this as is, you have to be good at doing math in your head. If you take two from* 156 *and add it to the* 98*, however, you should be able to do the math in your head easily.*

$$98 + 156 = 100 + 154 = \mathbf{254}$$

This uses the associative property of addition. To see this, we will write out more steps:

$$\begin{aligned}
98 + 156 &= 98 + (2 + 154) \\
&= (98 + 2) + 154 \\
&= 100 + 154 \\
&= \mathbf{254}
\end{aligned}$$

Example 10. *Simplify* $3 - 8 + 2 - 6 + 7 - 4$.

> **Solution 10.** *First, let us change all the subtractions to additions so we may use the commutative and associative properties of addition. Then, in a case like this where you are adding several positive and negative numbers, you may find it easier to group all the positive numbers together, and all the negative numbers together as the sign rule for addition says that when you are adding numbers of the same sign, you add their absolute values and the sum keeps the common sign. We will also rearrange the numbers in each group to make "nice sums". For example, instead of* $3+2+7$, *you may prefer* $3+7+2$, *since* $3+7=10$ *and* 10 *is an easy number to add.*

$$\begin{aligned} 3 - 8 + 2 - 6 + 7 - 4 &= (3 + 7 + 2) + [(-6) + (-4) + (-8)] \\ &= 12 + (-18) \\ &= -\mathbf{6} \end{aligned}$$

This same idea of rearranging to simplify the problem may also work when you are doing several multiplications.

Example 11. *Multiply* $2 \cdot 7 \cdot 6 \cdot 5$

> **Solution 11.** *As is, this problem may be difficult to do in your head, but by utilizing the commutative and associative properties of multiplication, it can be made easier.*

$$\begin{aligned} 2 \cdot 7 \cdot 6 \cdot 5 &= (7 \cdot 6) \cdot (2 \cdot 5) \\ &= 42 \cdot 10 \\ &= \mathbf{420} \end{aligned}$$

SECTION 1.13 EXERCISES

(Answers are found on page 382.)

Find the additive inverse of the following numbers.

1. 3
2. -8
3. 2.9
4. -4.1
5. $\frac{1}{6}$
6. $-\frac{2}{17}$

Find the multiplicative inverse of the following numbers.

7. -2
8. 18
9. $-\frac{1}{4}$
10. $\frac{13}{15}$
11. $-\frac{2}{3}$
12. 45

State which property of real numbers is being used.

13. $(5+3)+17=5+(3+17)$
14. $6+(-6)=0$
15. $5(3-4)=5\cdot 3-5\cdot 4$
16. $-6.5\cdot 1=-6.5$
17. $8\cdot 3=3\cdot 8$
18. $\frac{1}{17}+\left(-\frac{1}{17}\right)=0$
19. $(2\cdot 5)\cdot 3=2\cdot(5\cdot 3)$
20. $7+0=7$
21. $-3.45+16=16+(-3.45)$
22. $6(4+7)=6\cdot 4+6\cdot 7$
23. $-3\cdot\left(-\frac{1}{3}\right)=1$
24. $2.8\cdot(-1.5)=(-1.5)\cdot 2.8$
25. $(2+16)+3=2+(16+3)$
26. $\frac{2}{3}+9=9+\frac{2}{3}$
27. $\frac{3}{8}\cdot\frac{8}{3}=1$
28. $3\cdot(4\cdot 100)=(3\cdot 4)\cdot 100$
29. $-8.9\cdot 1=-8.9$
30. $-8.9+0=-8.9$

Add or multiply as indicated. If you use the real number properties to make the problems easier, you should be able to do the math in your head.

31. $0.99 + 0.99 + 1.19 + 0.99 + 0.99$
32. $199 + 312$
33. $15 - 18 + 12 - 2$
34. $9 \cdot 5 \cdot 3 \cdot 2$
35. $97 + 98 + 106$
36. $5 \cdot 5 \cdot 5 \cdot 2$

Chapter 2

Algebra

2.1 Variables and Algebraic Expressions

Introduction to Variables and Algebraic Expressions

Let us start with a definition.

Definition: A variable is a letter or symbol used to represent a number or group of numbers.

Using a letter, particularly an English letter, to be a variable is very common as letters are easy to write and easy to recognize. Still, there is no real limit on what you may use when choosing a variable, and if you are willing to take the time and effort, you may draw pictures of cows or aliens or whatever.

There are two main reasons for using variables:

1. to show a *pattern* or represent numbers in a *general* setting. Mathematicians often use the letters a, b or c in these instances.

2. to represent an unknown number or numbers which we are trying to find or waiting to be given. Mathematicians often use the letters x, y or z in these instances.

Example 1. *When we presented the additive identity in section 1.13, we wrote it as:*

$$a + 0 = 0 + a = a$$

for any real number a. Here the a is representing a general pattern. For a specific example of the additive identity property, we need to replace a with an actual number.

Example 2. *When we defined the set of rational numbers, we used the expression:*

$$\frac{a}{b}$$

for any integers a and b except $b \neq 0$. This is a general expression, and to see a specific rational number, we must replace a and b with integers, say $a = 2$ and $b = 3$ to get $\frac{2}{3}$ is a rational number.

So far, we haven't seen the second use of variables, and this is where algebraic expressions and equations come into play.

Definition: An algebraic expression is a combination of mathematical operations on numbers and variables. Sometimes we will simply say an expression.

Example 3. *The following are all algebraic expressions.*

1. $7 - 3 \cdot x$
2. $x^2 + y \div z$
3. $6 \cdot x^4 - 3 \cdot x^2 + 5 \cdot x - 18$

In algebraic expressions, multiplication signs between a number and a variable, or between two variables are usually hidden, and division is typically turned into fractions.

Example 4. *The algebraic expressions from last example are more commonly written:*

1. $7 - 3x$
2. $x^2 + \frac{y}{z}$
3. $6x^4 - 3x^2 + 5x - 18$

Definition: A term is a number or the product of a number and variables raised to powers.

In an algebraic expression, terms are truly only separated by a plus sign - any subtraction should be changed to addition. It may be easier to learn that terms are separated by plus or minus signs, but you must remember to treat the minus sign as a negative (or opposite) in the following term. *Multiplication and division* do NOT separate terms, rather these operations are part of terms.

Example 5. *State how many terms the following algebraic expressions have, and list what they are.*

1. $7 - 3x$
2. $x^2 + \frac{y}{z}$
3. $6x^4 - 3x^2 + 5x - 18$

Solution 5. *We have:*

1. $7 - 3x$*; this expression has two terms.*
   The first term is: 7.
   The second term is: $-3x$.
2. $x^2 + \frac{y}{z}$*; this expression has two terms.*
   The first term is: x^2.
   The second term is: $\frac{y}{z}$.
3. $6x^4 - 3x^2 + 5x - 18$*; this expression has four terms.*
   The first term is: $6x^4$.
   The second term is: $-3x^2$.
   The third term is: $5x$.
   The fourth term is: -18.

Definition: A term which does not contain a variable is called a constant term, or simply a constant.

Definition: A term which contains one or more variables is called a variable term. Every variable term has a multiplying number associated with it, this number is called the coefficient of the variable. If there is no obvious number, recall that you may always write $1\cdot$ in front of any expression (multiplicative identity), so the coefficient will be 1 (or -1 if there is a "$-$" sign). Note, typically the coefficient is written before the variables in the term.

Example 6. *In the following algebraic expressions, state whether each term is a constant term or variable term, and if the term is a variable term, state its coefficient.*

1. $7 - 3x$
2. $x^2 + \frac{y}{z}$
3. $6x^4 - 3x^2 + 5x - 18$

Solution 6. *We have:*

1. $7 - 3x$*; this expression has two terms.*
   The first term is: 7*, and this is a* ***constant term****.*
   The second term is: $-3x$*, and this is a* ***variable term*** *with coefficient* $= \mathbf{-3}$*.*
2. $x^2 + \frac{y}{z}$*; this expression has two terms.*
   The first term is: x^2*, and this is a* ***variable term*** *with coefficient* $= \mathbf{1}$*.*
   The second term is: $\frac{y}{z}$*, and this is a* ***variable term*** *with coefficient* $= \mathbf{1}$ $(\frac{y}{z} = 1 \cdot \frac{y}{z})$*.*
3. $6x^4 - 3x^2 + 5x - 18$*; this expression has four terms.*
   The first term is: $6x^4$*, and this is a* ***variable term*** *with coefficient* $= \mathbf{6}$*.*
   The second term is: $-3x^2$*, and this is a* ***variable term*** *with coefficient* $= \mathbf{-3}$*.*
   The third term is: $5x$*, and this is a* ***variable term*** *with coefficient* $= \mathbf{5}$*.*
   The fourth term is: -18*, and this is a* ***constant term****.*

Evaluating Algebraic Expressions

To *evaluate* an algebraic expression,

- you *must be given* the algebraic expression and values for every variable which appears in the expression.
- you are required to:
  1. replace all variables with their numerical value. A good habit to get into is to always use parentheses when replacing the variables. Sometimes this will be necessary (usually if the number you are replacing the variable with is negative), and sometimes not, but it never hurts to do so.
  2. simplify according to the order of operations.

Example 7. *Evaluate the expression* $x^2 - 2yz$ *for* $x = -2$, $y = -1$ *and* $z = 5$.

Solution 7. *Replacing the variables with their values gives:*

$$\begin{aligned} x^2 - 2yz &= (-2)^2 - 2(-1)(5) \\ &= 4 - 2(-1)(5) \\ &= 4 - (-2)(5) \\ &= 4 - (-10) \\ &= 4 + 10 \\ &= \mathbf{14} \end{aligned}$$

Example 8. *Evaluate the algebraic expression* $-x$ *when:*

1. $x = 4$

2. $x = 0$

3. $x = -6$

Solution 8. *Substituting in for x gives:*

1. $-x = -(4) = \mathbf{-4}$
2. $-x = -(0) = \mathbf{0}$
3. $-x = -(-6) = \mathbf{6}$

One comment on the last example. The expression $-x$ is often read as "negative x", but technically it is "the opposite of x". After all, we saw that if we evaluate the expression for $x =$ a negative number, the expression became positive. Hence, "negative x" need not be negative. This illustrates an important point, *be careful about making assumptions of the sign of a term involving a variable.* There will certainly be times when you are confident (we will see cases later involving absolute values or raising variables to even powers), but it certainly is not as obvious as looking at the sign of the coefficient. Note here that $-x = -1 \cdot x$ (either by changing "the opposite of" to multiplication or recalling what we do when no multiplying number is obvious on a variable term) so the coefficient of $-x$ is -1. This is a fact we are confident in; we just cannot extend this to the sign of the overall expression.

Example 9. *Evaluate each of the following expressions for $x = -4$ and $y = -2$.*

1. $x^2 - y^2$
2. $|y| - |x|$
3. $5 + \frac{y}{x}$
4. $11 - xy + 5y$

Solution 9. *Substituting in and applying the order of operations gives:*

1.
$$\begin{aligned} x^2 - y^2 &= (-4)^2 - (-2)^2 \\ &= 16 - 4 \\ &= \mathbf{12} \end{aligned}$$

2.

$$\begin{aligned} |y| - |x| &= |-2| - |-4| \\ &= 2 - 4 \\ &= \mathbf{-2} \end{aligned}$$

3.

$$\begin{aligned} 5 + \tfrac{y}{x} &= 5 + \tfrac{-2}{-4} \\ &= 5 + \tfrac{1}{2} \\ &= \mathbf{5\tfrac{1}{2}} \quad \mathit{or} \quad \mathbf{\frac{11}{2}} \end{aligned}$$

4.

$$\begin{aligned} 11 - xy + 5y &= 11 - (-4)(-2) + 5(-2) \\ &= 11 - 8 - 10 \\ &= 3 - 10 \\ &= \mathbf{-7} \end{aligned}$$

Like Terms versus Unlike Terms

Definition: Terms with exactly the same variables raised to exactly the same powers are called like terms, or similar terms. Terms which are not like terms are called unlike terms. Notice that only the variables of the terms matter - the coefficients have no effect on whether two terms are like terms or not.

Example 10. *In each of the following cases, you are given two terms separated by a comma. State whether the terms are like terms or unlike terms.*

1. $3x,\ -8x$
2. $3x,\ 3y$
3. $7,\ -23$
4. $2xy^2,\ -5x^2y$
5. $3x^2y,\ 7yx^2$

Solution 10. *Comparing the variables of each term:*

1. $3x$, $-8x$ *are* ***like terms****.*
2. $3x$, $3y$ *are* ***unlike terms****.*
3. 7, -23 *are* ***like terms****. These are both constants and have the same variables (none).*
4. $2xy^2$, $-5x^2y$ *are* ***unlike terms****.*
5. $3x^2y$, $7yx^2$ *are* ***like terms****. This last one didn't fool you did it? Don't forget that multiplication is commutative and associative, so it doesn't matter the order in which we write the parts of a term. Thus,* $7yx^2 = 7x^2y$ *and this term has* x *squared and* y *to the first power, same as the other term. As a side note, it is standard that when we are using English letters to write them in alphabetical order in the term, so we would normally write* $7x^2y$*, but it is not incorrect to rearrange the letters.*

Given that terms may be like terms or unlike terms, why does it matter? It turns out that like terms which are being added or subtracted my be combined, while unlike terms may not.

Algebraic Expressions Tool 1

- When adding or subtracting like terms, you may just add or subtract the coefficients, and the result stays the same type of term.
- When adding or subtracting unlike terms, there is no way of combining the terms, so you cannot simplify.

Example 11. *Simplify the following expressions by combining like terms if possible.*

1. $5x - 14x$
2. $3x^2 - 2x$

Solution 11. *Simplifying gives:*

1. $5x - 14x = \mathbf{-9x}$
2. $3x^2 - 2x = \mathbf{3x^2 - 2x}$

Simplifying like terms is actually an application of the distributive property. Refer back to page 217 to see all the various manifestations of the distributive property. Currently, we are mostly concerned with using the distributive property to remove parentheses. There will be times, however, like this one, where we would rather use the distributive property to factor out a common factor. Notice that since like terms have exactly the same variables to exactly the same powers, all of the variables may be factored out. What remains behind in the parentheses will be the sum or difference of regular numbers, which of course may be added or subtracted. Looking back at the first part of the last example:

$$5x - 14x = (5 - 14)x = -9x$$

Unlike terms, on the other hand, have different variables or different amounts of the same variables. Therefore, you will NOT be able to factor out all of the variables, and the parentheses will include more than just numbers. Look at the second part of the last example:

$$3x^2 - 2x = (3x - 2)x$$

You may wish to keep this in mind when you are simplifying, but you should learn and use Tool 1 as it is quick and simple.

Example 12. *Simplify each of the following by combining like terms.*

1. $4a - 2ab + 5ab - 7b + b - 9a$
2. $x^2 + 2x + 7 + 14 - 10x - 5x^2$

3. $5m - 11n + 2n + 18m$
4. $8x^2 - 17x - 5 - 2x + 3 - 15x^2$

Solution 12. *Grouping together like terms and simplifying gives:*

1.
$$\begin{aligned} 4a - 2ab + 5ab - 7b + b - 9a &= (4a - 9a) + (5ab - 2ab) + (b - 7b) \\ &= \mathbf{-5a + 3ab - 6b} \end{aligned}$$

2.
$$\begin{aligned} x^2 + 2x + 7 + 14 - 10x - 5x^2 &= (x^2 - 5x^2) + (2x - 10x) + (7 + 14) \\ &= \mathbf{-4x^2 - 8x + 21} \end{aligned}$$

3.
$$\begin{aligned} 5m - 11n + 2n + 18m &= (5m + 18m) + (2n - 11n) \\ &= \mathbf{23m - 9n} \end{aligned}$$

4.
$$\begin{aligned} 8x^2 - 17x - 5 - 2x + 3 - 15x^2 &= (8x^2 - 15x^2) + (-17x - 2x) + (3 - 5) \\ &= \mathbf{-7x^2 - 19x - 2} \end{aligned}$$

Simplifying Algebraic Expressions

We have just seen that you may simplify an algebraic expression by combining like terms, but there is one other way in which we will want to simplify, and that is getting rid of parentheses. To do so, we will use the distributive property (see page 217 if you need to review). Don't forget that if there is no obvious number in front of the opening parenthesis, you may always write in a $1\cdot$ in front (or $-1\cdot$ if there is a "$-$" in front).

Algebraic Expressions Tool 2

If an algebraic expression contains parentheses which include addition and/or subtraction, you may get rid of them by applying the distributive property.

Example 13. *Simplify the following algebraic expressions.*

1. $3(4m-2)-2(3m+2n-5)$
2. $5(2x+y+1)-3(x-2y)$
3. $-2(3a-5)-(a+10)$
4. $-4(7x-3)-(x-5)$

Solution 13. *Getting rid of parentheses and combining like terms:*

1.

$$\begin{aligned} 3(4m-2)-2(3m+2n-5) &= 12m-6-6m-4n+10 \\ &= (12m-6m)-4n+(10-6) \\ &= \mathbf{6m-4n+4} \end{aligned}$$

2.

$$\begin{aligned} 5(2x+y+1)-3(x-2y) &= 10x+5y+5-3x+6y \\ &= (10x-3x)+(5y+6y)+5 \\ &= \mathbf{7x+11y+5} \end{aligned}$$

3.

$$\begin{aligned} -2(3a-5)-(a+10) &= -6a+10-a-10 \\ &= (-6a-a)+(10-10) \\ &= \mathbf{-7a} \end{aligned}$$

4.

$$\begin{aligned} -4(7x-3)-(x-5) &= -28x+12-x+5 \\ &= (-28x-x)+(12+5) \\ &= \mathbf{-29x+17} \end{aligned}$$

SECTION 2.1 EXERCISES

(Answers are found on page 383.)

For the following algebraic expressions, state whether each term is a constant term or a variable term, and if the term is a variable term, state its coefficient.

1. $3x^5 - 4x^4 + x^2$
2. $xy - 2yz$
3. $5 - x$
4. $13x^2 - 11x + 1$
5. $x + y + xy$
6. $8 + 3x^2y - 2xy^2$

Evaluate $-x - 3$, *when:*

7. $x = 3$
8. $x = -1$
9. $x = -2$
10. $x = 5$

Evaluate each of the following algebraic expressions when $x = -2$ *and* $y = -3$.

11. $|x| - |y|$
12. $xy + 5$
13. $3x - y$
14. $y - 5x$
15. $x^2 + y^2$
16. $|x| + y$
17. $\dfrac{x}{y}$
18. $x^2 - y^2$
19. $1 - x - y$
20. $x - y$

Simplify the following algebraic expressions by getting rid of any parentheses and combining like terms.

21. $2(3x - 1) + 5x - 2$
22. $2(m - n) + 3(2n - m)$
23. $5x - (3x + y - 2)$
24. $-(x - y) - 3y$
25. $7(x + 2) - 3(5x - 1)$
26. $5x^2 - 2x + 3 - 7x^2 + 3x - 13$
27. $2(4x + 5) - (2x + 7) - 6x$
28. $4(a - 3) - 3a + 2(5a + 6)$
29. $7(x - 3) + 3(8 - x)$
30. $xy - 9x + 2y + 3x - 5xy$
31. $4(3x - 2) - (5x + 6)$
32. $-(3x + 5) - (x - 2)$
33. $8m - 2n + 6m + 3n - 9m$
34. $6(2 - x) + 7(3 + x)$

2.2 Linear Equations

Introduction to equations

Definition: An equation is a mathematical statement equating two algebraic expressions.

You may think of an equation as a balance, like you may use in a chemistry or physics course.

Important Notation

Throughout this book, we will use the following notation:

LHS = left-hand side of the equation (or later, inequality),
RHS = right-hand side of the equation (or later, inequality).

Example 1. $4x - 3 = 12 - x$ *is an equation with:*
$LHS = 4x - 3$*, and*
$RHS = 12 - x$

The most interesting equations involve at least one variable (how exciting is it writing $2 = 2$ or $8 = 8$?). When the equation contains one or more variables, not every value of the variables will make the equation true.

Definition: A solution of an equation is any value for the variable(s) which make the equation true.

Example 2. *For the equation* $4x - 3 = 12 - x$*:*

1. *show that* $x = 1$ *is NOT a solution.*
2. *show that* $x = 3$ *IS a solution.*

Solution 2. *Looking at the LHS and RHS separately, we get:*

1. *When* $x = 1$*:*

$$\begin{aligned} LHS &= 4(1) - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

While:

$$\begin{aligned} RHS &= 12 - 1 \\ &= 11 \end{aligned}$$

Since $1 \neq 11$*,* $x = 1$ *is NOT a solution of the equation* $4x - 3 = 12 - x$*.*

2. *When* $x = 3$*:*

$$\begin{aligned} LHS &= 4(3) - 3 \\ &= 12 - 3 \\ &= 9 \end{aligned}$$

While:

$$\begin{aligned} RHS &= 12 - 3 \\ &= 9 \end{aligned}$$

Since $9 = 9$*,* $x = 3$ *IS a solution of the equation* $4x - 3 = 12 - x$*.*

Example 3. *For the equation* $5x + 3 = 15 - x$*:*

1. *show that* $x = -1$ *is NOT a solution.*
2. *show that* $x = 2$ *IS a solution.*

Solution 3. *Looking at the LHS and RHS separately, we get:*

1. *When* $x = -1$*:*

$$\begin{aligned} LHS &= 5(-1) + 3 \\ &= -5 + 3 \\ &= -2 \end{aligned}$$

While:

$$\begin{aligned} RHS &= 15 - (-1) \\ &= 16 \end{aligned}$$

Since $-2 \neq 16$*,* $x = -1$ *is NOT a solution of the equation* $5x + 3 = 15 - x$*.*

2. *When $x = 2$:*

$$\begin{aligned} LHS &= 5(2) + 3 \\ &= 10 + 3 \\ &= 13 \end{aligned}$$

While:

$$\begin{aligned} RHS &= 15 - 2 \\ &= 13 \end{aligned}$$

Since $13 = 13$, $x = 2$ *IS a solution of the equation* $5x + 3 = 15 - x$.

Unfortunately, trying to find solutions (even if you know how many total there should be) by guessing is not practical except in the simplest cases. What we would like to have is an algebraic way to *find* any and all solutions. To aid us, we have the two Algebraic Expressions Tools from last section, but we need more help.

Equation Tools

There are two equation tools which we will find useful.

Equations Tool 1

Additive Property of Equality:
For any real numbers a, b and c:

$$\text{If} \quad a = b \quad \text{then} \quad a + c = b + c.$$

Since c may be negative, this really means that you may *add* or *subtract* any number from both sides of an equation, and you will get an equivalent equation.

Definition: Two equations are said to be equivalent if they are both true or both false for a given value of the variable(s).

Equations Tool 2

Multiplicative Property of Equality:
For any real numbers a, b and c, *except* $c \neq 0$,

$$\text{If} \quad a = b \quad \text{then} \quad a \cdot c = b \cdot c.$$

Since c is any nonzero real number, we may change the multiplication to division by changing c to its reciprocal, so this really means you may multiply or divide both sides of an equation by any nonzero number and get an equivalent equation.

With these two equation tools, and the two algebraic expression tools from last section, we are ready to solve, well - not ALL equations - but at least a certain type of equation.

Linear Equations in One Variable

Definition: A linear equation in one variable is an equation which may be written in the form:

$$ax + b = c$$

for any real numbers a, b and c, *except* $a \neq 0$.

They are also called first-degree equations, because the variable is raised to the first power. For this whole section, whenever we say linear equation, we will mean linear equation in one variable. Before we illustrate how to solve them, we will first present an important fact.

Important Linear Equation Fact:

A linear equation in one variable always has *exactly one* solution.

So, how do we find this solution? You will be able to solve every linear equation if you follow these steps:

To solve a linear equation in one variable:

1. Remove all parentheses by using the distributive property.
2. Clear any fractions (or decimals) by multiplying every term by the LCD (or appropriate power of 10), if desired.
3. Combine any like terms on the LHS and RHS separately.
4. Move all x-terms to one side of the equation, and all constants to the other side by using the additive property of equality.
5. At this point, you should have a single x-term on one side, and a single constant on the other. If the coefficient of the x-term is not 1, divide both sides of the equation by this coefficient.
6. Check that you answer is a solution to the original equation as we did back in examples 2 and 3, if you have time.

Some comments on these rules:

- Step 2 is optional. If you are using a calculator, you may not mind having decimals or fractions, and if this is the case, you may leave them in. We will do an example of a linear equation involving decimals both by clearing them and by leaving them in. We will always clear fractions.
- If you do decide to get rid of any fractions or decimals, you may switch steps 1 and 2 if you wish. You will just have to be wary when any fraction or decimal is *inside* parentheses. To avoid complications, we will always follow the rules as listed.
- Step 2 uses the multiplicative property of equality, and it illustrates an important difference between expressions and equations. If you are given an *expression* with fractions, all you may do is make equivalent fractions to simplify. If, instead, you are given an *equation*, you may multiply the fractions by the LCD (just numerator, not numerator and denominator) to get rid of the fractions. Remember, an equation is

a balance scale; if we have five pounds on each side of the scale (so the scale is balanced), and we triple the weight on each side to get fifteen pounds on each, the scale stays balanced. Sure we changed the amounts on both sides, but we maintained the balance (made an equivalent equation).

- The first three steps are all "tidying up" steps. After you have done all three, your equation should have the form:

  $$Ax + B = Dx + C$$

  for some numbers (possibly zero) A, B, C and D. The possibly zero indicates that some of the terms may be missing, but the most you should have is a single x-term and a single constant on both sides. This seems like the simplest, most basic way of writing a linear equation, so you may wonder why our definition only had an x-term on the LHS. We shall see why later in this section.

- When doing step 4, some people prefer to always take the x-terms to the LHS, and the constants to the RHS, while others prefer to take the x-terms to whichever side makes the coefficient positive. We will do an example each way, but you should find and do what works best for you. After step 4, your equation should have the form:

  $$ax = b$$

  We switched from capital letters to small case as the coefficient on the x and the constant may have no relation to the capital A and B we used earlier.

- After step 5, your equation should have the form:

  $$x = c$$

  for some real number c. This is your solution.

- Step 6 is an optional check, but you should always check your answers provided you have the time. Of course, if you are running out of time on a test, this step will be skipped. We will check our first few examples, but we will then leave the checks to you for practice.

- There is one last equation fact which you may find useful. If you think of an equation as a balance scale, what happens if you turn the scale around? If it was balanced before, it still is, while if it was unbalanced before, it is still unbalanced. This means you may, at any time, flip the LHS and RHS without changing any signs.

Important General Equation Fact:

If $a = b$ then $b = a$.

- Finally, do NOT connect equivalent equations with an equals sign. Usually, it is best to write equivalent equations underneath one another, but if you need to write them side-by- side, use an arrow instead. For example,

  $$x + 2 = 5 \longrightarrow x = 3$$

  A string of equal signs (as we have used before) indicates equivalent expressions. Recall that with equivalent equations, the algebraic expressions are being changed; the equality is just being maintained.

Ok, let us see some examples.

Example 4. *Solve for x:* $3x - 7 = 5$

Solution 4. *No parentheses, no fractions, no decimals, and the LHS and RHS are both simplified, so we may skip the first three steps. Next we should get all the x-terms to the left - done already. With all the x-terms on the left, we want all the constants on the right. To move the 7 over, we will add 7 to both sides:*

$$\begin{aligned}
3x - 7 &= 5 \\
3x - 7 + 7 &= 5 + 7 \\
3x &= 12 \\
\frac{3x}{3} &= \frac{12}{3} \\
x &= 4
\end{aligned}$$

Therefore, the solution is $\mathbf{x = 4}$.
Check:

$$\begin{aligned} LHS &= 3(4) - 7 \\ &= 12 - 7 \\ &= 5 \end{aligned}$$

While:

$$RHS = 5$$

So, $LHS = RHS$ ✓

Example 5. *Solve* $6x + 3 = 7$

Solution 5. *No parentheses, no fractions, no decimals, and the LHS and RHS are both simplified, so we may skip the first three steps. Next we should get all the* x*-terms to the left - done already. With all the* x*-terms on the left, we want all the constants on the right. To move the* 3 *over, we will subtract* 3 *from both sides:*

$$\begin{aligned} 6x + 3 &= 7 \\ 6x + 3 - 3 &= 7 - 3 \\ 6x &= 4 \\ \frac{6x}{6} &= \frac{4}{6} \\ x &= \tfrac{2}{3} \end{aligned}$$

Therefore, the solution is $\mathbf{x = \frac{2}{3}}$.
Check:

$$\begin{aligned} LHS &= \tfrac{6}{1} \cdot \tfrac{2}{3} + 3 \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

While:

$$RHS = 7$$

So, $LHS = RHS$ ✓

As mentioned earlier, for the sake of space, we will stop checking our solutions, but you are encouraged to do so for practice. More examples:

Example 6. *Solve for x:* $2(x-4) = 5x + 7$

Solution 6. *There are parentheses, so starting with step* 1, *we get:*

$$\begin{aligned} 2(x-4) &= 5x+7 \\ 2x-8 &= 5x+7 \end{aligned}$$

Here is a situation where some people prefer to take the x-terms to the LHS (always) and some prefer to take the x-terms to the RHS (to get a positive coefficient). In this example, we will take them to the left, in the next example, we will go for the positive coefficient.

$$\begin{aligned} 2(x-4) &= 5x+7 \\ 2x-8 &= 5x+7 \\ 2x-5x-8 &= 5x-5x+7 \\ -3x-8 &= 7 \\ -3x-8+8 &= 7+8 \\ -3x &= 15 \\ \frac{-3x}{-3} &= \frac{15}{-3} \\ x &= -5 \end{aligned}$$

Therefore, the solution is $\mathbf{x = -5}$.

Example 7. *Solve* $7-(2x+4) = 3+2(x-5)$

Solution 7. *Starting with step* 1, *and taking the x-terms to the* *right this time:*

$$\begin{aligned}
7-(2x+4) &= 3+2(x-5)\\
7-2x-4 &= 3+2x-10\\
3-2x &= 2x-7\\
3-2x+2x &= 2x+2x-7\\
3 &= 4x-7\\
3+7 &= 4x-7+7\\
10 &= 4x\\
4x &= 10\\
\frac{4x}{4} &= \frac{10}{4}\\
x &= \tfrac{5}{2}
\end{aligned}$$

Therefore, the solution is $\mathbf{x=\frac{5}{2}}$ *or* $\mathbf{2\frac{1}{2}}$

Now for some examples involving fractions which we wish to clear.

Example 8. *Solve for x:* $\frac{1}{2}(x-3)=\frac{x}{4}+7$

Solution 8. *This is a situation where you may clear the fractions first if you like, but we will do our steps in order.*

$$\begin{aligned}
\frac{1}{2}(x-3) &= \frac{x}{4}+7\\
\frac{x}{2}-\frac{3}{2} &= \frac{x}{4}+7\\
\frac{4}{1}\cdot\frac{x}{2}-\frac{4}{1}\cdot\frac{3}{2} &= \frac{4}{1}\cdot\frac{x}{4}+4\cdot 7\\
2x-6 &= x+28\\
2x-x-6 &= x-x+28\\
x-6 &= 28\\
x-6+6 &= 28+6\\
x &= 34
\end{aligned}$$

Therefore, the solution is $\mathbf{x=34}$.

In the last example, notice that we multiplied every term by the LCD. When the term we were multiplying by was a fraction, we made the LCD a fraction by putting it over 1 (so we could multiply fraction times fraction). When the term was an integer, we left the LCD as an integer as well.

Example 9. *Solve* $\dfrac{4x}{3} - \dfrac{2}{3} = \dfrac{3x}{5} + 2$

Solution 9. *No parentheses, so our first step will be to clear the fractions.*

$$\begin{aligned}
\frac{4x}{3} - \frac{2}{3} &= \frac{3x}{5} + 2 \\
\frac{15}{1} \cdot \frac{4x}{3} - \frac{15}{1} \cdot \frac{2}{3} &= \frac{15}{1} \cdot \frac{3x}{5} + 15 \cdot 2 \\
20x - 10 &= 9x + 30 \\
20x - 9x - 10 &= 9x - 9x + 30 \\
11x - 10 + 10 &= 30 + 10 \\
11x &= 40 \\
\frac{11x}{11} &= \frac{40}{11} \\
x &= \frac{40}{11}
\end{aligned}$$

Therefore, the solution is $\mathbf{x} = \mathbf{\dfrac{40}{11}}$ *or* $\mathbf{3\dfrac{7}{11}}$

Example 10. *Solve for x:* $\dfrac{2x+3}{2} = \dfrac{x}{4} - 1$

Solution 10. *When you have a more complicated fraction like the one on the LHS here, you have several options. You could rewrite it as* $\frac{1}{2}(2x+3)$ *and handle it as we did in the last example, but even easier is to split the fraction. You already know the* *rule that when you are adding or subtracting fractions with the same denominator, you may make a single fraction by adding or subtracting the numerators and putting them over the common denominator. You may also use that rule in reverse. When there are plus or minus signs in the numerator of a fraction, you may break the fractions across the plus or minus signs by putting the separate terms all over the common denominator. This is what we will do here.*

$$\begin{aligned}
\frac{2x+3}{2} &= \frac{x}{4} - 1 \\
\frac{2x}{2} + \frac{3}{2} &= \frac{x}{4} - 1 \\
4 \cdot x + \frac{4}{1} \cdot \frac{3}{2} &= \frac{4}{1} \cdot \frac{x}{4} - 4 \cdot 1 \\
4x + 6 &= x - 4 \\
4x - x + 6 &= x - x - 4 \\
3x + 6 - 6 &= -4 - 6 \\
3x &= -10 \\
\frac{3x}{3} &= \frac{-10}{3} \\
x &= -\frac{10}{3}
\end{aligned}$$

Therefore, the solution is $\mathbf{x} = -\mathbf{\frac{10}{3}}$ *or* $\mathbf{-3\frac{1}{3}}$

If there are decimals in the equation, we mentioned before that it is your option whether to keep them or get rid of them. With a calculator, decimal operations are straight-forward, so we will do the next example both ways, and you may decide which method you prefer. As for technique, the rules state that you are to multiply by the appropriate power of ten, but what is the appropriate power of ten? Since you will have already done step 1 and removed all of the parentheses, it will be easy to distinguish the terms in the equation. Look at each term, and note what is the largest number of decimal places which appear. You will want to multiply every term by 1 followed by that many zeros. For example, if the number of decimal places in the terms of an equation are: none, one, two and two, the most is two, so you would multiply all the terms by 100. Let us see an example:

Example 11. *Solve* $0.2(x-3)+0.53(2x+1)=0.56$

Solution 11. *First, we will solve by clearing the decimals. After we distribute, you will notice that the most number of decimal places will be two, so we will multiply every term by* 100*.*

$$\begin{aligned}
0.2(x-3)+0.53(2x+1) &= 0.56\\
0.2x-0.6+1.06x+0.53 &= 0.56\\
100\cdot 0.2x-100\cdot 0.6+100\cdot 1.06x+100\cdot 0.53 &= 100\cdot 0.56\\
20x-60+106x+53 &= 56\\
126x-7 &= 56\\
126x-7+7 &= 56+7\\
126x &= 63\\
\frac{126x}{126} &= \frac{63}{126}\\
x &= \frac{1}{2}
\end{aligned}$$

Therefore, the solution is $\mathbf{x}=\mathbf{\frac{1}{2}}$

Repeating the last example:

Example 12. *Solve* $0.2(x-3)+0.53(2x+1)=0.56$

Solution 12. *Now, let us leave the decimals in and solve. You* *may need to use a calculator or scratch paper.*

$$\begin{aligned}
0.2(x-3)+0.53(2x+1) &= 0.56\\
0.2x-0.6+1.06x+0.53 &= 0.56\\
1.26x-0.07 &= 0.56\\
1.26x-0.07+0.07 &= 0.56+0.07\\
1.26x &= 0.63\\
\frac{1.26x}{1.26} &= \frac{0.63}{1.26}\\
x &= 0.5
\end{aligned}$$

Therefore, the solution is $\mathbf{x = 0.5}$

Notice that the last two examples are the same, we simply left the answer in the first proper form we came to (remember to NOT suggest a fraction like $\frac{0.63}{1.26}$ for an answer!).

Example 13. *Solve* $0.34(2x-5)+0.2=-0.4(x-3)$

Solution 13. *We will clear the decimals again here. After distributing, you will see that the most number of decimal places will be two, so we will again multiply every term by* 100.

$$\begin{aligned}
0.34(2x-5)+0.2 &= -0.4(x-3)\\
0.68x-1.7+0.2 &= -0.4x+1.2\\
100\cdot 0.68x-100\cdot 1.7+100\cdot 0.2 &= 100\cdot -0.4x+100\cdot 1.2\\
68x-170+20 &= -40x+120\\
68x-150 &= -40x+120\\
68x+40x-150 &= -40x+40x+120\\
108x-150+150 &= 120+150\\
108x &= 270\\
\frac{108x}{108} &= \frac{270}{108}\\
x &= 2.5
\end{aligned}$$

Therefore, the solution is $\mathbf{x = 2.5}$

They look like linear equations, but ....

Some equations are obviously not linear. They may include an x^2 term or a $\sqrt{x}$ term or etc. There are some equations, though, which at first glance *seem* to be linear, but, in reality, are not. For example:

Example 14. *Solve for x:* $3(x+5) - 8(x+3) = -5(x+2)$

Solution 14. *Following our rules, we get:*

$$\begin{aligned} 3(x+5) - 8(x+3) &= -5(x+2) \\ 3x + 15 - 8x - 24 &= -5x - 10 \\ -5x - 9 &= -5x - 10 \\ -5x + 5x - 9 &= -5x + 5x - 10 \\ -9 &= -10 \end{aligned}$$

This is obviously nonsense, so there is ***no solution****.*

When we tried to take all of the x-terms to the LHS, the x-terms not only canceled on the RHS, but on the LHS as well. If this ever happens, you are left with an equation involving two numbers. This kind of equation will either be nonsense, like our last example, or obviously true, like $5 = 5$. It is for this very reason that in the definition of a linear equation (page 244), there was only a single x-term in the equation, and it was stated that its coefficient could not be zero.

In the case where you get nonsense, there is **no solution**. This is annoying, in that if you wish to check your answer, you can only check your work.

In the case where you get something which is obviously true, your solution is **x may be any real number**. This type of equation is called an identity equation. This solution may be checked as all you need to do is show that *two* separate values for x ($x = 0$ and $x = 1$ are nice, easy choices) are both solutions. Let us see an example of this kind:

Example 15. *Solve for x:* $\frac{1}{3}(6x - 7) = \frac{1}{2}(4x - 5) + \frac{1}{6}$

Solution 15. *Again, this equation looks like a linear equation. We will distribute and clear fractions to start.*

$$\begin{aligned}
\frac{1}{3}(6x-7) &= \frac{1}{2}(4x-5)+\frac{1}{6} \\
2x-\frac{7}{3} &= 2x-\frac{5}{2}+\frac{1}{6} \\
6\cdot 2x-\frac{6}{1}\cdot\frac{7}{3} &= 6\cdot 2x-\frac{6}{1}\cdot\frac{5}{2}+\frac{6}{1}\cdot\frac{1}{6} \\
12x-14 &= 12x-15+1 \\
12x-14 &= 12x-14 \\
12x-12x-14 &= 12x-12x-14 \\
-14 &= -14
\end{aligned}$$

Obviously true, so the answer is ***x may be any real number****.*

Beware, though, it does not matter if the constants cancel; only the variable terms.

Example 16. *Solve* $5(x-3)=3(x-5)$

Solution 16. *Following our steps:*

$$\begin{aligned}
5(x-3) &= 3(x-5) \\
5x-15 &= 3x-15 \\
5x-3x-15 &= 3x-3x-15 \\
2x-15+15 &= -15+15 \\
2x &= 0 \\
\frac{2x}{2} &= \frac{0}{2} \\
x &= 0
\end{aligned}$$

Thus, $\mathbf{x=0}$

Zero is a perfectly fine solution, and this last example WAS a linear equation.

SECTION 2.2 EXERCISES
(Answers are found on page 384.)

1. A student correctly applies all properties to get to $2x = 4x$ in a problem. The students then divides both sides by x obtaining $2 = 4$ and writes no solution as his/her answer. Is this correct?

Solve the following equations.

2. $2x + 3 - x = 4x - 5$

3. $\dfrac{4(5-x)}{3} = -x$

4. $\dfrac{2}{3}x - \dfrac{5}{6}x - 3 = \dfrac{1}{2}x - 5$

5. $2x - 5 = 3(x + 6)$

6. $7 - 5(x - 3) = 2(4 - 3x)$

7. $4(x + 2) = 2(2x - 1) + 10$

8. $\dfrac{x+3}{2} = \dfrac{3x-1}{4}$

9. $0.03(2x + 7) = 0.16(5 + x) - 0.13$

10. $4(x + 3) = 6(2x + 8) - 4$

11. $5(x - 6) = 2(x + 4) + 3x$

12. $7x - 3(4x - 2) + 6x = 8 - 9(3x + 2)$

13. $-8(5x-3)+\frac{1}{2}(4x-3)=5$

14. $\frac{3}{5}\left(7-3x\right)+\frac{1}{2}\left(3x+7\right)=\frac{3}{4}\left(4x-2\right)$

15. $7x-4(2x+1)=5(x+3)-6x$

16. $3x-2\left[2x-4\left(x-5\right)\right]=7$

17. $x-\left[2-3\left(2x+4\right)\right]=61$

18. $\frac{2}{3}\left(3x-4\right)+\frac{1}{3}=\frac{1}{9}x$

19. $\frac{5}{6}x-\left(x-\frac{1}{2}\right)=\frac{1}{4}\left(x+1\right)$

20. $4\left(x+3\right)-8\left(x-3\right)=6\left(2x-1\right)-10$

21. $3(2x-5)-5(6-4x)=2+3(x-3)$

22. $\frac{3x-7}{4}=\frac{8-5x}{3}$

23. $\frac{2}{3}\left(x+12\right)+\frac{1}{6}\left(x+2\right)=2x-3$

24. $\frac{7-3x}{3}=\frac{2}{5}$

25. $9(2x+3)-3x=5-3(2x-5)$

26. $\frac{1}{4}(8x-1)+\frac{9}{4}=\frac{1}{2}(4x+5)-\frac{1}{2}$

27. $\frac{1}{4}\left(x-12\right)+\frac{1}{2}\left(x+2\right)=2x+4$

28. $\frac{2}{3}x - \left(x + \frac{1}{4}\right) = \frac{1}{12}\left(x + 4\right)$

29. $\frac{1}{2}(2x - 3) + 5 = \frac{1}{3}(3x + 4)$

30. $-2(2x - 4) - 8 = -3(4x + 4) - 1$

31. $\frac{3x - 5}{4} = -2$

32. $.12(x - 6) + .06x = .08x - .07(10)$

2.3 Problem Solving

Word problems have become well-known as students' worst nightmares. The problems often sound so complicated and difficult that you feel like you need not only a calculator, but a high-powered computer and rocket scientist as well. They are further complicated by the fact that there isn't really a single set of rules which always work. Instead, we will list some general strategies to keep in mind, and encourage you to get plenty of practice.

Strategies for Solving Word Problems

- Don't get intimidated when you first read the problem! Word problems may sound incredibly difficult, but the trick is to keep reading and rereading the problem, and break it up into manageable bits.
- Start at the end. Usually it is the last sentence where you find out what you are looking for. Once you have that figured out, try to put the rest of the problem in perspective.
- Decide if a picture or drawing will help. Sometimes they will, sometimes not.
- Define your variable. We highly recommend taking the time to write out what your variable stands for. This will make your work more understandable to your instructor and to yourself. You may always use x for a variable, or you may use something which indicates what you are looking for, e.g. a for the number of apples, c for a child's ticket price, etc.
- Recall that you have only learned how to solve linear equations in one variable! You may be asked to find two or three different things, but you don't yet know how to solve equations involving two or more variables. This means that one of the things you are looking for may be the variable; the others will have to be algebraic expressions involving this variable. Hint: the object which keeps getting referred back to is the one you want to make the basic variable.
- Make a mathematical equation. OK, this is the tricky one. If you complete this step properly, the rest of the problem will be all downhill. If you do this part wrong, you will get the wrong answer, and since you will usually check your answer with the equation, it will seem right.

- After you find an answer, reconsider your problem. Sometimes the answer will need units. Sometimes certain answers must be thrown out - you don't want the length of something to be a negative number, or you can't have a fractional number for a page number.

Before we do some problems, you will also need to know how some basic English words and phrases translate into mathematical operations. There are words we will consider "math" words when it comes to translation: sum (+), difference (-), product (·) and quotient (÷). These words not only indicate a specific operation, but they also indicate grouping. They keep statements in order. Consider the translation of these statements below, where we have let $x =$the number.

- The sum of 2 and a number: $(2 + x)$.
- The difference of a number and 3: $(x - 3)$.
- The product of 2 and a number: $(2 \cdot x)$.
- The quotient of a number and 7: $(x \div 7)$.

Notice that sum, difference, product and quotient are always followed by (something) <u>and</u> (something else). The number or variable before the "and" goes before the operation (although this only matters for subtraction and division).

There are other words we use in every day language that translate into mathematical expressions. These words take statements out of order when we express them in a mathematical sentence and they do not group.

- Three more than a number: $x + 3$
- Four less than a number: $x - 4$

Some other mathematically translated words are:

- Double a number: $2 \cdot x$

- Twice a number: $2 \cdot x$
- Triple a number: $3 \cdot x$
- Subtract 5 from a number: $x - 5$

Let us see some examples.

Example 1. *Triple a number is the same as double the sum of the number and three. Find the number.*

Solution 1. *First, what are we looking for? Rereading the problem, we see: blah, blah, blah,... Find the number. OK, so :*

$$\text{Let } x = \text{the number.}$$

The rest of the problem is giving information which helps us find the number. We need to change the English into math, so we will take it piece by piece.

$$\underbrace{\textit{Triple}}_{3\cdot}\quad\underbrace{\textit{a number}}_{x}\quad\underbrace{\textit{is the same as}}_{=}\quad\underbrace{\textit{double}}_{2\cdot}\quad\underbrace{\textit{the sum of the number and 3}}_{(x+3)}$$

Now all we need to do is solve this linear equation.

$$\begin{aligned} 3x &= 2(x+3) \\ 3x &= 2x+6 \\ 3x-2x &= 2x-2x+6 \\ x &= 6 \end{aligned}$$

Before we circle our answer, we should reconsider the problem. We were just looking for "a number", so there are not any units and no bad possible answers ("a number" may be positive, negative, zero, an integer, a fraction, etc.). Therefore, the only thing else you may wish to do is check the answer with the original equation, and check that the original equation matches the word problem properly. After doing so, we are confident:
The number is* 6*.

Example 2. *Twice the sum of a number and four is equal to the sum of triple the number and eight. Find the number.*

Solution 2. *Same as before, we start with:*

$$\text{Let } x = \text{the number.}$$

Now we need to translate the word problem into a mathematical equation.

$$\underbrace{\text{Twice}}_{2\cdot} \quad \underbrace{\text{the sum of a number and } 4}_{(x+4)} \quad \underbrace{\text{is equal to}}_{=} \quad \underbrace{\text{the sum of triple the number and } 8.}_{(3x+8)}$$

Now we solve:

$$\begin{aligned} 2(x+4) &= (3x+8) \\ 2x+8 &= 3x+8 \\ 2x-2x+8 &= 3x-2x+8 \\ 8 &= x+8 \\ 8-8 &= x+8-8 \\ 0 &= x \\ x &= 0 \end{aligned}$$

Again, we were just looking for "a number", so we do not need to add units or worry about inappropriate answers. Thus,
The number is 0.

Example 3. *A 22 foot board is cut into two pieces such that the second piece is two foot less than twice the length of the first. Find the lengths of the two pieces.*

Solution 3. *This problem is not just matter of translating English into math; instead of looking for just a number, we are looking for lengths of boards. First, whenever there are units involved, you want to make sure all the units match. Here, we are only referring to lengths, and the only unit mentioned is feet, so* *we are safe. Had the board length been given in feet, but the difference of the two pieces been in inches, we would have wanted to convert one measurement to the other units.*

Since the units all match, the next step is to define our variable. Looking at the last sentence, we see that we are looking for TWO things - the length of the first piece and the length of the second piece. We will let x represent one of the lengths, but which one? Rereading the problem, we see that the length of the second piece refers back to the length of the first (the second piece is two foot less than twice the length of the first). Therefore, it will be best to let the variable be the length of the first piece. So we have:

x	=	*the length of the first piece*
?	=	*the length of the second piece*

So what do we call the length of the second piece? Going back to the problem and translating the bit where the length of the second is referred back to the first, we get:

x	=	*the length of the first piece*
2x - 2	=	*the length of the second piece*

Now, we need to make an equation to solve. The only other piece of information in the problem is the total length of the board. If you stop and think about it (or perhaps make a picture), except for a tiny sliver which gets turned to sawdust by the saw, the length of the two pieces should add up to the original length. Thus,

$$\begin{aligned} x + (2x - 2) &= 22 \\ x + 2x - 2 &= 22 \\ 3x - 2 &= 22 \\ 3x - 2 + 2 &= 22 + 2 \\ 3x &= 24 \\ \frac{3x}{3} &= \frac{24}{3} \\ x &= 8 \end{aligned}$$

We wanted two lengths, so finding x is not enough. To find the length of the second piece, plug our solution for x into the algebraic expression which represents the length of the second *piece.*

$$2(8) - 2 = 16 - 2 = 14$$

Besides checking our work, we also need to remember that these are lengths measured in feet, and we need to make sure our answers are not inappropriate. A length of a board may be an integer or a fraction or decimal, but it MUST be a positive number. Both of our lengths are fine, so:
The length of the first piece is **8** ***feet, and the length of the second piece is*** **14** ***feet.***

Example 4. *A* 25 *foot board is cut into three pieces such that the first piece is two foot more than twice the length of the second piece, and the third piece is three foot longer than the second piece. Find the lengths of all three pieces.*

Solution 4. *The units all match (feet), and the lengths of the first and third pieces refer back to the second piece. Translating as before, we get:*

$$\begin{array}{ccc} 2x + 2 & = & \textit{the length of the first piece} \\ x & = & \textit{the length of the second piece} \\ x + 3 & = & \textit{the length of the third piece} \end{array}$$

Since the three lengths should add up to 25 *feet, we get:*

$$\begin{aligned} (2x+2) + x + (x+3) &= 25 \\ 2x + 2 + x + x + 3 &= 25 \\ 4x + 5 &= 25 \\ 4x + 5 - 5 &= 25 - 5 \\ 4x &= 20 \\ \frac{4x}{4} &= \frac{20}{4} \\ x &= 5 \end{aligned}$$

This is the length of the second piece. The first piece will be:

$$2(5) + 2 = 10 + 2 = 12$$

and the third:

$$5 + 3 = 8$$

All answers are appropriate for lengths of boards, so after adding on units and checking our work, we can say:
The first piece is* 12 *feet long, the second piece is* 5 *feet long, and the third piece is* 8 *feet long.

Example 5. *You open a book to a random spot, and notice that the left- and right-hand page numbers add up to* 329*. What are the two page numbers?*

Solution 5. *No units this time, but we are looking for two different page numbers. One may be the variable, but we need to find a relation between the two in order to make an algebraic expression for the other. You could use that the sum of the two pages is* 329*, but we are going to use this to make our equation; there must be another relation between the two. If you are having trouble figuring out what the other relation must be, try opening a book to random pages and see if you can spot a relationship between the left-hand and right-hand page numbers. Could one be* 125 *and the other* 367*, for example? No, the page numbers must be consecutive, as the right-hand page number is always one bigger than the left. Thus, we may define our variables as:*

$$\begin{aligned} x &= \text{the left-hand page number,} \\ x+1 &= \text{the right-hand page number.} \end{aligned}$$

Since the sum of the two pages is 329*, we get:*

$$\begin{aligned} x + (x+1) &= 329 \\ x + x + 1 &= 329 \\ 2x + 1 &= 329 \\ 2x + 1 - 1 &= 329 - 1 \\ 2x &= 328 \\ \frac{2x}{2} &= \frac{328}{2} \\ x &= 164 \end{aligned}$$

Substituting this into the expression for the right-hand page number gives:

$$164 + 1 = 165$$

No units to worry about, and a page number should be a positive integer, but both of our answers are, so:
The page numbers are 164 and 165.

Example 6. *The boss makes* \$40,000 *more than the secretary. Together, they make* \$130,000. *How much does each one earn?*

Solution 6. *Our units are dollars, and all the numbers match. We are looking for two salaries, but notice that the boss's salary refers back to the secretary's. Therefore, we will let the secretary's salary be the variable, and use the fact that the boss's salary is* \$40,000 *more to make an algebraic expression involving the variable:*

$$\begin{array}{ccc} x & = & \textit{the secretary's salary,} \\ x + \textit{40,000} & = & \textit{the boss's salary.} \end{array}$$

Since the sum of the their salaries is \$130,000*, we get:*

$$\begin{aligned} x + (x + 40,000) &= 130,000 \\ x + x + 40,000 &= 130,000 \\ 2x + 40,000 &= 130,000 \\ 2x + 40,000 - 40,000 &= 130,000 - 40,000 \\ 2x &= 90,000 \\ \frac{2x}{2} &= \frac{90,000}{2} \\ x &= 45,000 \end{aligned}$$

The boss's salary would then be:

$$45,000 + 40,000 = 85,000$$

Both answers are appropriate numbers for salaries (positive numbers), so adding on dollar signs gives:
The boss is making* \$85,000*, and the secretary is making* \$45,000*.

SECTION 2.3 EXERCISES

(Answers are found on page 385.)

Write an equation for the following problems and solve.

1. The difference of five and twice a number is the same as three times the difference of the number and five. Find the number.

2. The sum of two times a number and seven is equal to the difference of the number and six. Find the number.

3. The product of five and a number is the same as two times the difference of the number and three. Find the number.

4. Twice the difference of a number and three is equal to the difference of three and the number. Find the number.

5. A fifteen foot board is cut into two pieces such that the second piece is three foot less than twice the length of the first piece. Find the lengths of the two pieces.

6. A twelve foot board is cut into two pieces such that the first piece is five times the length of the second piece. Find the lengths of the two pieces.

7. A twelve foot board is cut into three pieces such that the first piece is one foot less than twice the length of the second piece, while the third piece is one foot more than three times the length of the second piece. Find the lengths of the three pieces.

8. A seventeen foot board is cut into three pieces such that the second piece is one foot less than twice the length of the first piece, while the third piece is two foot longer than the first piece. Find the lengths of the three pieces.

9. The cook makes $\$7,000$ more than the waitress. Together they make $\$57,000$. How much does each make?

10. The owner of a business earns twice as much as the secretary. Together they make $\$81,000$. How much does each make?

11. Opening a book to a random place, you notice that the left-hand and right-hand page numbers add up to 297. What are the page numbers?

12. After buying two CD's, you notice that the second CD was \$5 more than the first, while the two together cost \$29. How much did each CD cost?

13. You bowl two games for a total score of 241. If your score for the second game was nine points better, what did you bowl in each game?

14. After golfing eighteen holes, your total score is 86. If your score for the front nine was four points higher than your score for the back nine, how much did you shoot on each nine?

2.4 Proportions and Conversion Factors

Ratios

Definition: A ratio is a quotient of two quantities.

You may see a ratio written in several different ways. The ratio of a to b may be written:

$$\frac{a}{b}, \quad a \text{ to } b, \quad \text{or } a : b$$

We will prefer to write ratios as fractions, as fractions are easy to perform mathematical operations on. For example, we have seen how to compare two fractions (see page 118). This comes in handy as one common use of ratios is to determine the better buy between two products.

Example 1. *The local store sells salsa in two sizes: there is a* 15 *oz. jar for* \$1.35*, and a* 32 *oz. jar for* \$2.76*. Which jar is the better buy?*

Solution 1. *Right off the bat, we need to be a bit careful. We wish to make ratios to compare, but if we just throw numbers together, we might not be comparing the same things. To determine the better buy, we will make ratios of price per volume for the two jars. We will leave the units on for the first step, to make sure we have the fractions the right way round, but then ignore them temporarily while we do the math. Anyway, we wish to compare:*

$$\frac{\$1.35}{15\ oz} \quad \frac{\$2.76}{32\ oz}$$

We could compare as we did back on page 118, but now that we are using calculators, there is an easier way. Let us use the Fundament Principle of Fractions (version 2*) and make equivalent fractions by dividing each numerator and denominator by that fraction's denominator. So, for the* 15 *oz. jar:*

$$\begin{aligned}\frac{1.35}{15} &= \frac{1.35 \div 15}{15 \div 15} \\ &= \frac{0.09}{1}\end{aligned}$$

While, for the 32 oz. jar:

$$\begin{aligned}\frac{2.76}{32} &= \frac{2.76 \div 32}{32 \div 32} \\ &= \frac{0.08625}{1}\end{aligned}$$

This goes back to idea that it is easy to compare fractions if they have the same denominator, but instead of making equivalent fractions with a denominator of the product or LCD of the denominators, we make equivalent fractions with denominators equal to 1. This is easy when we use a calculator and are unafraid of ugly decimals.

So, how do we compare? Here we must ask ourselves what these fractions represent, and this means going back and examining the units. Our fractions here have units of price (dollars) per amount (ounces), so these fractions are telling us how much it costs for 1 ounce (this is called unit pricing). Hence, for the 15 ounce jar, we would be paying 9 cents per ounce, while for the 32 ounce jar we would be paying just under 9 cents per ounce. Thus, the better buy is ***the 32 ounce jar.***

Had we put ounces in the numerators and dollars in the denominators, then after making equivalent fractions with a 1 denominator, we would have seen how many ounces each would be giving for 1 dollar, and the better buy would be the larger amount. As a check, try making the fractions this way to see that the 32 ounce is the slightly better buy.

Example 2. *The same store sells coffee in two sizes: a 13 ounce jar for \$2.75, and a 34 ounce jar for \$7.34. Which is the better buy?*

Solution 2. *Again making equivalent fractions of price per amount, we get that the 13 ounce jar is:*

$$\begin{aligned}\frac{2.75}{13} &= \frac{1.35 \div 13}{13 \div 13} \\ &\approx \frac{0.211538}{1}\end{aligned}$$

While, for the 34 oz. jar:

$$\begin{aligned} \frac{7.34}{34} &= \frac{7.34 \div 34}{34 \div 34} \\ &\approx \frac{0.215882}{1} \end{aligned}$$

Therefore, the better buy is ***the*** *13* ***ounce jar.***

Proportions

Definition: A proportion is an equation which equates two ratios.

Therefore, a proportion is an equation of the form:

$$\frac{a}{b} = \frac{c}{d}$$

Notice that this is just a special case of an equation involving fractions, which we have already seen. In general, we learned to clear the fractions by multiplying every term by the LCD, but look at what happens here if we multiply both fractions by the common multiple of the product:

$$\begin{aligned} \frac{a}{b} &= \frac{c}{d} \\ \frac{b \cdot d}{1} \cdot \frac{a}{b} &= \frac{b \cdot d}{1} \cdot \frac{c}{d} \\ \frac{\cancel{b} \cdot d}{1} \cdot \frac{a}{\cancel{b}} &= \frac{b \cdot \cancel{d}}{1} \cdot \frac{c}{\cancel{d}} \\ d \cdot a &= b \cdot c \end{aligned}$$

Thus, once we cleared the fractions, we ended up with each numerator being multiplied by the other fraction's denominator being equal. Since this is always true (provided nothing is undefined!), this is considered a shortcut to solving proportions.

Cross-multiplying:

To simplify a proportion, you may multiply each numerator by the other denominator and set the products equal to each other.

Be careful, though, cross-multiplying only works on proportions; i.e. a single fraction equalling a single fraction. If there is another term, like a $+7$ say, you must go back to the general method of multiplying every term by the LCD.

Proportions are useful anytime you know you want two ratios to represent the same amount. We recommend that you keep the units in the first step to make sure you are setting up the ratios in the right way, drop the units to do the math, and then make sure your final answer has the right units again at the end. Let us see some examples:

Example 3. *You have just invented a new recipe to enter in the local cooking contest. As invented, the recipe calls for* 2 *cups of onions for three servings. Contest rules state that the recipe must make four servings. How many cups of onions must be used for the recipe to make four servings?*

Solution 3. *Just as we did when solving general word problems, we will start by defining our variable to be what we are looking for:*

$$x = \textit{cups of onions needed in recipe for four servings}$$

Using our units to guide us, we set up the proportion:

$$\frac{2 \textit{ cups onions}}{3 \textit{ servings}} = \frac{x \textit{ cups onions}}{4 \textit{ servings}}$$

Now we will cross-multiply and solve:

$$\begin{aligned} \frac{2}{3} &= \frac{x}{4} \\ 8 &= 3x \\ \frac{8}{3} &= \frac{3x}{3} \\ x &= \frac{8}{3} \end{aligned}$$

Therefore, to make four servings, the recipe requires $\mathbf{2\frac{2}{3}}$ *cups of onions.*

Example 4. *A recipe that makes* 3 *dozen peanut butter cookies requires* $1\frac{1}{4}$ *cups of flour. How much flour would you require for* 5 *dozen cookies?*

Solution 4. *First we define our variable:*

$$x = \textit{amount of flour for } 5 \textit{ dozen cookies}$$

Set up the proportion:

$$\frac{3 \textit{ dozen cookies}}{1\frac{1}{4} \textit{ cups of flour}} = \frac{5 \textit{ dozen cookies}}{x \textit{ cups of flour}}$$

Now let's cross-multiply and solve:

$$\begin{aligned} \frac{3}{\frac{5}{4}} &= \frac{5}{x} \\ 3x &= \frac{5}{1} \cdot \frac{5}{4} \\ 3x &= \frac{25}{4} \\ \frac{3x}{3} &= \frac{25}{4} \cdot \frac{1}{3} \\ x &= \frac{25}{12} \end{aligned}$$

Therefore, to make 5 *dozen cookies, you need to use* $\mathbf{2\frac{1}{12}}$ *cups of flour.*

Example 5. *The driver at pump* 2 *filled her tank with* 12.5 *gallons of regular unleaded gas for* \$33*. How many gallons of regular unleaded gas did the driver at pump* 6 *get if his total was* \$42.90*?*

Solution 5. *First we define our variable:*

$$x = \textit{gallons of gas bought by driver at pump } 6$$

Set up the proportion:

$$\frac{12.5 \textit{ gallons}}{33 \textit{ dollars}} = \frac{x \textit{ gallons}}{42.9 \textit{ dollars}}$$

Now let's cross-multiply and solve:

$$\begin{aligned}
\frac{12.5}{33} &= \frac{x}{42.9} \\
42.9 \cdot 12.5 &= 33x \\
536.25 &= 33x \\
\frac{536.25}{33} &= \frac{33x}{33} \\
x &= 16.25
\end{aligned}$$

Therefore, the driver at pump 6 *got* **16.25** *gallons of gas.*

Example 6. *Akron, OH is located approximately* 60 *miles west of the Pennsylvania border. An Ohio map represents this distance as* 3 *inches. On the same map, Youngstown, OH is represented by approximately* $\frac{11}{20}$ *of an inch from the Pennsylvania border. How far is Youngstown, OH actually from the Pennsylvania border?*

Solution 6. *First we define our variable:*

$$x = \textit{distance Youngstown is from PA}$$

Set up the proportion:

$$\frac{60 \textit{ miles}}{3 \textit{ inches}} = \frac{x \textit{ miles}}{\frac{11}{20} \textit{ inches}}$$

Now let's cross-multiply and solve:

$$\begin{aligned} \frac{60}{3} &= \frac{x}{\frac{11}{20}} \\ \frac{11}{20} \cdot \frac{60}{1} &= 3x \\ 33 &= 3x \\ \frac{33}{3} &= \frac{3x}{3} \\ x &= 11 \end{aligned}$$

Therefore, Youngstown is **11** *miles from the PA border.*

Conversion Factors

Another use for ratios is conversion factors. Conversion factors relate a measurement of one type to its equivalent measure of a second type (e.g. inches and feet). When we write conversion factors as fractions (ratios), the numerator and denominator are different ways of writing the same amount, and thus are just fancy ways of writing the number one. Therefore, any multiplication (or division) we do with conversion factors are simply changing units; the amount remains unchanged.

To start, we present two tables of conversion factors. The first deals with the English system for length, conversions between English and metric for length and weight (sort of), and time.

Length (English)	**Length or Weight**	**Time**
1 ft $=$ 12 in	1 km $\approx$ 0.62 mi	1 min $=$ 60 s
1 yd $=$ 3 ft	1 in $\approx$ 2.54 cm	1 hr $=$ 60 min
1 mi $=$ 5,280 ft	1 kg $\approx$ 2.2 lb	1 day $=$ 24 hr

The "weight" is actually incorrect in that kilograms (kg) are a measure of mass, while pounds (lb) are a measure of weight. For people on the surface of the Earth, mass and weight may be considered similar, and this conversion is often used.

The other table we will present is a table for some of the metric prefixes. The very nice property of the metric system is that all the conversion factors are powers of ten. For example, 100 centimeters (cm) = 1 meter (m) and $1,000$ m = 1 kilometer (km), as opposed to the English system where 12 inches (in) = 1 foot (ft) and $5,280$ ft = 1 mile (mi). The U.S. has been slow to adopt the metric system, but it is used often in the industrial and medical fields.

Prefix	**Symbol**	**Factor**
giga-	G	$1,000,000,000$
mega-	M	$1,000,000$
kilo-	k	$1,000$
centi-	c	0.01
milli-	m	0.001
micro-	μ or mc	0.000 001
nano-	n	0.000 000 001

The factor indicates how many of the base unit is equal to one of the prefixed unit. For the really small factors, we included a break after every three zeros to make the numbers easier to read. For example:

1. 1 megabyte = $1,000,000$ bytes.

2. 1 microgram = 0.000 001 grams.

Note that we included two symbols for the prefix "micro-": the Greek letter mu, μ, and the English letters, mc. Mu is the official symbol, but the mc is used as much as, if not more than, it - look at drug prescriptions, for example. Also, note that small case versus upper case letters matter; mg represents milligrams, while Mg represents megagrams.

In all of the following examples and exercises, we will use the conversions listed in the two tables. If we ever need another conversion factor, it will be listed in the example or exercise.

A typical conversion, no matter how it is done, will ultimately require multiplication and/or division, but it can be hard to guess (based only on intuition) which one and when. The following method for doing conversions is simple and takes away the need for guess work!

The Multiplication Method for Conversion Factors.

1. Make the number you are trying to convert into a fraction, including units.
2. Plan out what changes you need to make, and find the appropriate conversion factors.
3. Set up each conversion factor so that the undesired units will cancel away when multiplied by the conversion factor. You will be able to cancel units just as you cancel common factors.
4. Do the final calculation.

Example 7. *How much does a* 200 *pound man weight in kg? Round your final answer to the nearest tenth.*

Solution 7. *First, change the* 200 *pounds to a fraction.*

$$200\ lb = \frac{200\ lb}{1}$$

Notice that the denominator is unitless. Now let us plan our changes. Can we go directly from pounds to kilograms? Yes, we have a conversion factor which lets us do this, so we will need to only multiply by the one conversion factor. The big question is which way do we write the conversion factor. Well, the pound units, which we don't want, are in the numerator, so we need the *pound units of our conversion factor to be in the denominator so that the two units will cancel. Thus we get:*

$$\frac{200\ lb}{1} = \frac{200\ \cancel{lb}}{1} \cdot \frac{1\ kg}{2.2\ \cancel{lb}} \approx 90.9\ kg$$

Therefore, the man weighs **9.9 kg**.

Example 8. *Convert* 287 *micrograms to grams.*

Solution 8. *First, let us make* 287 *micrograms a fraction:*

$$287\ mcg = \frac{287\ mcg}{1}$$

Now we ask ourselves, can we go directly from mcg to g? Yes, this is a simple changing of a metric prefix to the base unit. When doing metric conversions, the fancy prefix always gets a 1*, while the base unit gets the factor listed on the table. So:*

$$\frac{287\ mcg}{1} = \frac{287\ \cancel{mcg}}{1} \cdot \frac{0.000\ 001\ g}{1\ \cancel{mcg}} = \mathbf{0.000287\ g}$$

One-step conversions like the last two examples may also be done using proportions. However, when there are two or more conversions to be accomplished, like the next three examples, it is generally best to use this multiplication method.

Example 9. *Convert* 0.0000813 *kilometers to millimeters.*

Solution 9. *This conversion could be done in one step if you are comfortable enough with the metric prefixes to do so, but we will just use the conversion factors listed on the table, so we wish to convert:*

$$kilometers \longrightarrow meters \longrightarrow millimeters$$

Each arrow represents a conversion factor. So, we get:

$$\begin{aligned} 0.0000813\ km &= \frac{0.0000813\ \cancel{km}}{1} \cdot \frac{1{,}000\ \cancel{m}}{1\ \cancel{km}} \cdot \frac{1\ mm}{0.001\ \cancel{m}} \\ &= \mathbf{81.3\ mm} \end{aligned}$$

Example 10. *The height of Mount Everest is* $8,848$ *meters. Find the height of Mount Everest to the nearest foot.*

Solution 10. *This is a conversion between the metric system and the English system. Looking at our tables, we see that we don't have a direct meter to feet conversion, so we will have to use one of the two which are listed. Let us choose to use the* 1 *kilometer* ≈ 0.62 *mile conversion. Then, we need to convert:*

$$m \longrightarrow km \longrightarrow mi \longrightarrow ft$$

We get:

$$\begin{aligned} 8{,}848\ m &= \frac{8{,}848\ \cancel{m}}{1} \cdot \frac{1\ \cancel{km}}{1{,}000\ \cancel{m}} \cdot \frac{0.62\ \cancel{mi}}{1\ \cancel{km}} \cdot \frac{5{,}280\ ft}{1\ \cancel{mi}} \\ &= \mathbf{28{,}965\ ft} \end{aligned}$$

Example 11. *The speed of sound in air is approximately* 344 *m/s. Find the speed of sound to the nearest mi/hr.*

Solution 11. *Units written with a slash "/" or with the word "per" in them mean that the first part of the unit is in the numerator, while the other part is in the denominator. In some cases, like this one, we may wish to convert both units. In that case, figure out which conversions need to be done on the numerator and which need to be done on the denominator. Here, for our numerator, we wish to change:*

$$m \longrightarrow km \longrightarrow mi$$

while for the denominator:

$$s \longrightarrow min \longrightarrow hr$$

Therefore, we will need a total of four conversions. We will start with the numerator:

$$344\, m/s \quad = \quad \frac{344 \ \cancel{m}}{1\ s} \cdot \frac{1 \ \cancel{km}}{1,000 \ \cancel{m}} \cdot \frac{0.62\, mi}{1 \ \cancel{km}}$$

If we would stop here, the units would be mi/s, which is not what we want. Thus, we now need to convert the denominator.

$$\begin{aligned} 344\, m/s \quad &= \quad \frac{344 \ \cancel{m}}{1\ s} \cdot \frac{1 \ \cancel{km}}{1,000 \ \cancel{m}} \cdot \frac{0.62\, mi}{1 \ \cancel{km}} \\ &= \quad \frac{344 \ \cancel{m}}{1 \ \cancel{s}} \cdot \frac{1 \ \cancel{km}}{1,000 \ \cancel{m}} \cdot \frac{0.62\, mi}{1 \ \cancel{km}} \cdot \frac{60 \ \cancel{s}}{1 \ \cancel{min}} \cdot \frac{60 \ \cancel{min}}{1\, hr} \\ &= \quad \mathbf{768\ mi/hr} \end{aligned}$$

Occasionally you may see units with powers. For example, a unit of area is ft^2, or a unit of volume is m^3. To do the conversions in these cases, you have two options:

1. write out the units as products rather than powers and apply the conversion factor as many times as is necessary to cancel all the undesired units,
2. raise your conversion factor to the appropriate power to get a new conversion factor for the higher power units.

We will do the following two examples both ways to illustrate:

Example 12. *Convert* $587\ in^2$ *to* ft^2*. Round your answer to the nearest hundredth.*

Solution 12. *First we will convert by changing* in^2 *to* $in \cdot in$*.*

$$\begin{aligned} 587\, in^2 \quad &= \quad \frac{587 \ \cancel{in} \cdot \ \cancel{in}}{1} \cdot \frac{1\, ft}{12 \ \cancel{in}} \cdot \frac{1\, ft}{12 \ \cancel{in}} \\ &= \quad \mathbf{4.08\ ft^2} \end{aligned}$$

OR:

We notice that our units are squared, so we will square our conversion factor:

$$12\ in = 1\ ft \implies (12\ in)^2 = (1\ ft)^2 \implies 144\ in^2 = 1\ ft^2$$

Thus, we have a new conversion factor for the squared units:

$$\begin{aligned} 587\ in^2 &= \frac{587\ \cancel{in}^2}{1} \cdot \frac{1\ ft^2}{144\ \cancel{in}^2} \\ &= \mathbf{4.08\ ft^2} \end{aligned}$$

Example 13. *Convert* $0.045\ m^3$ *to* cm^3.

Solution 13. *Using the first method of changing the power to a product:*

$$\begin{aligned} 0.045\ m^3 &= \frac{0.045\ \cancel{m}\cdot \cancel{m}\cdot \cancel{m}}{1} \cdot \frac{1\ cm}{0.01\ \cancel{m}} \cdot \frac{1\ cm}{0.01\ \cancel{m}} \cdot \frac{1\ cm}{0.01\ \cancel{m}} \\ &= \mathbf{45,000\ cm^3} \end{aligned}$$

OR:

We notice that our units are cubed, so we will cube our conversion factor:

$$1\ cm = 0.01\ m \implies (1\ cm)^3 = (0.01\ m)^3 \implies 1\ cm^3 = 0.000\,001\ m^3$$

Thus, we have a new conversion factor for the squared units:

$$\begin{aligned} 0.045\ m^3 &= \frac{0.045\ \cancel{m}^3}{1} \cdot \frac{1\ cm^3}{0.000\,001\ \cancel{m}^3} \\ &= \mathbf{45,000\ cm^3} \end{aligned}$$

SECTION 2.4 EXERCISES

(Answers are found on page 386.)

Find the better buy.

1. Yogurt: 8 ounces for \$1.29 or 14 ounces for \$2.39.
2. Cereal: 45 ounces for \$3.49 or 64 ounces for \$4.69.
3. Eggs: 8 for \$0.59 or 12 for \$0.85.
4. Walnuts: 8 ounces for \$3.99 or 12 ounces for \$6.09.

All of the following may be solved using proportions or conversion factors.

5. A recipe for 36 cookies requires 2 cups of chocolate chips. How many cups are requires for 90 cookies?
6. If 7 gallons of premium gasoline costs \$22.50, how much would it cost to fill up with 16 gallons of premium gasoline? Round to two decimal places.
7. If it takes 15 minutes to burn 6 CD's, how long will it take to burn 10 CD's?
8. A recipe which makes 4 dozen oatmeal raisin cookies requires $1\dfrac{1}{2}$ cups of flour. How much flour is required to make 5 dozen cookies?
9. The architect has drawn up the blueprints for your new house. In the drawing, $\dfrac{1}{2}$ inch represents 3 feet. The length of the house is 42 feet. How many inches would this represent on the blueprint?
10. A family purchases 2 gallons of milk every 3 weeks. At this rate, how many gallons of milk will the family purchase in a year (52 weeks)? Round to the nearest whole gallon.
11. The weight of an object on the moon is proportional to its weight on Earth. If a 144 pound woman weighs 24 pounds on the moon, what would an 84 pound boy weigh on the moon?
12. The property tax for a home with an assessed value of $\$145,000$ is $\$1,373.88$. What is the property tax for a home with an assessed value of $\$124,000$? Round to the nearest cent.

13. Convert $8,365\ mcm$ to cm.
14. Convert $2.3\ mi$ to yd.
15. Convert $2000\ m$ to mi.
16. Convert $90\ km/hr$ to mi/hr.
17. Convert $200\ m/s$ to mi/hr.
18. Convert $45\ kg$ to lb.
19. Convert $0.15\ m^2$ to mm^2.
20. Convert $36,000\ in^3$ to yd^3. Round to the nearest hundredth.

2.5 Linear Inequalities

Definition: A linear inequality in one variable is an inequality that can be written in one of the following forms:

$$ax + b < c, \quad ax + b \leq c, \quad ax + b > c, \quad \text{or} \quad ax + b \geq c$$

where a, b, and c are real numbers with $a \neq 0$. A solution of an inequality in one variable is a number that makes the inequality true when all the occurrences of the variable in the inequality are replaced with that number. To solve an inequality means to find *all* of its solutions.

The solution of a linear inequality can be written in two different ways: inequality or interval notation. Before we start solving linear inequalities, we need to discuss these two different ways.

An *inequality* uses the inequality signs along with a number or numbers which represent bounds for the solution. For example, $x \leq 3$ represents all of the real numbers less than or equal to 3. Here, 3 is considered an upper bound of the solution.

Interval Notation may also be used to describe a set of numbers. For interval notation, the bounds become endpoints. The interval always consists of an opening parenthesis or bracket, the lower endpoint (lower bound), a comma, the upper endpoint (upper bound) and a closing parenthesis or bracket. A parenthesis is used to denote that the endpoint is not included in the interval; whereas, a bracket is used to denote that the endpoint is included in the interval. It is important that the smallest endpoint be written on the left and the largest endpoint be written on the right. If there is no smallest endpoint of the interval, we use the symbol $-\infty$ (negative infinity). The symbol $-\infty$ is not a number, but indicates that the interval continues indefinitely through all the negative real numbers. Likewise, if there is no largest endpoint of the interval, we use the symbol ∞ (infinity). Once again, ∞ is not a number, but is used to indicate that the interval continues indefinitely through all the positive real numbers.

Example 1. *Rewrite the following inequalities in interval notation.*

1. $x \geq 7$
2. $-4 \leq x < 9$
3. $x < 14$
4. $\frac{4}{5} < x \leq \frac{9}{2}$

Solution 1. *We get:*

1. *Notice that there is no largest endpoint and also that* 7 *is included in the interval. Hence, the interval notation is* $\mathbf{[7, \infty)}$
2. *Notice that* -4 *is included in the interval, while* 9 *is not included in the interval. Therefore, the interval notation is* $\mathbf{[-4, 9)}$
3. *Here there is no smallest endpoint, and* 14 *is not included. So, the interval notation is* $\mathbf{(-\infty, 14)}$
4. *Finally,* $\frac{4}{5}$ *is the smallest endpoint but it is not included in the interval. The largest endpoint is* $\frac{9}{2}$ *and it is included in the interval. Thus, the interval notation is* $\mathbf{\left(\frac{4}{5}, \frac{9}{2}\right]}$

The following chart summarizes various inequalities in interval notation.

Inequality	Interval Notation
$x < b$	$(-\infty, b)$
$x \leq b$	$(-\infty, b]$
$x > a$	(a, ∞)
$x \geq a$	$[a, \infty)$
$a < x < b$	(a, b)
$a \leq x < b$	$[a, b)$
$a < x \leq b$	$(a, b]$
$a \leq x \leq b$	$[a, b]$

Solving a linear inequality in one variable is very similar to solving a linear equation in one variable with two exceptions. The Algebraic Expression Tools don't change, as they only replace expressions with equivalent expressions, so it doesn't matter if the expressions are in an equation or in an inequality. The Equation Tools, however, must be replaced with Inequality Tools.

The Inequality Tools

Inequality Tool 1

Additive Property of Inequality:
For any real numbers a, b and c,

$$\text{If} \quad a < b \quad \text{then} \quad a + c < b + c.$$

Inequality Tool 2

Multiplicative Property of Inequality:
For any real numbers a, b and c, *except* $c \neq 0$.

1. If $a < b$ and $c > 0$ then $ac < bc$.

2. If $a < b$ and $c < 0$ then $ac > bc$.

Before we comment, we also wish to convert the Important General Equation Fact to an Important General Inequality Fact.

Important General Inequality Fact:

$$\text{If} \quad a < b \quad \text{then} \quad b > a.$$

Some comments:

- In both of the Tools and the Fact, we started with a $<$ sign, but these rules may be used with $>$, $\leq$ and $\geq$ as well. The trick is to notice whether the rule indicates that the sign needs to be flipped. Notice that the additive property shows that you may add or subtract any number to both sides of an inequality, and the inequality sign stays the same. The multiplicative property shows that if you wish to multiply or divide both sides of an inequality by a positive number, the sign stays the same, BUT if you choose to multiply or divide both sides of an inequality by a negative number, the inequality sign flips. The fact indicates that if you ever wish to switch the left side for the right side, of course you flip the sign in this case as well.

- Other than the two cases where you may have to flip the sign, we will solve a linear inequality by following the exact same rules as we used to solve a linear equation. In fact, we can say that the only step which will differ is step 5. Here is where you need to be careful when the coefficient of x is a negative number, to remember to flip the sign.

Solving Linear Inequalities

As mentioned, we will be using the same steps to solve linear inequalities as we did to solve linear equations (see page 245). We only have to worry about flipping the sign if we decide to switch the LHS and RHS, or when we get to step 5 and the x-coefficient is negative.

Let us see some examples. Note that for inequalities, we will always wish for your final answer to be in interval notation.

Example 2. *Solve for x:* $3(2x-4) < 5(3x-1)$

Solution 2. *Following the rules for solving a linear equation (but* *keeping in mind the two cases where we need to flip the sign):*

$$\begin{aligned}
3(2x-4) &< 5(3x-1) \\
6x-12 &< 15x-5 \\
6x-12-15x &< 15x-15x-5 \\
-9x-12 &< -5 \\
-9x-12+12 &< -5+12 \\
-9x &< 7 \\
\frac{-9x}{-9} &> \frac{7}{-9} \\
x &> -\frac{7}{9}
\end{aligned}$$

Therefore, the solution set is $\left(-\mathbf{\frac{7}{9}}, \infty\right)$

In the last example, note that we did need to switch the sign when we divided both sides of the equation by -9.

By the way, if you wish to check your solution, you may of course check your work, but you can also do a rough check by picking a value for x in the interval and making sure this value of x makes the initial inequality true, and picking a value for x outside the interval and making sure this value of x makes the initial inequality false. For example, if you try $x = 0$ in the original inequality in the last example, you will get $-12 < -5$ which is true ✓; whereas if try $x = -1$ in the original inequality, you will get $-18 < -20$ which is false ✓. This is only a rough check as notice if we made a minor mistake and the answer should have been:

$$\left(-\frac{5}{9}, \infty\right)$$

instead, those same two check values would still give the same results since 0 is in this new interval and -1 is not. Still, this rough check is usually easy to do (you will often be able to choose integers for the test values), and it will catch some mistakes like forgetting to flip your inequality sign.

Example 3. *Solve* $5 - 2(3x+1) \geq 4x + 3$

Solution 3. *Following the rules for solving a linear equation (but keeping in mind the two cases where we need to flip the sign):*

$$\begin{aligned}
5-2(3x+1) &\geq 4x+3 \\
5-6x-2 &\geq 4x+3 \\
-6x+3 &\geq 4x+3 \\
-6x+6x+3 &\geq 4x+6x+3 \\
3-3 &\geq 10x+3-3 \\
0 &\geq 10x \\
10x &\leq 0 \\
\frac{10x}{10} &\leq \frac{0}{10} \\
x &\leq 0
\end{aligned}$$

Therefore, the solution set is $(-\infty, \mathbf{0}]$

Here, we prefer to have the x-terms on the left eventually, but we had originally taken the x-terms to the right side in order to keep the coefficient on the x-term positive. Hence, at one point we simply switched the sides of the inequality (along with the inequality sign).

Example 4. *Solve* $\frac{1}{3}(x+5) \leq \frac{x}{2}+1$

Solution 4. *We see fractions which we want to get rid of, but* *first, we will distribute:*

$$
\begin{aligned}
\frac{x}{3}+\frac{5}{3} &\leq \frac{x}{2}+1 \\
\frac{6}{1}\cdot\frac{x}{3}+\frac{6}{1}\cdot\frac{5}{3} &= \frac{6}{1}\cdot\frac{x}{2}+6\cdot 1 \\
2x+10 &\leq 3x+6 \\
2x-3x+10 &\leq 3x-3x+6 \\
-x+10-10 &\leq 6-10 \\
-x &\leq -4 \\
\frac{-x}{-1} &\geq \frac{-4}{-1} \\
x &\geq 4
\end{aligned}
$$

So, the solution set is $[\mathbf{4},\infty)$.

Compound Linear Inequalities

Not every linear inequality has a left and right hand side only. In fact, some consist of three parts – left hand side, middle, and right hand side (we saw a couple of these when we introduced interval notation). The example, $-2 < x \leq 5$, is considered to be a compound inequality (also called a double inequality) because it is the combination of two inequalities. Namely, $-2 < x \leq 5$ says that $-2 < x$ *AND* $x \leq 5$ *at the same time*. To solve a three part, or compound inequality, we need to isolate the variable in the middle part (as opposed to the left-hand side as we did before). The Algebraic Expression Tools don't change, but the additive and multiplicative properties of inequalities need to be generalized for compound inequalities. Instead of adding, subtracting, multiplying or dividing both sides by a number, you now must do the same operation on all three parts.

Example 5. *Solve* $-9 < 3\,(2x+1) < 21$

Solution 5. *Following our rules:*

$$\begin{array}{rcccl} -9 & < & 3\,(2x+1) & < & 21 \\ -9 & < & 6x+3 & < & 21 \\ -9-3 & < & 6x+3-3 & < & 21-3 \\ -12 & < & 6x & < & 18 \\ \\ \dfrac{-12}{6} & < & \dfrac{6x}{6} & < & \dfrac{18}{6} \\ \\ -2 & < & x & < & 3 \end{array}$$

So, the solution set is $(-\mathbf{2},\mathbf{3})$

It is standard to write compound inequalities as "lower bound $< x <$ upper bound", so that it matches interval notation. If the inequality signs go the other direction, you need to flip the inequality around (remember to flip inequality signs as well). For example $5 > x > -3 \longrightarrow -3 < x < 5$. You should NEVER write a compound inequality with a mix of signs. For example, $5 > x < 8$ is nonsense.

Example 6. *Solve* $-5 \leq 2\,(4-3x) < 16$

Solution 6. *Following our rules:*

$$\begin{array}{rcccl} -5 & \leq & 2\,(4-3x) & < & 16 \\ -5 & \leq & 8-6x & < & 16 \\ -5-8 & \leq & 8-8-6x & < & 16-8 \\ -13 & \leq & -6x & < & 8 \\ \\ \dfrac{-13}{-6} & \geq & \dfrac{-6x}{-6} & > & \dfrac{8}{-6} \\ \\ \dfrac{13}{6} & \geq & x & > & -\dfrac{4}{3} \\ \\ -\dfrac{4}{3} & < & x & \leq & \dfrac{13}{6} \end{array}$$

So, the solution set is $\left(-\mathbf{\frac{4}{3}},\mathbf{\frac{13}{6}}\right]$

In three part inequalities, we can still eliminate fractions by multiplying every term in all three parts by the LCD.

Example 7. *Solve* $3 \leq \dfrac{2x+1}{3} \leq 9$

Solution 7. *We will start with breaking the fraction up across its plus sign:*

$$\begin{array}{rcccl}
3 & \leq & \dfrac{2x+1}{3} & \leq & 9 \\[2ex]
3 & \leq & \dfrac{2x}{3} + \dfrac{1}{3} & \leq & 9 \\[2ex]
3 \cdot 3 & \leq & \dfrac{3}{1} \cdot \dfrac{2x}{3} + \dfrac{3}{1} \cdot \dfrac{1}{3} & \leq & 3 \cdot 9 \\[2ex]
9 & \leq & 2x+1 & \leq & 27 \\
9-1 & \leq & 2x+1-1 & \leq & 27-1 \\
8 & \leq & 2x & \leq & 26 \\[2ex]
\dfrac{8}{2} & \leq & \dfrac{2x}{2} & \leq & \dfrac{26}{2} \\[2ex]
4 & \leq & x & \leq & 13
\end{array}$$

So the solution set is $[\mathbf{4}, \mathbf{13}]$

Applications

Linear inequalities also arise in application problems.

Example 8. *Jill scored an 88, 94, and 82 on her first three math tests. An average of at least 90 will earn her an A in the course. What does Jill need to score on her fourth and final test to earn an A in the course?*

Solution 8. *Let $x =$ Jill's fourth test score.*

Then since her average needs to be at least a 90, we have

$$\begin{aligned} 90 &\leq \frac{88+94+82+x}{4} \\ 4(90) &\leq 4\left(\frac{88+94+82+x}{4}\right) && \textit{multiply by } 4 \\ 360 &\leq 88+94+82+x \\ 360 &\leq 264+x && \textit{simplify} \\ 360-264 &\leq 264+x-264 && \textit{subtract } 264 \\ 96 &\leq x \\ &[96, \infty) \end{aligned}$$

Therefore, in order for Jill to receive an A in the course, she must score at least a 96 on the fourth test.

Example 9. *The local grocery store rents carpet cleaners for* \$25 *plus* \$5.50 *per hour. If Chrissy can spend no more than* \$80 *to clean her carpets, how long can she rent the carpet cleaner? (Assume all portions of an hour are prorated.)*

Solution 9. *Let $x =$ the number of hours carpet cleaner is rented.*

This means that her cost for renting the carpet cleaner is given by $25 + 5.5x$. Therefore,

$$\begin{aligned} 25+5.50x &\leq 80 \\ 25+5.50x-25 &\leq 80-25 && \textit{subtract } 25 \\ 5.50x &\leq 55 \\ \frac{5.50x}{5.50} &\leq \frac{55}{5.50} && \textit{divide by } 5.50 \\ x &\leq 10 \end{aligned}$$

Hence, she can rent the carpet cleaner for a maximum of 10 hours.

SECTION 2.5 EXERCISES

(Answers are found on page 386.)

Rewrite the following inequalities in interval notation.

1. $-3 \leq x$
2. $2 < x \leq 8$
3. $x < 7$
4. $3 \leq x \leq \frac{11}{2}$
5. $x \geq \frac{3}{4}$
6. $14 \geq x$

Solve each linear inequality. Give the solution set in interval notation.

7. $3x - 7 \leq 5x + 3$
8. $7x + 8 \geq 4 - 3x$
9. $\dfrac{1}{2}x + 3 \leq \dfrac{2}{3}x - 4$
10. $4(x-3) + 5 \leq 9 + 2x$
11. $3 + 2x > 5 + 3(2x - 7)$
12. $2(5x - 4) \leq 3(6x + 2)$
13. $7(2x - 1) + 4 < 3(4x + 3)$
14. $4x + 6(5 - 2x) \geq -4(3 - 5x) + 7$
15. $\dfrac{7}{5}(10x - 1) \leq \dfrac{2}{3}(6x + 5)$
16. $-4(2x - 3) + 7 > 3(2x - 5) - 9(3x - 2)$
17. $-\dfrac{2}{3}(x - 7) + \dfrac{1}{6}(3x - 4) \geq -3$

18. $-2(4+7x)-7+6x \leq 5-9x$

19. $-(9+2x)-5+4x \geq 4+5x$

20. $-\frac{1}{4}(x+6)+\frac{3}{2}(2x-5)<10$

21. $4 \leq 3x-5 \leq 7$

22. $-3<5(4-3x) \leq 12$

23. $-7 \leq 3(2x-5)<10$

24. $-2 \leq \frac{15-6x}{5}<5$

25. $-2 \leq \frac{4+5x}{7}<6$

26. $\frac{2}{3} \leq \frac{4}{5}(2x-3)<5$

27. The Jones Family is renting a van for their 7 day family vacation. Auto Depot rents minivans for \$35 per day and \$0.02 per mile. Car Zone rents the same minivan for \$43 per day with no additional charge for mileage. How many miles would the Jones Family need to drive for the the Car Zone rental to be the cheapest?

28. You are offered two different real estate positions. Job 1 pays \$250 per week plus a 6% commission on sales. Job 2 pays 9% commission only. What do the average weekly sales need to be in order for Job 1 to be the better deal?

29. The local parking garage calculates the cost for parking using the formula $c = 0.50 + 1.25h$ where c is the cost in dollars and h is the number of hours parked rounded to the next highest integer.

    (a) When will the cost be \$5.50?

    (b) When will the cost be more than \$12.00?

    (c) When will the cost be less than \$9.00?

Chapter 3

Graphing and Lines

3.1 Cartesian Coordinate System

In a previous chapter, we solved linear equations in one variable. Recall that a solution of an equation in one variable is a number that makes the equation true when the variable is replaced by that number. For an equation having two variables, two numbers will be required for a solution, one for each variable.

Example 1. *(a) Determine whether $x = 2$ and $y = 3$ is a solution of the equation $3x + y = 9$.*

(b) Determine whether $x = 3$ and $y = 2$ is a solution of the equation $3x + y = 9$.

Solution 1. *(a) Substituting $x = 2$ and $y = 3$ into the equation we have*

$$\begin{aligned} 3(2) + (3) &\stackrel{?}{=} 9 \\ 6 + 3 &\stackrel{?}{=} 9 \\ 9 &= 9\checkmark \end{aligned}$$

Since the substitution results in a true statement, $x = 2$ and $y = 3$ is a solution to $3x + y = 9$.

(b) Substituting $x = 3$ and $y = 2$ into the equation we have

$$\begin{aligned} 3(3) + 2 &\stackrel{?}{=} 9 \\ 9 + 2 &\stackrel{?}{=} 9 \\ 11 &\neq 9 \end{aligned}$$

*Hence, $x = 3$ and $y = 2$ is **not** a solution to $3x + y = 9$.*

We see from Example 1 that in a solution of an equation with two variables, it is important to be careful about which number replaces which variable. So we say that a **solution of an equation in two variables** is a pair of numbers, one number assigned to each variable, such that the equation is true when each occurrence of a variable in the equation is replaced by its assigned number. We can keep things organized by establishing an order for the variables. This allows us to abbreviate a solution of an equation in two variables as an **ordered pair** of numbers, that is, two numbers in a particular order, written (a, b), where a is the first number and b is the second. Using ordered pairs, we know from Example 1 that $(2, 3)$ is a solution of $3x + y = 9$ while $(3, 2)$ is not.

Example 2. *Determine whether $\left(-\dfrac{1}{2}, 3\right)$ is a solution of $5y = 7 - 14x$.*

Solution 2. *Replacing x with $-\frac{1}{2}$ and y with 3, we obtain*

$$\begin{aligned} 5(3) &= 7 - 14\left(-\frac{1}{2}\right) \\ 15 &= 7 + 7 \\ 15 &\neq 14. \end{aligned}$$

Hence, $\left(-\frac{1}{2}, 3\right)$ is not a solution to $5y = 7 - 14x$.

Examples 1 and 2 are examples of a special kind of equation in two variables called linear equations. We will see some nonlinear later in this section. A **linear equation in two variables** is an equation that can be written in the form

$$ax + by = c$$

where a, b and c are real numbers such that a and b are not both zero. If a and b are both not zero, it is easy to fill in a missing component of an ordered pair so that it becomes a solution of the linear equation as the next example illustrates.

Example 3. *Given $3x - 2y = 7$ complete the following ordered pairs to find a solution to the given equation.*

(a) $(-1, ?)$

(b) $\left(?, \frac{5}{2}\right)$

Solution 3. *(a) We start by replacing x with -1 and solve for y. This yields*

$$\begin{aligned} 3(-1) - 2y &= 7 \\ -3 - 2y &= 7 \\ -2y &= 10 \\ y &= -5. \end{aligned}$$

Hence, the ordered pair solution is $(-1, -5)$.

(b) This time, we start by replacing y with $\frac{5}{2}$ and solve for x. This yields

$$\begin{aligned} 3x - 2\left(\frac{5}{2}\right) &= 7 \\ 3x - 5 &= 7 \\ 3x &= 12 \\ x &= 4. \end{aligned}$$

Hence, the ordered pair solution is $\left(4, \frac{5}{2}\right)$.

In the previous example we solved for the missing coordinate in an ordered pair solution. However, suppose that we were interested in solving for the missing term in several ordered pair solutions. Instead of writing each ordered pair, we could organize them in a chart or table.

Example 4. *Given $2x - 4y = -3$, complete the following table.*

x	y
-2	
	$\frac{1}{2}$
0	
	0

Solution 4. *We begin by substituting the given value into the specified variable. Therefore, replacing x with -2 we get*

$$\begin{aligned} 2(-2) - 4y &= -3 \\ -4 - 4y &= -3 \\ -4y &= 1 \\ y &= -\frac{1}{4}. \end{aligned}$$

When $y = \frac{1}{2}$, we obtain

$$\begin{aligned} 2x - 4\left(\frac{1}{2}\right) &= -3 \\ 2x - 2 &= -3 \\ 2x &= -1 \\ x &= -\frac{1}{2}. \end{aligned}$$

For $x = 0$, we have

$$\begin{aligned} 2(0) - 4y &= -3 \\ -4y &= -3 \\ y &= \frac{3}{4} \end{aligned}$$

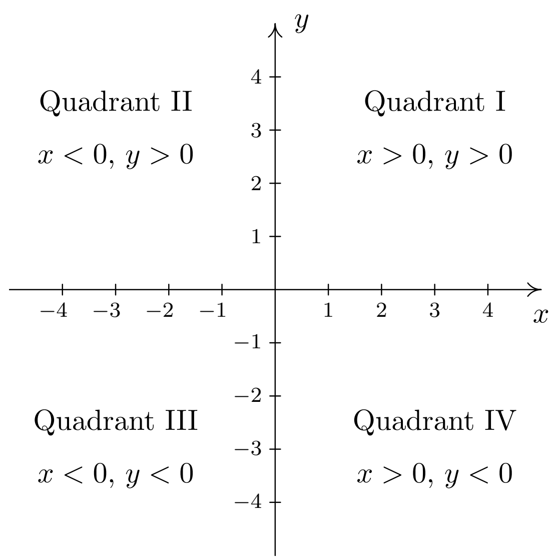
Finally, replacing y with 0, we get

$$\begin{aligned} 2x - 4(0) &= -3 \\ 2x &= -3 \\ x &= -\frac{3}{2}. \end{aligned}$$

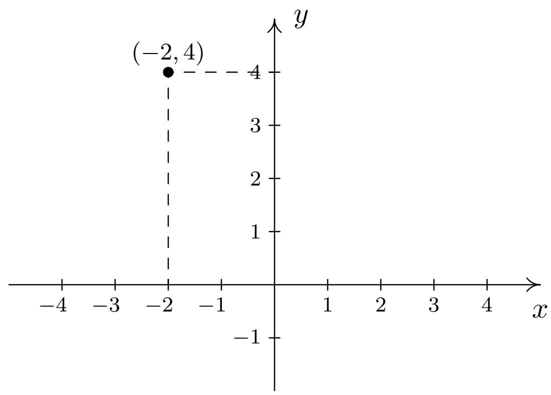
Thus, the table of solutions becomes

x	y
-2	$-\frac{1}{4}$
$-\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{3}{4}$
$-\frac{3}{2}$	0

Suppose we now wanted to plot the solutions to a linear equation in two variables. We will need to have two axes – one for the x variable and another for the y variable. Together these axes will form the **Rectangular Coordinate System**, or **Cartesian Coordinate System**. The horizontal axis is the $x-$axis and the vertical axis is the $y-$axis. These two axes divide the $xy-$plane into four **quadrants** and the intersection of the two axes is called the **origin** and is denoted $(0, 0)$. See the following diagram.

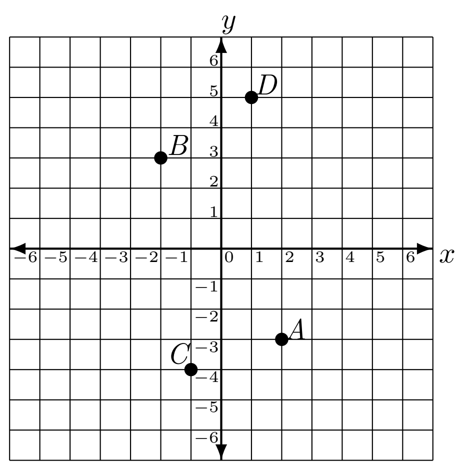


To plot an ordered pair on the Cartesian coordinate system we start at the origin and move the number of places determined by the x and y coordinates. For example, to plot $(-2, 4)$, starting at the origin, we move 2 units to the left, and then four units up on a line parallel to the $y-$axis, as illustrated in the following diagram.



Example 5. *Plot the following points on the same set of axes:* $A = (2, -3)$, $B = (-2, 3)$, $C = (-1, -4)$, *and* $D = (1, 5)$.

Solution 5.



The next example is a prelude to graphing linear equations in Section 3.2.

Example 6. *Find* 3 *solutions of* $y = 3x + 1$ *and plot them.*

Solution 6. *As with Example 4, we will organize our results in a table. However, unlike Example 4, we are not provided with any coordinates. Hence we choose any number for either* x *or* y *and solve for its corresponding coordinate. Consider the following:*

x	y
-1	
0	
1	

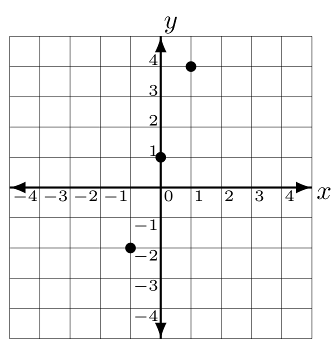
To finish the problem, we substitute each of these values into x and solve for the corresponding $y-$variable.

$$\text{If } x=-1, \text{ then} \qquad y = 3(-1)+1 = -3+1 = -2.$$
$$\text{If } x=0, \text{ then} \qquad y = 3(0)+1 = 0+1 = 1.$$
$$\text{If } x=1, \text{ then} \qquad y = 3(1)+1 = 3+1 = 4.$$

Therefore, three solutions of $y = 3x+1$ are

x	y
-1	-2
0	1
1	4

We now plot them to obtain



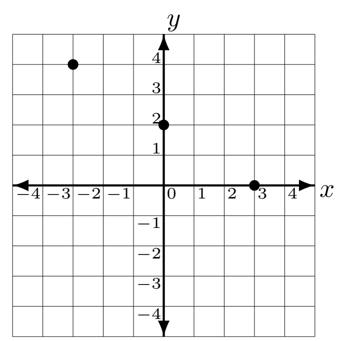
In the previous example, we randomly chose numbers for x and solved for their corresponding $y-$values. We could also have chosen values for y and solved for their corresponding $x-$values. Furthermore, we could also have chosen many different values of both x and y and complete the ordered pairs.

Example 7. *Find three solutions for $2x + 3y = 6$ and plot them.*

Solution 7. *Instead of organizing our results in a table, let us consider a few ordered pairs to complete. For this example, consider* $(0, \ )$*,* $(\ , 0)$*, and* $(-3, \ )$*. Hence, substituting the values in for the correct variable we see that*

$$\begin{array}{llcrlr} \textit{when } x = 0, & 2(0) + 3y = 6 & \Longrightarrow & 3y = 6 & \textit{or} & y = 2 \\ \textit{when } y = 0, & 2x + 3(0) = 6 & \Longrightarrow & 2x = 6 & \textit{or} & x = 3 \\ \textit{when } x = -3, & 2(-3) + 3y = 6 & \Longrightarrow & 3y = 12 & \textit{or} & y = 4 \end{array}$$

Hence, three solutions for $2x + 3y = 6$ are $(0, 2)$, $(3, 0)$, and $(-3, 4)$. The following graph shows these solutions plotted on the same set of axes.



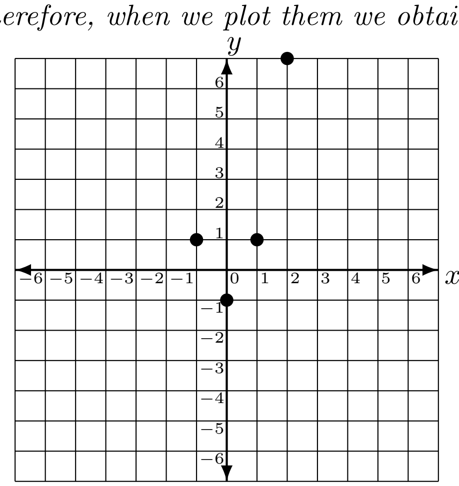
Notice in the previous two examples that the three solutions appear to lie in a straight line. Will this always happen when we graph a linear equation in two variables? Yes, but this will be discussed in Section 3.2. For our final examples, we turn our attention away from linear equations in two variables.

Example 8. *Find four solutions of $y = 2x^2 - 1$ and plot them.*

Solution 8. *Consider the following ordered pairs* $(0, \ )$*,* $(-1, \ )$*,* $(2, \ )$*, and* $(1, \ )$*. Substituting the values in for the correct variable we* *see that*

$$\begin{aligned}
&\textit{when } x = 0, && y = 2(0)^2 - 1 = 0 - 1 = -1\\
&\textit{when } x = -1, && y = 2(-1)^2 - 1 = 2 - 1 = 1\\
&\textit{when } x = 2, && y = 2(2)^2 - 1 = 8 - 1 = 7\\
&\textit{when } x = 1, && y = 2(1)^2 - 1 = 2 - 1 = 1
\end{aligned}$$

Hence, four solutions of $y = 2x^2 - 1$ *are* $(0, -1)$, $(-1, 1)$, $(2, 7)$, *and* $(1, 1)$. *Therefore, when we plot them we obtain*



Note that for the first time, the solutions do not lie in a straight line. What is different between the equation given in the above example compared to the two previous examples? If you said the exponent, you are right.

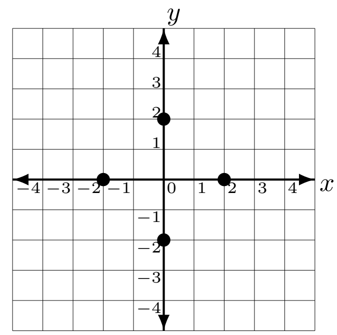
Let us consider another example whose graph is not a straight line. Can you see the difference in this example?

Example 9. *Find four solutions of* $x^2 + y^2 = 4$.

Solution 9. *We start by considering the following ordered pairs:* $(0, \quad)$ *and* $(\quad, 0)$. *We see that*

$$\begin{aligned}
&\textit{when } x = 0, && (0)^2 + y^2 = 4 \implies y^2 = 4 \quad \textit{or} \;\; y = \pm 2\\
&\textit{when } y = 0, && x^2 + (0)^2 = 4 \implies x^2 = 4 \quad \textit{or} \;\; x = \pm 2.
\end{aligned}$$

Hence, we have four solutions; namely, $(0, 2)$, $(0, -2)$, $(2, 0)$, *and* $(-2, 0)$. *When we plot these four solutions on the same axes we obtain*



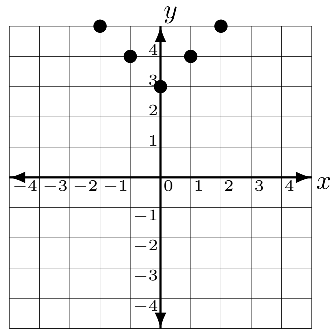
In a later course it will be discussed that $x^2 + y^2 = r^2$ is the equation of a circle centered at the origin with radius r. Hence, $x^2 + y^2 = 4$ is the circle centered at the origin with radius 2. For our last example, recall that the absolute value of a, denoted $|a|$, is the distance a number is from zero on the real number line.

Example 10. *Find five solutions of* $y = |x| + 3$ *and plot them*

Solution 10. *We will consider the following ordered pairs:* $(0, \quad)$, $(1, \quad)$, $(-1, \quad)$, $(2, \quad)$, *and* $(-2, \quad)$. *Hence,*

$$\begin{aligned}
&\textit{when } x = 0, & y &= |0| + 3 = 3\\
&\textit{when } x = 1, & y &= |1| + 3 = 1 + 3 = 4\\
&\textit{when } x = -1, & y &= |-1| + 3 = 1 + 3 = 4\\
&\textit{when } x = 2, & y &= |2| + 3 = 2 + 3 = 5\\
&\textit{when } x = -2, & y &= |-2| + 3 = 2 + 3 = 5
\end{aligned}$$

Therefore, the graph of these five solutions is



SECTION 3.1 EXERCISES

(Answers are found on page 388.)

1. Determine whether $x = 3$ and $y = -2$ is a solution of $4x - 2y = 16$.

2. Determine whether $(3, 1)$ is a solution of $5x + 3y = 12$

3. Determine whether $\left(-2, \dfrac{1}{2}\right)$ is a solution of $3x + 4y = 4$.

4. Determine whether $(2, -1)$ is a solution of $5x + 3y = 7$.

5. Complete the following ordered pairs to find solutions of $5x - 2y = 9$.

   (a) $(-4, \quad)$
   (b) $\left(\quad, \dfrac{1}{2}\right)$
   (c) $(0, \quad)$
   (d) $(\quad, 0)$

6. Complete the following ordered pairs to find solutions of $7x + 3y = 8$.

   (a) $(\quad, -1)$
   (b) $(2, \quad)$
   (c) $\left(\dfrac{2}{7}, \quad\right)$
   (d) $\left(\quad, \dfrac{1}{3}\right)$

7. Complete the following ordered pairs to find solutions of $-6x + 5y = -7$.

   (a) $(0, \quad)$
   (b) $(2, \quad)$
   (c) $(\quad, -1)$
   (d) $\left(\quad, \dfrac{2}{5}\right)$

8. Complete the following ordered pairs to find solutions of $\frac{1}{2}x + \frac{4}{3}y = 2$.

   (a) $(\ , 0)$

   (b) $(-4, \ )$

   (c) $(\ , 6)$

   (d) $\left(\ , \frac{1}{2}\right)$

9. Given $8x - 3y = 1$ complete the following table of values:

x	y
1	
$\frac{1}{4}$	
-2	

10. Given $7x + 2 = 3y$ complete the following table of values:

x	y
0	
-3	
$\frac{1}{2}$	

11. Given $-2x + 3y = 4$ complete the following table of values:

x	y
	-2
	-4
	$\frac{2}{3}$
	6

12. Given $x - 3y = 12$ complete the following table of values:

x	y
0	
	2
	$-\frac{2}{3}$
5	

13. Given $3x = 8 + 2y$ complete the following table of values:

x	y
	-3
-3	
4	
	$-\frac{5}{2}$

14. Given $2x - 5y = 7$ complete the following table of values:

x	y
	-2
0	
	1
-4	

15. Plot the following points on the same coordinate plane:

$$A = (3, 1),\ B = (-5, 3),\ C = \left(\frac{1}{2}, -4\right),\ D = (-2, -5)$$

16. Find three solutions of $y = 5x - 1$ and plot them.

17. Find three solutions of $2x + 4y = 8$ and plot them.

18. Find three solutions of $6y = -5x + 2$ and plot them.

19. Find three solutions of $y = 3x^2 + 2$ and plot them.

20. Find four solutions of $x = -y^2 + 3$ and plot them.

21. Find four solutions of $x^2 + 4y^2 = 16$ and plot them.

22. Find four solutions of $x^2 + y^2 = 25$ and plot them.

23. Find five solutions of $y = 3|x| - 2$ and plot them.

24. Find five solutions of $y = 2|x - 1| + 3$ and plot them.

25. Find five solutions of $y = -2|x + 1| + 3$ and plot them.

3.2 Graphing Linear Equations

Now that we know how to find the solutions of a equation, we want to discuss the graph of a linear equation in two variables. The **graph of a equation** is the set of all ordered pairs (x, y) that are a solution to the equation. The graph of a linear equation in two variables will always be a straight line. In Section 3.1, we were able to find the solutions of a linear equation. To complete the graph, we connect these solutions with a straight line. It is true that two points determine a line; therefore, it is necessary to find two solutions in order to graph the line. However, it is recommended that a third point be graphed as a check point.

Example 1. *Graph $x + 2y = 1$ by plotting points.*

Solution 1. *We begin by finding three solutions of $x + 2y = 1$. If we let $x = 0$, we get*

$$\begin{aligned} 0 + 2y &= 1 \\ 2y &= 1 \\ y &= \frac{1}{2}. \end{aligned}$$

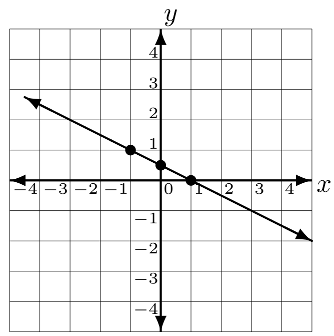
If we let $x = 1$, we find that

$$\begin{aligned} 1 + 2y &= 1 \\ 2y &= 0 \\ y &= 0. \end{aligned}$$

Finally, if we let $x = -1$, we obtain

$$\begin{aligned} -1 + 2y &= 1 \\ 2y &= 2 \\ y &= 1. \end{aligned}$$

Therefore, $\left(0, \frac{1}{2}\right)$, $(1, 0)$, and $(-1, 1)$ are solutions of $x+2y = 1$. Graphing these three points on a common set of axes and connecting them with a straight line, we obtain the following graph of $x + 2y = 1$.



Example 2. *Graph $3x - 2y = 6$ by plotting points.*

Solution 2. *We begin by finding three solutions of $3x - 2y = 6$. If we let $x = 0$, then*

$$\begin{aligned} 3(0) - 2y &= 6 \\ -2y &= 6 \\ y &= -3. \end{aligned}$$

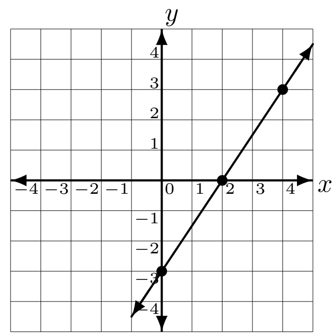
So, $(0, -3)$ is a solution of our equation. If we let $y = 0$, then

$$\begin{aligned} 3x - 2(0) &= 6 \\ 3x &= 6 \\ x &= 2. \end{aligned}$$

Thus, $(2, 0)$ is also a point on the graph. Finally, if we let $y = 3$, then

$$\begin{aligned} 3x - 2(3) &= 6 \\ 3x - 6 &= 6 \\ 3x &= 12 \\ x &= 4. \end{aligned}$$

So, a third point on the graph is $(4, 3)$. If we plot $(0, -3)$, $(2, 0)$, and $(4, 3)$ on a common set of axes and connect them with a straight line, we obtain the following graph of $3x - 2y = 6$.



In the previous example, $(0, -3)$ is the y-intercept and $(2, 0)$ is the x-intercept of $3x - 2y = 6$. The x**-intercept** is the point where the graph crosses the x-axis. Likewise, the y**-intercept** is the point where the graph crosses the y-axis. How can we identify the x-intercept(s) and y-intercept(s) without graphing? Well, all the points on the x-axis can be written as ordered pairs $(x, 0)$. Therefore, in order to find the x-intercept of an equation, we let $y = 0$ and solve for x. Likewise, all the points on the y-axis can be written as ordered pairs $(0, y)$. So, to find the y-intercept, we let $x = 0$ and solve for y. These results are summarized below.

Finding x-intercepts and y-intercepts of a graph

- <u>To find the x-intercept</u>: Let $y = 0$ and solve for x.

- <u>To find the y-intercept</u>: Let $x = 0$ and solve for y.

Since only two points determine a line, instead of finding arbitrary solutions of a linear equation, we can also graph a linear equation by finding the x-intercept and y-intercept and connecting them with a straight line. We illustrate this in the next example.

Example 3. *Find the x-intercept and y-intercept of $5x + 2y = 10$, and graph.*

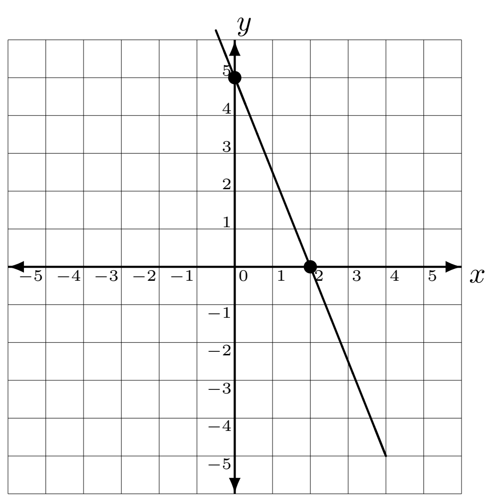
Solution 3. *To find the x-intercept we substitute $y = 0$ into the* *equation and solve for x. Thus,*

$$\begin{aligned} 5x + 2(0) &= 10 \\ 5x &= 10 \\ x &= 2. \end{aligned}$$

To find the y-intercept we substitute $x = 0$ into the equation and solve for y. This gives us

$$\begin{aligned} 5(0) + 2y &= 10 \\ 2y &= 10 \\ y &= 5. \end{aligned}$$

Therefore, $(2, 0)$ is the x-intercept and $(0, 5)$ is the y-intercept. Plotting both of these on a common set of axes and connecting them with a straight line, we obtain the following graph of $5x + 2y = 10$.



Example 4. *Graph $-3x + 4y = 12$ using the intercepts.*

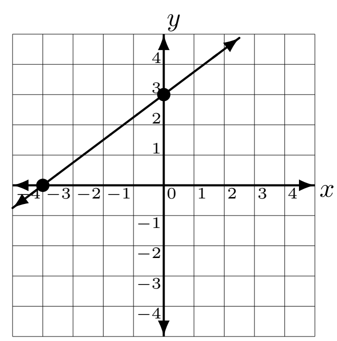
Solution 4. *First, we will find the y-intercept by letting $x = 0$. This gives us*

$$\begin{aligned} -3(0) + 4y &= 12 \\ 4y &= 12 \\ y &= 3. \end{aligned}$$

To find the x-intercept, we let $y = 0$. This gives us

$$\begin{aligned} -3x + 4(0) &= 12 \\ -3x &= 12 \\ x &= -4. \end{aligned}$$

Therefore, $(-4, 0)$ is the x-intercept and $(0, 3)$ is the y-intercept. Plotting both of these on a common set of axes and connecting them with a straight line, we obtain the following graph of $-3x + 4y = 12$.



Although the two previous examples were easily graphed using the intercepts, relying solely on the intercepts for graphing linear equations can lead to a problem. Consider the following example.

Example 5. *Find the intercepts of $3x - 5y = 0$ and graph.*

Solution 5. *To find the x-intercept, we let $y = 0$ and solve for x. This yields*

$$\begin{aligned} 3x - 5(0) &= 0 \\ 3x &= 0 \\ x &= 0. \end{aligned}$$

To find the y-intercept, we let $x = 0$ and solve for y. Thus,

$$\begin{aligned} 3(0) - 5y &= 0 \\ -5y &= 0 \\ y &= 0. \end{aligned}$$

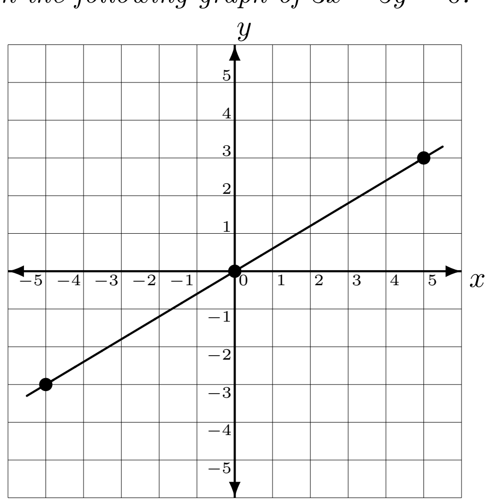
Therefore, we find that $(0, 0)$ is both the x-intercept and the y-intercept. Therefore, it would be impossible to graph this line using only the intercepts. We will finish by finding two additional points. If we let $x = 5$, we find

$$\begin{aligned} 3(5) - 5y &= 0 \\ 15 - 5y &= 0 \\ -5y &= -15 \\ y &= 3. \end{aligned}$$

So, $(5, 3)$ is a point on the graph. If we let $y = -3$, then

$$\begin{aligned} 3x - 5(-3) &= 0 \\ 3x + 15 &= 0 \\ 3x &= -15 \\ x &= -5. \end{aligned}$$

Thus, $(-5, -3)$ is an additional point on the graph. Plotting $(0, 0)$, $(5, 3)$, and $(-5, -3)$ and connecting them with a straight line, we obtain the following graph of $3x - 5y = 0$.



In the previous example, we were able to avoid graphing any solutions containing fractions by choosing appropriate x and y values. However, it can sometimes be difficult to see which values of x and y will avoid fractions. To help with this, it is sometimes better to rewrite the linear equation by solving for y first. This is illustrated in the next example.

Example 6. *Graph* $3x - 2y = 4$.

Solution 6. *Note that for this linear equation the* x*-intercept is* $\left(\frac{4}{3}, 0\right)$ *which can be difficult to graph. In order to try to avoid fractions, we first solve the equation for* y *obtaining*

$$\begin{aligned} 3x - 2y &= 4 \\ -2y &= -3x + 4 \\ y &= \frac{3}{2}x - 2. \end{aligned}$$

Now we see that in order to avoid fractions we must choose values of x *that are divisible by* 2. *So, if we let* $x = 0$, *we obtain*

$$\begin{aligned} y &= \frac{3}{2}(0) - 2 \\ y &= -2. \end{aligned}$$

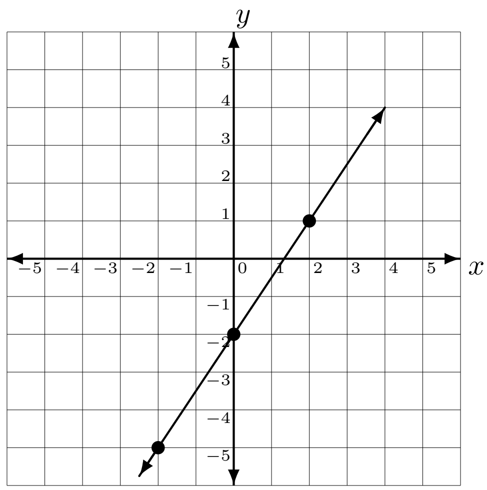
If we let $x = -2$, *we obtain*

$$\begin{aligned} y &= \frac{3}{2}(-2) - 2 \\ y &= -3 - 2 \\ y &= -5. \end{aligned}$$

Finally, if we let $x = 2$, *we obtain*

$$\begin{aligned} y &= \frac{3}{2}(2) - 2 \\ y &= 3 - 2 \\ y &= 1. \end{aligned}$$

Therefore, $(0, -2)$, $(-2, -5)$, *and* $(2, 1)$ *are points on the graph. Plotting these points and connecting them with a straight line, we obtain the following graph of* $3x - 2y = 4$.



Solving a linear equation for y places the equation in the form $y = mx + b$ which is called the **slope-intercept form** of a linear equation. This will be discussed in Section 3.4.

Example 7. *Graph* $2x + 3y = 9$.

Solution 7. *First, we solve the equation for* y *to obtain*

$$\begin{aligned} 2x + 3y &= 9 \\ 3y &= -2x + 9 \\ y &= -\frac{2}{3}x + 3. \end{aligned}$$

So, in order to avoid fractions, we will choose values of x *that are divisible by* 3*. If* $x = 0$*, then*

$$\begin{aligned} y &= -\frac{2}{3}(0) + 3 \\ y &= 3. \end{aligned}$$

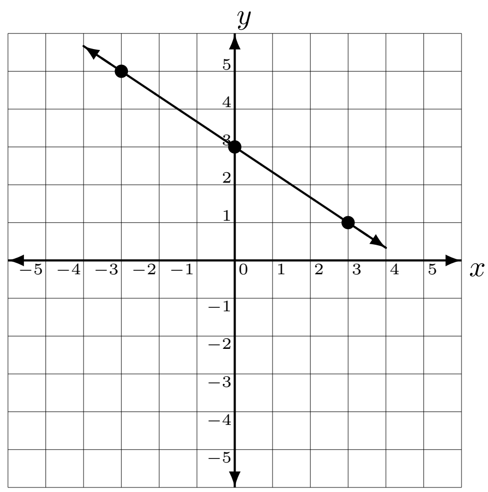
If $x = -3$*, then*

$$\begin{aligned} y &= -\frac{2}{3}(-3) + 3 \\ y &= 2 + 3 \\ y &= 5. \end{aligned}$$

Finally, if $x = 3$*, then*

$$\begin{aligned} y &= -\frac{2}{3}(3) + 3 \\ y &= -2 + 3 \\ y &= 1. \end{aligned}$$

Therefore, $(0, 3)$*,* $(-3, 5)$*, and* $(3, 1)$ *are all points on the graph. Plotting these and connecting them with a straight line we obtain the following graph of* $2x + 3y = 9$*.*



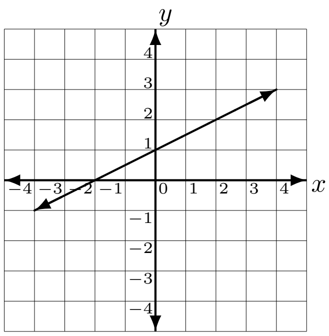
Plotting points or using the intercepts are not the only ways a linear equation can be graphed. In Section 3.4, we will discuss graphing a line using a point and the slope.

SECTION 3.2 EXERCISES

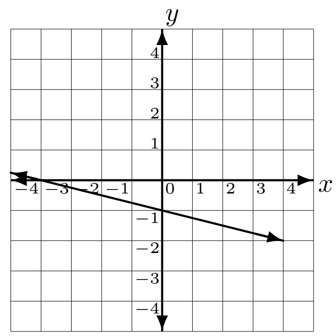
(Answers are found on page 391.)

Find the intercepts of each graph. Write answers as ordered pairs.

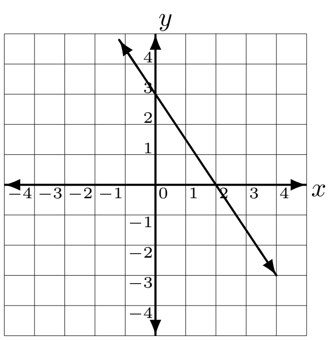
1.



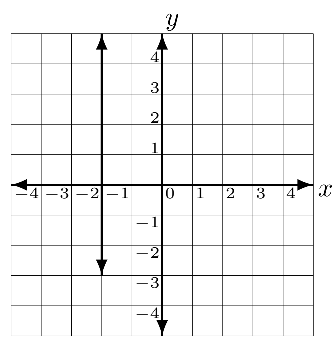
4.



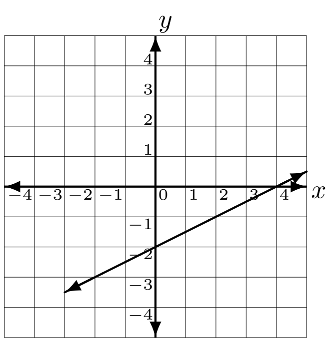
2.



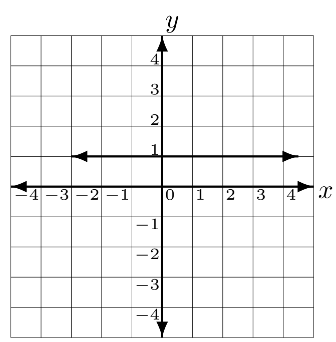
5.



3.



6.



Find the intercepts of each equation. Write answers as ordered pairs.

7. $2x - y = 4$
8. $y + 3x = -6$
9. $4x + 2y = 8$
10. $3x - 6y = 6$
11. $2x + 5y = 4$
12. $3x - 4y = 4$
13. $3x - 5y = 1$
14. $-2x + 3y = -10$
15. $4x - 5y = 12$
16. $6x - 5y = 3$
17. $8x - 2y = 4$
18. $-7x + 3y = 2$
19. $6x - 3y = 5$
20. $9x + 2y = 11$

Graph each linear equation by using the intercepts

21. $x + y = 3$
22. $2x - y = 6$
23. $-x + 3y = 6$
24. $-2x - y = 4$
25. $-y + 3x = 6$
26. $4x + 2y = 8$
27. $3x - 4y = -12$
28. $3x - 6y = 6$
29. $-4x + y = 4$
30. $2x - 3y = 6$
31. $3x - 2y = 12$
32. $5x - 2y = -10$

Graph each linear equation by plotting points.

33. $y = 2x - 4$
34. $y = -3x + 2$
35. $y = \frac{1}{3}x - 2$
36. $y = -\frac{2}{3}x + 2$
37. $y = \frac{3}{2}x - 2$
38. $x = 3$
39. $y = -2$
40. $y = 2x$
41. $y = -3x$
42. $y = x$
43. $3x - y = 4$
44. $2x + 3y = 9$
45. $-4x + y = 3$
46. $6x - 2y = -4$
47. $x + 3y = 5$
48. $4x - 3y = 2$

3.3 Function Notation

Each of the nonvertical lines graphed in Section 3.2 is an example of a function. A **function** f is a process that takes each element of a set A and transforms it into a unique element of another set B. The elements of A are the **input** and the elements of B are the **output**. Since a function assigns to each input a unique output, you can think of the output as being dependent on the input. Therefore, we call the input of a function the **independent variable** and the output the **dependent variable**.

In the previous section we graphed linear equations in two variables x and y. The ordered pairs (x, y) can be thought of as

$$\text{(independent variable, dependent variable)}$$

or (input, output).

Because functions are important in mathematics is is vital to be able to represent when a relationship is a function. To use the x and y variables that we have used before, we could say that

$$y \text{ is a function of } x.$$

This can be transformed into

$$y \text{ is } f \text{ of } x.$$

Using function notation, this becomes $y = f(x)$. The symbol $f(x)$, read "f of x," is called the **value of the function at** x and is equated with the variable y. In other words, we write $y = f(x)$. So, the exercise $y = 2x + 4$ from the previous section, can be written as $f(x) = 2x + 4$ using function notation. It is important to note here that $f(x)$ does NOT mean f times x. Instead, it means f evaluated at x.

Example 1. *Suppose the cost C, in dollars, of producing x beach balls is given by $C(x) = 2x + 300$.*

(a) *Identify the independent variable (input).*

(b) *Identify the dependent variable (output).*

(c) *Find the cost of producing 50 beach balls. Write answer using function notation.*

(d) *Find $C(25)$, and interpret its meaning in the context of the problem.*

Solution 1. *(a) The independent variable is x beach balls*

(b) The dependent variable is cost C, because cost depends on the number of beach balls.

(c) To find the cost of producing 50 *beach balls, we substitute* 50 *in for x in $C(x) = 2x + 300$. Therefore,*

$$C(50) = 2(50) + 300 = 100 + 300 = 400.$$

So, the cost of producing 50 *beach balls is* \$400.

(d) To find $C(25)$, we substitute 25 *into x to get*

$$C(25) = 2(25) + 300 = 50 + 300 = 350.$$

Therefore, the cost of producing 25 *beach balls is* \$350.

Example 2. *Suppose a car travels at a rate of* 65 *miles per hour. The distance D, in miles, that the car has traveled after t hours is given by $D(t) = 65t$.*

(a) Identify the independent variable (input).

(b) Identify the dependent variable (output).

(c) Find the distance traveled after 3 *hours. Write answer using function notation.*

(d) Find $D(2.5)$, and interpret its meaning in the context of the problem.

Solution 2. *(a) The independent variable is time t.*

(b) The dependent variable is the distance D traveled, because the distance depends on the number of hours t.

(c) To find the distance traveled in 3 *hours, we replace t with* 3 *to get*

$$D(3) = 65(3) = 195 \textit{ miles.}$$

(d) To find $D(2.5)$, we substitute 2.5 *for t to get*

$$D(2.5) = 65(2.5) = 162.5.$$

Therefore, the distance traveled in 2.5 *hours is* 162.5 *miles.*

Example 3. *The profit P (in dollars) from the production and sale of golf hats is given by the function $P(x) = 20x - 4000$, where x is the number of hats produced and sold.*

(a) Identify the independent variable.

(b) Identify the dependent variable.

(c) What is $P(250)$? Interpret this result.

(d) What is the profit from the production and sale of 450 hats? Write this in function notation.

Solution 3. *(a) The independent variable is x, the number of hats produced and sold.*

(b) The dependent variable is P, since the profit depends on the number of hats sold.

(c) To find $P(250)$, we substitute 250 in for x to obtain

$$\begin{aligned} P(250) &= 20(250) - 4000 \\ &= 5000 - 4000 \\ &= 1000. \end{aligned}$$

Therefore, a profit of \$1000 is made from the production and sale of 250 golf hats.

(d) To find the profit from the production and sale of 450 hats, we substitute 450 in for x to obtain

$$\begin{aligned} P(450) &= 20(450) - 4000 \\ &= 9000 - 4000 \\ &= 5000. \end{aligned}$$

Thus, the production and sale of 450 hats yields a \$5000 profit.

Now that we can identify the input and output values, let us turn our attention to evaluating a function using function notation. Remember, the independent variable serves as a placeholder.

Example 4. *If $f(x) = 3x + 4$, find*

(a) $f(0)$ *(b)* $f(1)$ *(c)* $f(-2)$

Solution 4. *(a) To find $f(0)$, we replace the independent variable x with 0 to get*

$$\begin{aligned} f(0) &= 3(0) - 4 \\ &= -4 \end{aligned}$$

(b) To find $f(1)$, we replace the independent variable x with 1 obtaining

$$\begin{aligned} f(1) &= 3(1) - 4 \\ &= 3 - 4 \\ &= -1 \end{aligned}$$

(c) Finally, to find $f(-2)$, we replace the independent variable x with -2 which yields

$$\begin{aligned} f(-2) &= 3(-2) - 4 \\ &= -6 - 4 \\ &= -10 \end{aligned}$$

In Section 3.2 we graphed linear equations in two variables. The function $f(x) = 3x + 4$ in the previous example, is a linear equation where y has been replaced by the function notation $f(x)$. The values of the function found in the previous example, $f(0) = -4$, $f(1) = -1$, and $f(-2) = -10$, refer to the ordered pairs $(0, -4)$, $(1, -1)$, and $(-2, -10)$. Graphing these three points we would obtain the graph of the given function.

Although this chapter is restricted to discussing linear functions, not all functions are linear. The following example is a **quadratic function** that will be discussed in Core Mathematics III.

Example 5. *The height of a rocket is measured by the function*

$$H(t) = -16t^2 + 120t + 80,$$

where H is measured in feet and t is time in seconds.

(a) *Identify the independent variable.*

(b) *Identify the dependent variable.*

(c) *Evaluate $H(1)$ and interpret what it means in the context of the problem.*

(d) *Determine the height of the rocket at 4 seconds. Write answer using function notation.*

Solution 5. (a) *The independent variable is t measured in seconds.*

(b) *The dependent variable is H because the height of the rocket depends on the time.*

(c) *To find $H(1)$ we replace the independent variable t with 1 obtaining*

$$\begin{aligned} H(1) &= -16(1)^2 + 120(1) + 80 \\ &= -16 + 120 + 80 \\ &= 184. \end{aligned}$$

Since $H(1) = 184$, this means that the height of the rocket at 1 second is 184 feet.

(d) *To find the height of the rocket at 4 seconds, we replace the independent variable t with 4 to get*

$$\begin{aligned} H(4) &= -16(4)^2 + 120(4) + 80 \\ &= -16(16) + 480 + 80 \\ &= -256 + 480 + 80 \\ &= 304 \end{aligned}$$

Thus, the height of the rocket at 4 seconds is 304 feet.

Functions will be discussed in more detail in Core Mathematics II.

SECTION 3.3 EXERCISES

(Answers are found on page 396.)

1. The sales tax, in dollars, due on an item costing x dollars is given by $S(x) = 0.065x$.
   (a) Determine the independent variable.
   (b) Determine the dependent variable.
   (c) Find $S(12)$ and interpret in the context of the problem.
   (d) Find the sales tax due on a DVD costing \$20. Write answer in function notation.

2. Suppose a car travels at a rate of 55 miles per hour. The distance D, in miles, that the car has traveled after t hours is given by $D(t) = 55t$.
   (a) Identify the independent variable.
   (b) Identify the dependent variable.
   (c) Find the distance traveled after 2 hours. Write answer using function notation.
   (d) Find $D(3)$, and interpret its meaning.

3. The area of lawn A, in square yards, remaining to be mowed after h hours spent mowing is written as $A(h) = 1500 - 300h$.
   (a) Identify the independent variable.
   (b) Identify the dependent variable.
   (c) Find $A(2)$ and interpret it in the context of the problem.
   (d) Find the area of lawn remaining after 4 hours spent mowing. Write answer using function notation.

4. The local plumbing company charges C, in dollars, is given by the function $C(t) = 45t + 50$, where t is the number of hours on the job.
   (a) Identify the independent variable.
   (b) Identify the dependent variable.
   (c) Evaluate $C(2)$ and interpret what it means in the context of the problem.
   (d) Determine charges if the plumber is on the job for 3 hours. Write answer using function notation.

5. The manager of a flea market knows from past experience that if she charges d dollars for each rental space at the flea market, then the number N of spaces she can rent is given by the equation $N(d) = -4d + 200$.

   (a) Identify the independent variable.

   (b) Identify the dependent variable.

   (c) Find $N(10)$ and interpret it in the context of the problem.

   (d) Find the number of spaces the manager can rent if she charges \$20 for each rental space. Write answer using function notation.

6. A company buys and retails baseball caps. The total cost function is linear and is given by $C(x) = 12x + 280$, where x is the number of baseball caps, and C is the total cost in dollars.

   (a) Identify the independent variable.

   (b) Identify the dependent variable.

   (c) Find $C(400)$ and interpret it in the context of the problem.

   (d) Find the cost of buying 350 baseball caps. Write answer using function notation.

7. The profit P (in dollars) from the production and sale of Christmas ornaments is given by the function $P(x) = 15x - 1200$, where x is the number of ornaments produced and sold.

   (a) Identify the independent variable.

   (b) Identify the dependent variable.

   (c) What is $P(200)$? Interpret this result.

   (d) What is the profit from the production and sale of 350 ornaments? Write this in function notation.

8. A small appliance manufacturer finds that if he produces x microwaves in a month his production cost C, in dollars, is given by $C(x) = 6x + 3000$.

   (a) Identify the independent variable.

   (b) Identify the dependent variable.

   (c) Find $C(150)$ and explain what it means.

(d) Find the cost of producing 75 microwaves in a month and write this in function notation.

9. The revenue R, in dollars, from selling x computer is given by the function $R(x) = 1500x$.

   (a) Identify the independent variable.
   (b) Identify the dependent variable.
   (c) Find $R(8)$ and explain what it means.
   (d) Find the revenue from selling 10 computers and write this in function notation.

10. The water level W, measured in feet, in a reservoir is given by the function $W(t) = 5t + 30$, where t is the number of years since the reservoir was constructed.

    (a) Identify the independent variable.
    (b) Identify the dependent variable.
    (c) Find $W(4)$ and explain what it means.
    (d) Find the water level 7 years after the reservoir was constructed. Write your answer using function notation.

11. The perimeter P, measured in meters, of a garden with fixed length 14 meters is given by $P(w) = 28 + 2w$, where w is the width of the garden in meters.

    (a) Identify the independent variable.
    (b) Identify the dependent variable.
    (c) Evaluate $P(12)$ and interpret what it means in the context of the problem.
    (d) Determine the perimeter of the garden if the width is 20 meters. Write answer using function notation.

12. The local phone company's monthly bill is calculated using the function $A(m) = 0.09m + 5.25$, where A is the monthly cost in dollars, and m is the number of minutes of phone usage.

    (a) Identify the independent variable.
    (b) Identify the dependent variable.

(c) Evaluate $A(100)$ and interpret what it means in the context of the problem.

(d) Determine the monthly bill if the phone usage was 250 minutes. Write answer using function notation.

13. The height of a rocket is measured by the function

$$H(t) = -16t^2 + 40t + 100,$$

where H is measured in feet and t is time in seconds.

(a) Identify the independent variable.

(b) Identify the dependent variable.

(c) Evaluate $H(2)$ and interpret what it means in the context of the problem.

(d) Determine the height of the rocket at 3 seconds. Write answer using function notation.

14. The area of a rectangle A, in square feet, is measured by the function $A(w) = -w^2 + 8w$, where w is the width of the rectangle measured in feet.

(a) Identify the independent variable.

(b) Identify the dependent variable.

(c) Evaluate $A(4)$ and interpret what it means in the context of the problem.

(d) Determine the area of the rectangle if the width is 2 feet. Write answer using function notation.

15. The area of a square A, in square meters, is measured by the function $A(s) = s^2$, where s is the length of the side of the square measured in meters.

(a) Identify the independent variable.

(b) Identify the dependent variable.

(c) Evaluate $A(10)$ and interpret what it means in the context of the problem.

(d) Determine the area of a square with a side measuring 13 meters. Write answer using function notation.

16. If $f(x) = -3x$, find
    (a) $f(1)$ (b) $f(-2)$ (c) $f(4)$

17. If $A(w) = 12w$, find
    (a) $A(1)$ (b) $A(10)$ (c) $A(15)$

18. If $D(t) = 25t$, find
    (a) $D(2)$ (b) $D(10)$ (c) $D(-2)$

19. If $h(x) = 16x$, find
    (a) $h\left(\frac{1}{2}\right)$ (b) $h\left(-\frac{1}{4}\right)$ (c) $h\left(\frac{1}{8}\right)$

20. If $g(x) = 4x - 2$, find
    (a) $g(0)$ (b) $g(-3)$ (c) $g(2)$

21. If $P(n) = 12n + 35$, find
    (a) $P(10)$ (b) $P(-2)$ (c) $P(0)$

22. If $s(t) = -2t + 3$, find
    (a) $s(0)$ (b) $s(2)$ (c) $s(-5)$

23. If $C(x) = 4x - 3$, find
    (a) $C(-1)$ (b) $C\left(\frac{1}{2}\right)$ (c) $C\left(-\frac{1}{2}\right)$

24. If $P(w) = 30 + 2w$, find
    (a) $P(13)$ (b) $P\left(\frac{1}{2}\right)$ (c) $P(5)$

25. If $h(t) = |2 - t|$, find

    (a) $h(1)$ (b) $h(-7)$ (c) $h(4)$

26. If $Q(w) = 2\,|w + 3|$, find

    (a) $Q(-5)$ (b) $Q(2)$ (c) $Q(-6)$

27. If $F(p) = \dfrac{1}{p} + 2$, find

    (a) $F(1)$ (b) $F(2)$ (c) $F(-1)$

28. If $G(v) = 4 - \dfrac{1}{v}$, find

    (a) $G(1)$ (b) $G(2)$ (c) $G(-1)$

29. If $g(x) = x^2 - 4$, find

    (a) $g(3)$ (b) $g(-2)$ (c) $g(0)$

30. If $f(x) = -x^2 + 4$, find

    (a) $f(3)$ (b) $f(-2)$ (c) $f(0)$

3.4 Slope

An important part of a line is how much it slants or its steepness. The **slope** of a line measures its steepness. The slope, denoted by m, measures the vertical change and the horizontal change as we move along the line. The vertical change, also called the **rise**, is the difference between the y-coordinates. Therefore, rise is an up and down change. The horizontal change, also called the **run**, is the difference between the x-coordinates. Thus, run is a left and right change.

If the coordinates of two points on the line are known, we can use the **slope formula** to find the slope of the line.

Slope formula: The slope of the line through the points (x_1, y_1) and (x_2, y_2) is given by

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{change in } y}{\text{change in } x} = \frac{\text{rise}}{\text{run}}$$

Note that it does not matter if you start with y_1 or y_2 in the numerator. However, you must start with its corresponding x in the denominator.

Example 1. *Find the slope of the line passing through* $(-1, 3)$ *and* $(5, -2)$.

Solution 1. *Using the slope formula, we obtain*

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-2 - 3}{5 - (-1)} \\ &= -\frac{5}{6} \end{aligned}$$

Therefore, the slope of the line passing through $(-1, 3)$ *and* $(5, -2)$ *is* $m = -\frac{5}{6}$.

Example 2. *Find the slope of the line passing through* $(-9, 2)$ *and* $(-5, 5)$.

Solution 2. *Using the slope formula, we obtain*

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{5 - 2}{-5 - (-9)} \\ &= \frac{3}{4} \end{aligned}$$

So, the slope of the line passing through $(-9, 2)$ *and* $(-5, 5)$ *is* $m = \frac{3}{4}$.

Example 3. *Find the slope of the line passing through* $\left(\frac{1}{2}, 3\right)$ *and* $\left(-\frac{3}{4}, -\frac{1}{5}\right)$.

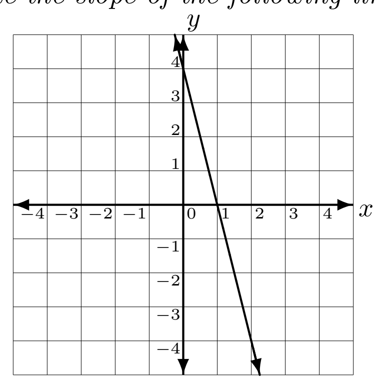
Solution 3. *Although some of the coordinates are fractions, we still substitute the given information into the slope formula. This yields*

$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-\frac{1}{5} - (3)}{-\frac{3}{4} - \frac{1}{2}} \\ &= \frac{-\frac{16}{5}}{-\frac{5}{4}} \\ &= -\frac{16}{5} \cdot -\frac{4}{5} \\ &= \frac{64}{25}. \end{aligned}$$

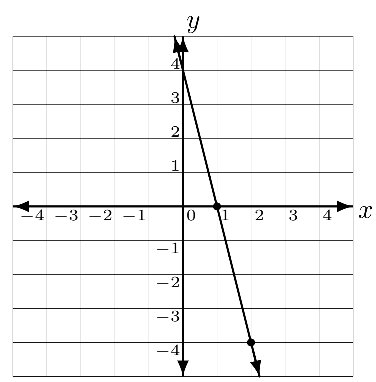
Thus, the slope is $m = \frac{64}{25}$.

What if we want to find the slope of a line where we are given the graph. We can still use the slope formula if we first identify two points on the line. However, we can also find the slope by reviewing slope as $\dfrac{\text{rise}}{\text{run}}$. Both methods will be illustrated in the next example.

Example 4. *Determine the slope of the following line.*



Solution 4. *To use the slope formula we will identify two points on the line. Note that* $(1, 0)$ *and* $(2, -4)$ *are two points on this line.*



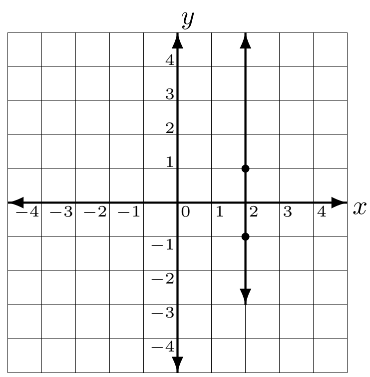
Using the slope formula with $(1, 0)$ *and* $(2, -4)$*, we obtain*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-4 - 0}{2 - 1} = \frac{-4}{1} = -4.$$

We can also find the answer to this problem by viewing slope as $\dfrac{\textit{rise}}{\textit{run}}$*. Starting at* $(1, 0)$*, we note the number of units and the* *direction we travel in order to get to the second point we identified as* $(2, -4)$. *Therefore, we travel* 4 *units down, so the rise is* -4. *Then we travel* 1 *unit to the right, so the run is* 1. *Thus,* $m = \frac{rise}{run} = \frac{-4}{1} = -4$.

Example 5. *Determine the slope of* $x = 2$.

Solution 5. *If we graph the line* $x = 2$, *we get the vertical line shown below.*



Therefore, since $(2, 1)$ *and* $(2, -1)$ *are points on this line, we can use the slope formula to find the value of the slope. Hence,*

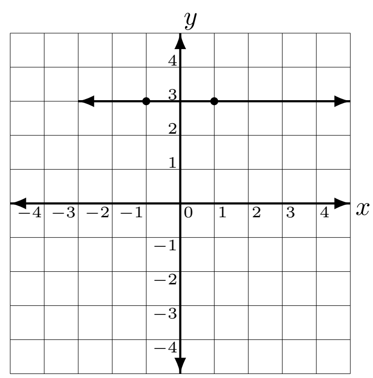
$$\begin{aligned} m &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{-1 - 1}{2 - 2} \\ &= \frac{-2}{0} \end{aligned}$$

Since $\frac{-2}{0}$ *is undefined, we find that the slope of* $x = 2$ *is undefined.*

In the last example, we found that the slope of the vertical line $x = 2$ was undefined. In fact, the slope of every vertical line $x = a$ is undefined. Is the same true for a horizontal line? The next example answers this question.

Example 6. *Determine the slope of $y = 3$.*

Solution 6. *If we graph the line $y = 3$, we get the horizontal line shown below.*



Therefore, since $(1, 3)$ and $(-1, 3)$ are points on this line, we can use the slope formula to find the value of the slope. Hence,

$$
\begin{aligned}
m &= \frac{y_2 - y_1}{x_2 - x_1} \\
&= \frac{3 - 3}{-1 - 1} \\
&= \frac{0}{-2} \\
&= 0.
\end{aligned}
$$

Thus, the slope of $y = 3$ is zero.

In fact, the slope of every horizontal line $y = b$ is zero. The equation of vertical and horizontal lines will be discussed in Section 3.5.

We have seen examples of lines with positive slope, negative slope, zero slope, and undefined slope. The following table summarizes information concerning the slope of a line.

- If the slope is positive ($m > 0$), then the line slants up $\nearrow$
- If the slope is negative ($m < 0$), then the line slants down $\searrow$
- If the slope is zero ($m = 0$), then the line is horizontal $\rightarrow$
- If the slope is undefined, then the line is vertical $\uparrow$

The slope formula is useful when we know two points located on the line. What if we are given the equation of the line? We could use the equation to locate two points on the line, and then use the slope formula as in the previous examples. However, we could also use the **slope-intercept form** of the line to identify the slope and y-intercept.

Slope-intercept form: The slope-intercept form of an equation with slope m and $y-$intercept b is given by

$$y = mx + b.$$

When identifying the slope and $y-$intercept using the slope-intercept form, remember to divide each term by the coefficient on y. The slope and $y-$intercept can only be identified once you have isolated y.

Example 7. *Identify the slope and the $y-$intercept of each line.*

(a) $3x - 2y = 6$

(b) $5x + 10y = -3$

Solution 7. *(a) In order to identify the slope and y-intercept, we need to solve the equation for y. This gives us*

$$\begin{aligned} 3x - 2y &= 6 \\ -2y &= -3x + 6 \\ y &= \frac{3}{2}x - 3. \end{aligned}$$

Next, the slope is the coefficient on x and the y-intercept is the constant term. Therefore, $m = \frac{3}{2}$ and the y-intercept is $(0, -3)$.

(b) *Solving for y, we get*

$$\begin{aligned} 5x + 10y &= -3 \\ 10y &= -5x - 3 \\ y &= -\frac{5}{10}x - \frac{3}{10} \\ y &= -\frac{1}{2}x - \frac{3}{10}. \end{aligned}$$

Therefore, $m = -\frac{1}{2}$ and the y-intercept is $\left(0, -\frac{3}{10}\right)$.

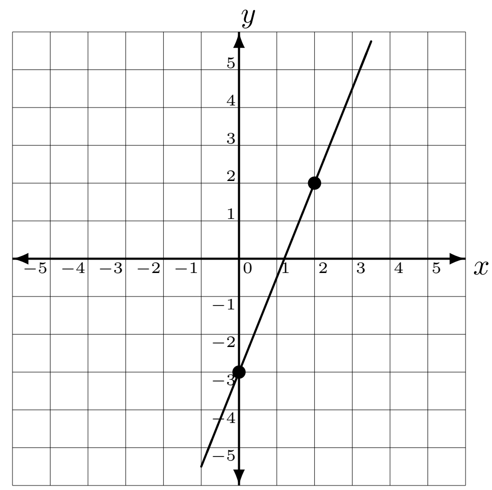
Now we return to graphing a linear equation. However, instead of graphing it by plotting points we will use the slope and y-intercept to graph the line.

Example 8. *Graph $3x - 2y = 6$ by finding the slope and y-intercept.*

Solution 8. *To find the slope and y-intercept of $5x - 2y = 6$, we first must solve for y. So,*

$$\begin{aligned} 5x - 2y &= 6 \\ -2y &= -5x + 6 \\ y &= \frac{5}{2}x - 3 \end{aligned}$$

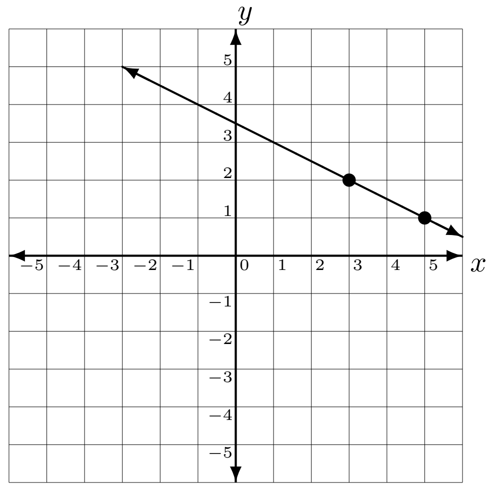
The slope of this line is $\frac{5}{2}$ and the y-intercept is $(0, -3)$. To graph the line using the slope and y-intercept we plot the y-intercept and then think of slope as $\frac{rise}{run} = \frac{5}{2}$. Since the rise is 5 and the run is 2, we start at the y-intercept and rise 5 units up and run 2 units to the right. This gives us a second point on the graph; namely, $(2, 2)$. Connecting these with a straight line, we obtain the following graph of $5x - 2y = 6$.



The advantage the above method has for graphing a line is that it does not require the equation of the line in order to graph it.

Example 9. *Graph the linear equation with* $m = -\dfrac{1}{2}$ *and which passes through* $(3, 2)$.

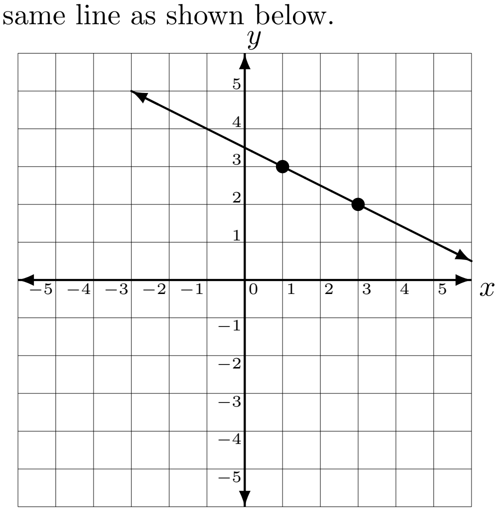
Solution 9. *In order to graph this by plotting points, we would first need to find the equation of the line. However, we can graph this line using just the slope and the given point. First, we plot* $(3, 2)$. *Again, viewing the slope as* $\dfrac{rise}{run} = -\dfrac{1}{2}$, *we see that the rise is* -1 *and the run is* 2. *Therefore, starting at* $(3, 2)$, *we go* 1 *unit down and* 2 *units to the right. Therefore, we end at* $(5, 1)$. *Connecting these two points with a straight line we obtain*



In the previous example, the slope was $-\dfrac{1}{2}$. We know that

$$-\frac{1}{2} = \frac{-1}{2} = \frac{1}{-2}.$$

Will we get a different line if we view the slope as $\dfrac{1}{-2}$? No, but we will get a different second point. To see this, the rise is now 1 so we rise one unit up, and the run is -2 so we go 2 units to the left from the given point $(3, 2)$. Therefore, the second point would be $(1, 3)$. Therefore, a different second point but still the same line as shown below.



Next, let us consider parallel and perpendicular lines. **Parallel lines** are two lines in the same plane that never intersect. Two lines are **perpendicular lines** if they intersect to form a $90°$ angle. However, since graphs can be deceiving, can we tell if two lines are parallel, perpendicular, or neither, without graphing them? The next two facts give us the answer.

Parallel and Perpendicular Lines

- **Parallel lines** have the same slope. So, $m_1 = m_2$.
- **Perpendicular lines** have negative reciprocal slopes. In other words, $m_1 \cdot m_2 = -1$.

Example 10. *Determine whether the following lines are parallel, perpendicular, or neither.*

$$3x - 5y = 10 \quad \textit{and} \quad 5x + 3y = 7$$

Solution 10. *First, we will rewrite each line in slope-intercept form. Once in this form, the slope is the coefficient on x. Solving each equation for y yields*

$$\begin{aligned} 3x - 5y &= 10 \\ -5y &= -3x + 10 \\ y &= \frac{3}{5}x - 2 \end{aligned} \qquad \begin{aligned} 5x + 3y &= 7 \\ 3y &= -5x + 7 \\ y &= -\frac{5}{3}x + \frac{7}{3} \end{aligned}$$

$$m = \frac{3}{5} \qquad m = -\frac{5}{3}$$

Since $\frac{3}{5} \cdot \left(-\frac{5}{3}\right) = -1$*, the two lines are perpendicular.*

Example 11. *Determine whether the following lines are parallel, perpendicular, or neither.*

$$2x + 5y = -9 \quad \textit{and} \quad 6x + 15y = 3$$

Solution 11. *Again, we rewrite each line in slope-intercept form. Solving for y gives us*

$$\begin{aligned} 2x + 5y &= -9 \\ 5y &= -2x - 9 \\ y &= -\frac{2}{5}x - \frac{9}{5} \\ m &= -\frac{2}{5} \end{aligned}$$

$$\begin{aligned} 6x + 15y &= 3 \\ 15y &= -6x + 3 \\ y &= -\frac{6}{15}x + \frac{3}{15} \\ y &= -\frac{2}{5}x + \frac{1}{5} \\ m &= -\frac{2}{5} \end{aligned}$$

Since the slopes are identical, the two lines are parallel.

SECTION 3.4 EXERCISES

(Answers are found on page 399.)

Graph the line that satisfies the following conditions.

1. $m = 2$ and passing through $(1, -1)$.
2. $m = -3$ and passing through $(-4, 3)$.
3. $m = -1$ and passing through $(-2, -3)$.
4. $m = 1$ and passing through $(1, 0)$.
5. $m = \frac{1}{2}$ and passing through $(-3, 1)$.

6. $m = -\frac{1}{2}$ and passing through $(-4, 2)$.

7. $m = -\frac{2}{3}$ and passing through $(0, 2)$.

8. $m = \frac{5}{2}$ and passing through $(0, -4)$.

9. $m = \frac{3}{2}$ and passing through $(-1, -3)$.

10. $m = -\frac{3}{4}$ and passing through $(-2, 1)$.

Calculate the slope of the line passing through the given points.

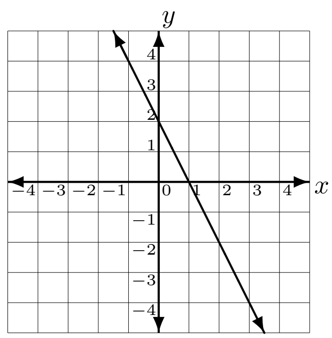
11. $(3, 4)$ and $(2, -1)$
12. $(7, 6)$ and $(-4, -3)$
13. $(2, 7)$ and $(2, 4)$
14. $(-2, -3)$ and $(-4, -5)$
15. $(7, -1)$ and $(-5, -1)$
16. $(-5, -3)$ and $(-7, 4)$
17. $(4, 1)$ and $(3, -6)$
18. $(-5, 2)$ and $(7, -4)$
19. $(1, 4)$ and $\left(\frac{1}{2}, -2\right)$
20. $\left(\frac{1}{2}, -1\right)$ and $\left(-\frac{1}{2}, 3\right)$
21. $\left(-4, -\frac{1}{3}\right)$ and $\left(2, -\frac{2}{3}\right)$
22. $\left(\frac{1}{2}, \frac{1}{4}\right)$ and $\left(-\frac{1}{4}, \frac{1}{2}\right)$

Calculate the slope and y-intercept (do not graph).

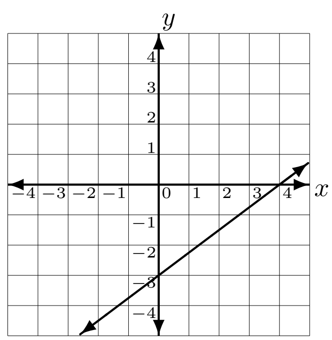
23. $3x + 2y = 10$
24. $-7x + 3y = 12$
25. $14x - 6y = 6$
26. $21x + 6y = 0$
27. $18x + 12y = 1$
28. $8x - 10y = 12$
29. $-3x + 5y = 4$
30. $7x - 9y = 1$
31. $5x + 2y = 13$
32. $-3x - 4y = 9$
33. $4x - 3y = 9$
34. $2x + 5y = 6$

Calculate the slope of the line.

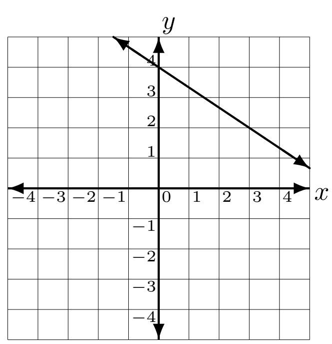
35.



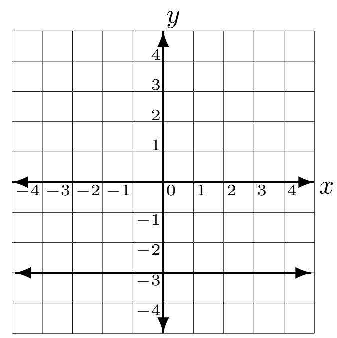
36.



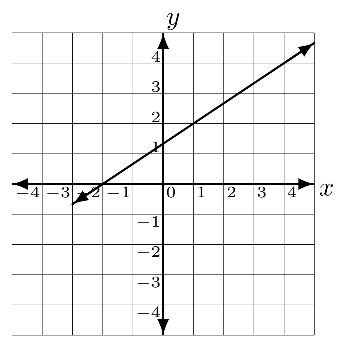
37.



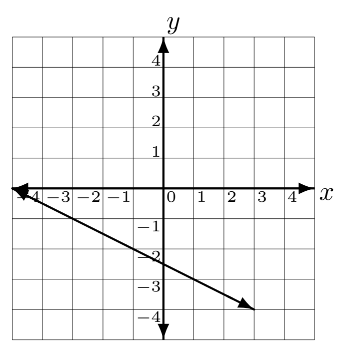
38.



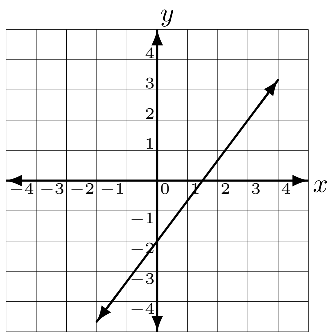
39.



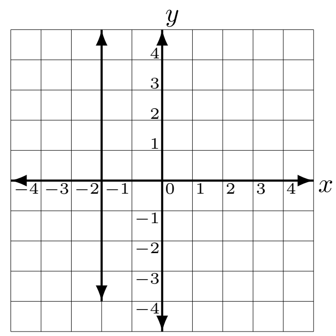
40.



41.



42.



Determine if the following lines are parallel, perpendicular, or neither.

43. $y = \frac{2}{3}x - 7$ and $y = -\frac{3}{2}x + 2$

44. $y = \frac{1}{7}x + 8$ and $y = 7x - 2$

45. $y = 2x - 1$ and $y = 2x + 3$

46. $y = x$ and $y = -x$

47. $y = 4$ and $x = 2$

48. $y = \frac{1}{2}x - 7$ and $y = 2x + 3$

49. $3x - 4y = 2$ and $4x + 3y = 1$

50. $2x - 6y = 3$ and $x - 3y = 4$

51. $-5x + 2y = 6$ and $10x - 4y = 3$

52. $6x + 2y = 4$ and $15x - 5y = 1$

3.5 Equations of Lines

Finally, we are ready to find the equation of a line. We have already noted that a linear equation in two variables can be written as $ax + by = c$ where a, b, and c are real numbers and a and b cannot both be zero. This form, $ax + by = c$, is called the **general form of a line**.

In order to find the equation of any line (that is not horizontal or vertical) we will always need two items: the slope and a point on the line. Once we have these two items, we need to use either the **slope-intercept form** or the **point-slope formula** to find the equation of the line. Although we have already discussed the slope-intercept form, it is stated here again for convenience.

Slope-intercept form: The slope-intercept form of an equation with slope m and $y-$intercept b is given by

$$y = mx + b.$$

Point-slope formula: The equation of the line with slope m and passing through (x_1, y_1) can be found using

$$y - y_1 = m(x - x_1).$$

Either formula can be used to solve for the equation of a line. The next example illustrates the use of each formula.

Example 1. *Find the equation of the line with slope $m = -3$ and which passes through $(5, -2)$.*

Solution 1. *We will illustrate how to find the equation of this line using both formulas. We start with the point-slope formula. Here, $m = -3$ and $(x_1, y_1) = (5, -2)$. Substituting these values* *into the point-slope formula, we obtain*

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - (-2) &= -3(x - 5) \\ y + 2 &= -3x + 15 \\ y &= -3x + 13 \end{aligned}$$

To use the slope-intercept formula, we know that $m = -3$, $x = 5$, and $y = -2$. Substituting these values into the slope-intercept formula, and solving for b, we get

$$\begin{aligned} y &= mx + b \\ -2 &= -3(5) + b \\ -2 &= -15 + b \\ 13 &= b \end{aligned}$$

Now, that we have the value of b, we substitute the value of m and b into the slope-intercept form to get $y = -3x + 13$ as the equation of the given line.

The answer to Example 1 could also be written in general form as $3x + y = 13$. However, we will write all answers using slope-intercept form because of its ease in identifying the slope and y-intercept.

Both the point-slope formula and the slope-intercept formula can be used to find the equation of a line. However, remember that with the slope-intercept formula, you must first find the value of b. So, the slope-intercept formula finds the equation of the line indirectly.

Example 2. *Find the equation of the line with $m = \frac{3}{4}$ and passing through $(-1, 2)$.*

Solution 2. *We will find the equation using the point-slope for-*

mula. Here, $m = \frac{3}{4}$ *and* $(x_1, y_1) = (-1, 2)$*. Thus,*

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \frac{3}{4}(x - (-1)) \\ y - 2 &= \frac{3}{4}x + \frac{3}{4} \\ y &= \frac{3}{4}x + \frac{11}{4} \end{aligned}$$

In order to use either the slope-intercept formula or the point-slope formula, we need to know the slope and a point on the line. However, what if we know only two points on the line? Well, this means that we first need to find the slope of the line by using the slope formula from Section 3.4.

Example 3. *Find the equation of the line passing through* $(-2, 3)$ *and* $(4, -5)$*.*

Solution 3. *First, we must find the slope of the line. The slope formula gives us*

$$m = \frac{-5 - 3}{4 - (-2)} = \frac{-8}{6} = \frac{-4}{3}.$$

Next, we substitute all known information into the point-slope formula to get

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{4}{3}(x - (-2)) \\ y - 3 &= -\frac{4}{3}x - \frac{8}{3} \\ y &= -\frac{4}{3}x + \frac{1}{3}. \end{aligned}$$

Example 4. *Find the equation of the line passing through* $(-7, 2)$ *and has a* $y-$*intercept at* 3*.*

Solution 4. *First, we must find the slope of the line. Remember that a $y-$intercept at 3 translates to the ordered pair $(0,3)$. Using the slope formula on $(-7,2)$ and $(0,3)$, we get*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-3}{-7-0} = \frac{-1}{-7} = \frac{1}{7}$$

Since we know that $m = \frac{1}{7}$ and the y-intercept $b = 3$, we use the slope-intercept formula to get

$$\begin{aligned} y &= mx + b \\ y &= \frac{1}{7}x + 3 \end{aligned}$$

Example 5. *Find the equation of the line which has an $x-$intercept at -2 and a $y-$intercept at 4.*

Solution 5. *Since the line has an x-intercept at -2 and a y-intercept at 4, this means that the line passes through $(-2,0)$ and $(0,4)$. Substituting these values into the slope formula gives*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-0}{0-(-2)} = \frac{4}{2} = 2$$

Again, we know $m = 2$ and the y-intercept $b = 4$, so using the slope-intercept formula gives

$$\begin{aligned} y &= mx + b \\ y &= 2x + 4. \end{aligned}$$

In Section 3.4 we found that the slope of a vertical line is undefined and the slope of a horizontal line is zero. For a every point on the graph of a vertical line, the x-coordinate is the same. Therefore, every **vertical line** has the equation $x = a$ where a is some constant. Likewise, every point on the graph of a horizontal line has exactly the same y-coordinate. Therefore, every **horizontal line** has the equation $y = b$ where b is some constant. These results are summarized below.

Vertical and Horizontal Lines

- **Vertical Lines** always have the equation $x = c$, for some constant c. For example, the equation of the vertical line through (a, b) is $x = a$.
- **Horizontal Lines** always have the equation $y = c$, for some constant c. For example, the equation of the horizontal line through (a, b) is $y = b$.

Example 6. *Find the equation of the vertical line through* $(-6, 3)$.

Solution 6. *Recall that all the* x*-coordinates on a vertical line are the same. Since this line passes through* $(-6, 3)$*, the* x*-coordinate must be* -6*. Hence, the equation of the line is* $x = -6$

Example 7. *Find the equation of the horizontal line through* $\left(\frac{2}{7}, 4\right)$.

Solution 7. *Since all the* y*-coordinates on a horizontal line are the same and this line passes through* $\left(\frac{2}{7}, 4\right)$*, the* y*-coordinate must be* 4*. Hence, the equation of the line is* $y = 4$.

Finally, we discuss finding the equation of parallel and perpendicular lines. In Section 3.4, it was noted that two lines are **parallel** if their slopes are equal and the lines are **perpendicular** if their slopes are negative reciprocals.

Example 8. *Find the equation of the line parallel to the* $2x - 3y = 10$ *and which passing through* $(-8, 3)$.

Solution 8. *First you need to find the slope of the line* $2x-3y = 10$ *by placing it in slope-intercept form. Solving for* y*, we obtain*

$$\begin{aligned} 2x - 3y &= 10 \\ -3y &= -2x + 10 \\ y &= \frac{2}{3}x - \frac{10}{3} \end{aligned}$$

Therefore, the slope of the given line is $m = \frac{2}{3}$. *Since we need a line parallel to this one, the slope of the line we need is also* $m = \frac{2}{3}$. *Substituting* $m = \frac{2}{3}$ *and* $(x_1, y_1) = (-8, 3)$ *into the point-slope formula, we obtain*

$$\begin{aligned} y - 3 &= \frac{2}{3}(x + 8) \\ y - 3 &= \frac{2}{3}x + \frac{16}{3} \\ y &= \frac{2}{3}x + \frac{25}{3}. \end{aligned}$$

Example 9. *Find the equation of the line perpendicular to* $2x + 3y = 8$ *passing through* $(-1, 4)$.

Solution 9. *First, we need to find the slope of the given line by placing it in slope-intercept form. Solving for* y *we get*

$$\begin{aligned} 2x + 3y &= 8 \\ 3y &= -2x + 8 \\ y &= -\frac{2}{3} + \frac{8}{3}. \end{aligned}$$

Therefore, the slope of this line is $m = -\frac{2}{3}$. *Since perpendicular lines have negative reciprocal slopes, the slope of the line perpendicular to* $2x + 3y = 8$ *is* $m = \frac{3}{2}$. *Now, substituting* $m = \frac{3}{2}$ *and*

$(x_1, y_1) = (-1, 4)$ *into the point-slope formula gives*

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= \frac{3}{2}(x - (-1)) \\ y - 4 &= \frac{3}{2}(x + 1) \\ y - 4 &= \frac{3}{2}x + \frac{3}{2} \\ y &= \frac{3}{2}x + \frac{11}{2}. \end{aligned}$$

Example 10. *Find the equation of the line passing through* $(-1, 4)$ *which is perpendicular to the line passing through* $(2, 3)$ *and* $(4, 2)$.

Solution 10. *First, we must find the slope of the line passing through* $(2, 3)$ *and* $(4, 2)$ *using the slope formula. This yields*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 2}{2 - 4} = \frac{1}{-2} = -\frac{1}{2}.$$

Therefore, the slope of the line perpendicular to this one is $m = 2$. *Using the point-slope formula with* $m = 2$ *and* $(x_1, y_1) = (-1, 4)$ *we get*

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 4 &= 2(x - (-1)) \\ y - 4 &= 2(x + 1) \\ y - 4 &= 2x + 2 \\ y &= 2x + 6. \end{aligned}$$

In terms of vertical and horizontal lines, a vertical line is parallel to another vertical line. Similarly, a horizontal line is parallel to another horizontal line. Furthermore, a vertical line is perpendicular to a horizontal line and vice versa.

Example 11. *Find the equation of the line parallel to $x = 3$ and passes through $(-9, 5)$*

Solution 11. *We know that $x = 3$ is a vertical line, and one vertical line is parallel to another vertical line. Therefore, the line that we are looking for is the vertical line through $(-9, 5)$. Therefore, the line we are looking for is $x = -9$.*

Example 12. *Find the equation of the line perpendicular to $y = -2$ through $(1, 5)$.*

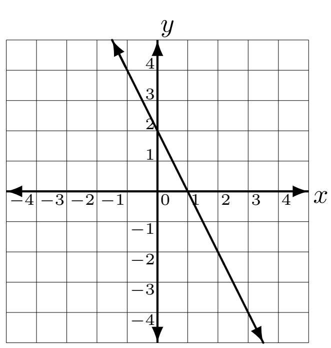
Solution 12. *We know that $y = -2$ is a horizontal line and a vertical line is perpendicular to a horizontal line. Therefore, we are looking for the vertical line through $(1, 5)$. The equation is $x = 1$.*

SECTION 3.5 EXERCISES

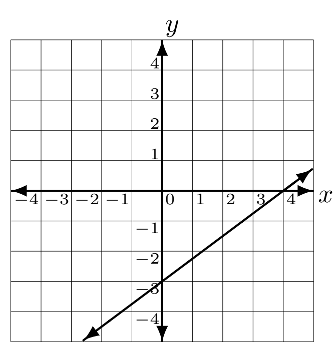
(Answers are found on page 401.)

Find the equation of each line. Write answers in slope-intercept form.

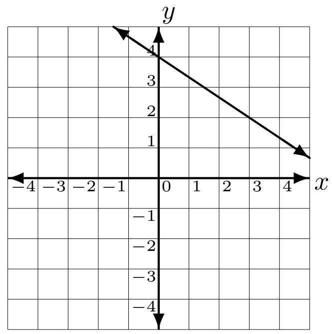
1.



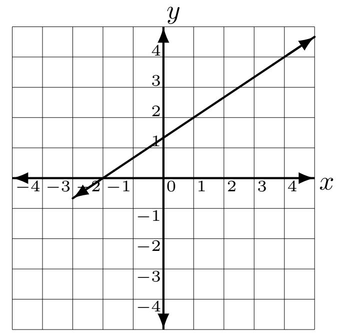
2.



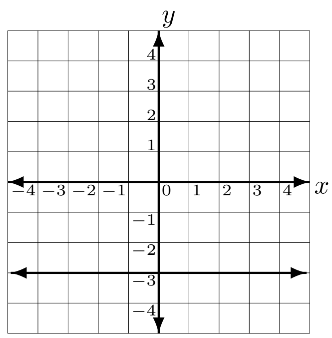
3.



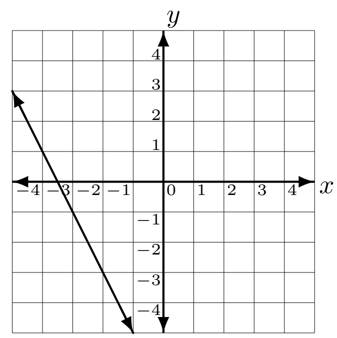
6.



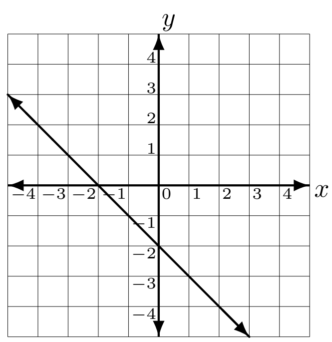
4.



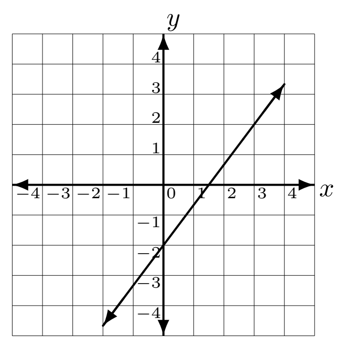
7.



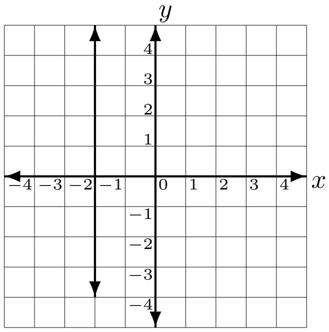
5.



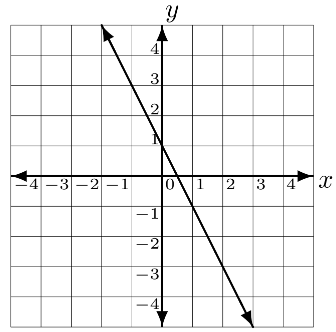
8.



9.



10.



Find the equation of the line having the following properties. Write answers in slope-intercept form.

11. $m = 4$ and passing through $(3, -2)$.

12. $m = -\dfrac{1}{3}$ and passing through $(6, -12)$.

13. $m = 6$; passing through $(2, 2)$

14. $m = -8$; passing through $(-1, -5)$

15. $m = \dfrac{1}{2}$; passing through $(5, -6)$

16. $m = \dfrac{2}{3}$; passing through $(-8, 9)$

17. Passing through the points $(5, -1)$ and $(-10, -7)$.

18. Passing through the points $(7, 5)$ and $(-7, 11)$.

19. passing through $(3, 2)$ and $(5, 6)$

20. passing through $(6, 2)$ and $(8, 8)$

21. passing through $(-1, 3)$ and $(-2, -5)$

22. passing through $(2, 3)$ and $(-1, -1)$

23. passing through $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{3}\right)$

24. vertical line through $(0, 2)$

25. horizontal line through $(1, 4)$

26. horizontal line through $\left(-4, \frac{1}{3}\right)$

27. vertical line through $\left(\frac{-7}{3}, \frac{1}{2}\right)$

28. parallel to $y = 5$, through $(1, 2)$

29. perpendicular to $y = 5$, through $(1, 2)$

30. perpendicular to $x = 3$, through $(-2, 5)$

31. parallel to $x = 7$, through $(-5, 2)$

32. Passing through $(5, 4)$ and parallel to $y = \frac{1}{5}x - 8$.

33. Passing through $(11, 4)$ and perpendicular to $y = \frac{11}{4}x + 3$.

34. Passing through $(-4, 23)$ and perpendicular to $y = x + 7$.

35. Passing through $(-8, -22)$ and parallel to $y = x$.

36. parallel to the line $2x - 3y = 10$, passing through $(-8, 3)$

37. perpendicular to the line $2x - 3y = 10$, passing through $(-8, 3)$

38. parallel to $3x - 4y = 5$, through $(2, -3)$

39. perpendicular to $2x + 3y = 8$, through $(1, -4)$

40. perpendicular to $2x - y = 7$, through $(8, 5)$

41. parallel to $-x + 2y = 10$, through $(-2, -2)$

42. parallel to $-3x + 4y = 10$, through $(-2, 0)$

43. perpendicular to $3x + 5y = 2$, through $(-1, 3)$

44. passing through $(2, 1)$, parallel to the line passing through $(0, 4)$ and $(2, 9)$

45. passing through $(4, -2)$, parallel to the line passing through $(-1, 3)$ and $(3, 2)$

46. passing through $(3, -1)$, perpendicular to the line passing through $(4, -2)$ and $(0, 6)$

47. passing through $(-3, 5)$, perpendicular to the line passing through $(-2, 1)$ and $(3, -2)$

Appendix A

Real Number Operations

This appendix is a summary of what to do when you are adding, subtracting, multiplying or dividing two real numbers. In all the following cases, we will assume you are operating on two numbers of the same type (both fractions, both decimals, etc.), and neither number is zero (since a zero in addition, subtraction and multiplication is easy, while a zero in division is either easy or undefined). Hence, if you wish to multiply a fraction and a decimal, change the fraction to a decimal first (or the decimal to a fraction). The only exceptions are if you are adding or subtracting a mixed number to either an integer or a proper fraction. This is covered in section 1.8 under the shortcuts for mixed number addition and subtraction.

Addition

If the numbers you are adding are:

- fractions (either proper or improper).

  1. Find the LCD of the denominators.
  2. Make both fractions have the LCD for a denominator by creating equivalent fractions.
  3. Combine into a single fraction over the common denominator.
  4. Add the numbers in the numerator (usually integers) according to the appropriate rules.

- integers, mixed numbers or decimals.
    1. Consult the "Sign Rules for Addition" Table to see whether your scratch work should involve addition or subtraction.
    2. Either add or subtract the absolute value of the numbers as indicated by the Table in step 1, and following the appropriate rules for the type of numbers.
    3. Make the sum have the appropriate sign.

Sign Rules for Addition

- If the numbers are both positive, or both negative:
    1. Add the absolute values of the numbers.
    2. The sum should have the same sign as the two addends did originally.
- If one number is positive and one is negative:
    1. Subtract the larger absolute value minus the smaller absolute value.
    2. The sum should have the same sign as the original sign of the number with the larger absolute value.

Subtraction

If the numbers you are subtracting are:

- fractions (either proper or improper).
    1. Find the LCD of the denominators.
    2. Make both fractions have the LCD for a denominator by creating equivalent fractions.
    3. Combine into a single fraction over the common denominator.
    4. Subtract the numbers in the numerator (usually integers) according to the appropriate rules.

- integers, mixed numbers or decimals AND the two numbers are both positive with the larger number first.

  1. Subtract as appropriate for the type of number.

- integers, mixed numbers or decimals AND one or both of the numbers is negative or they are both positive but the smaller number is first.

  1. Change the minus sign to a plus sign, and change the number following the sign to its opposite.
  2. Add according to the addition rules.

Multiplication

If the numbers you are multiplying are:

- mixed numbers.

  1. Change the mixed numbers to improper fractions.
  2. Multiply as listed below.

- integers, fractions or decimals.

  1. Multiply the absolute values of the numbers as appropriate for the type of numbers.
  2. Make the product have the proper sign as listed under the "Sign Rules for Multiplication" Table.

Sign Rules for Multiplication

- If the numbers are both positive, or both negative, the product is **positive**.
- If one number is positive and one is negative, the product is **negative**.

Division

If the numbers you are dividing are:

- mixed numbers.
    1. Change the mixed numbers to improper fractions.
    2. Divide as listed below.
- integers, fractions or decimals.
    1. Divide the absolute values of the numbers as appropriate for the type of numbers.
    2. Make the quotient have the proper sign as listed under the "Sign Rules for Division" Table.

Sign Rules for Division

- If the numbers are both positive, or both negative, the quotient is **positive**.
- If one number is positive and one is negative, the quotient is **negative**.

Appendix B

Multiplying or Dividing Mixed Numbers

When adding, subtracting, multiplying or dividing mixed numbers, you may always convert the mixed numbers into improper fractions. For addition and subtraction, however, we showed in section 1.8 "shortcuts" for adding and subtracting mixed numbers. When we went on to section 1.9, we did not list any such shortcut for multiplication or division of mixed numbers. In this appendix, we will examine what the shortcuts would be, and why they are never used.

<u>**Multiplication**</u>

Recall that the shortcuts arise from remembering the plus sign in the mixed number. We will put the plus signs back in, and then simplify. The following work is a bit beyond what you need to know at this point, but if you remember how to FOIL from high school, you can follow along. Otherwise, just take our word for the work, and see what the shortcut would be.

Example 1. *Multiply* $-2\frac{1}{2} \cdot 1\frac{1}{5}$

Solution 1. *We will ignore the signs for the moment, and just* *multiply the absolute values of the mixed numbers:*

$$\begin{aligned} 2\tfrac{1}{2}\cdot 1\tfrac{1}{5} &= \left(2+\tfrac{1}{2}\right)\cdot\left(1+\tfrac{1}{5}\right) \\ &= 2\cdot 1+2\cdot\tfrac{1}{5}+\tfrac{1}{2}\cdot 1+\tfrac{1}{2}\cdot\tfrac{1}{5} \\ &= 2+\tfrac{2}{1}\cdot\tfrac{1}{5}+\tfrac{1}{2}+\tfrac{1}{10} \\ &= 2+\tfrac{2}{5}+\tfrac{1}{2}+\tfrac{1}{10} \end{aligned}$$

So the shortcut would be to multiply the two integers, multiply the two fractions, and multiply each integer times the other fraction and add it all together. Of course we don't want to leave our answer like this, so we will find the LCD, make all the fractions have this denominator, and add.

$$\begin{aligned} 2\tfrac{1}{2}\cdot 1\tfrac{1}{5} &= 2+\tfrac{2}{5}+\tfrac{1}{2}+\tfrac{1}{10} \\ &= \tfrac{2}{1}+\tfrac{2}{5}+\tfrac{1}{2}+\tfrac{1}{10} \\ &= \tfrac{2\cdot 10}{1\cdot 10}+\tfrac{2\cdot 2}{5\cdot 2}+\tfrac{1\cdot 5}{2\cdot 5}+\tfrac{1}{10} \\ &= \tfrac{20}{10}+\tfrac{4}{10}+\tfrac{5}{10}+\tfrac{1}{10} \\ &= \tfrac{20+4+5+1}{10} \\ &= \tfrac{30}{10} \\ &= 3 \end{aligned}$$

Multiplying a negative number times a positive number gives a negative number.
Therefore, $2\frac{1}{2}\cdot 1\frac{1}{5} = \mathbf{-3}$

Compare this to example 11 on page 155 to see why no one in their right mind uses this "shortcut".

Division

The shortcut for division is even further beyond the scope of this book, but follow along the best you can.

Example 2. *Divide* $2\frac{1}{4} \div 3\frac{1}{3}$

Solution 2. *Putting the plus signs back in gives:*

$$2\tfrac{1}{4} \div 3\tfrac{1}{3} = \frac{2+\frac{1}{4}}{3+\frac{1}{3}}$$

This is a complex fraction. To simplify, we will multiply the big numerator and big denominator by the LCD of all the little fractions.

$$\begin{aligned} 2\tfrac{1}{4} \div 3\tfrac{1}{3} &= \frac{\left(2+\frac{1}{4}\right)\cdot 12}{\left(3+\frac{1}{3}\right)\cdot 12} \\ &= \frac{24+3}{36+4} \\ &= \frac{\mathbf{27}}{\mathbf{40}} \end{aligned}$$

Compare this example to example 12 on page 155, and again you will see why it is much better to just change the mixed numbers to fractions.

Appendix C

Answers to Exercises

Chapter 1

Section 1.2 (Exercises on page 33.)

1. $-8 \quad \leq \quad 4$
2. $22 \quad > \quad 0$
3. $-5 \quad \leq \quad -5$
4. $-18 \quad \leq \quad -7$
5. $56 \quad > \quad 43$
6. $-19 \quad \leq \quad -19$
7. $33 \quad \leq \quad 54$
8. $-14 \quad \leq \quad -4$
9. $197 \quad \leq \quad 197$
10. $-3 \quad > \quad -5$
11. $12 \quad \geq \quad -12$
12. $7 \quad \geq \quad 7$
13. $-3 \quad \geq \quad -9$
14. $15 \quad \geq \quad 13$
15. $0 \quad < \quad 10$
16. $-11 \quad < \quad 1$
17. $25 \quad < \quad 52$
18. $34 \quad \geq \quad 34$
19. $-9 \quad \geq \quad -10$
20. $-2 \quad < \quad -1$
21. 206
22. 34
23. 16
24. 0
25. 28
26. 17
27. 45
28. 91
29. 73

30. 22
31. -52
32. 17
33. 8
34. 0
35. -102
36. 1
37. 13
38. -37
39. -22
40. 9
41. 2
42. 2
43. -9
44. -7
45. -5
46. 3
47. 4
48. -18
49. -6
50. 32
51. $-(-5) \quad > \quad -|-5|$
52. $|-3| \quad \leq \quad |-4|$
53. $-0 \quad \leq \quad |0|$
54. $-(-8) \quad > \quad -7$
55. $-|-6| \quad \leq \quad -|6|$
56. $|-12| \quad > \quad -|12|$
57. $|-3| \quad \leq \quad |3|$
58. $-(-23) \quad > \quad -(-20)$
59. $|0| \quad \leq \quad |-1|$
60. $-|13| \quad \leq \quad -|-2|$
61. $-(-14) \quad \geq \quad -12$
62. $|-45| \quad \geq \quad -(-45)$
63. $-0 \quad \geq \quad |-0|$
64. $17 \quad \geq \quad |-17|$
65. $-|-8| \quad < \quad |-(-8)|$
66. $-1 \quad < \quad -(-1)$
67. $|28| \quad < \quad |-32|$
68. $|-12| \quad < \quad |-48|$
69. $-(-11) \quad \geq \quad -(-10)$
70. $|0| \quad < \quad |-109|$

Section 1.3 (Exercises on page 53.)

1. -3
2. -21
3. 6
4. -29

5. 5
6. −6
7. −14
8. 12
9. 1
10. 19
11. 36
12. −58
13. −58
14. −63
15. 32
16. 44
17. −15
18. −13
19. −36
20. 18
21. −37
22. −10
23. −55
24. 88
25. −35
26. 32
27. −33
28. 35
29. −15
30. −2
31. −36
32. 16
33. −64
34. 3
35. −4
36. −17
37. 50
38. 21
39. −34
40. −11

Section 1.4 (Exercises on page 61.)

1. −28
2. 48
3. −220
4. −66
5. 0
6. 18
7. 408
8. −90
9. 5
10. −8

11. 84
12. 20
13. 0
14. 72
15. −68
16. −42
17. 88
18. −2170
19. −693
20. −6
21. −5
22. 4
23. −3
24. −180
25. 0
26. 3
27. 1
28. undefined
29. −7
30. 2
31. −23
32. −19
33. undefined
34. 9
35. −42
36. 0
37. 18
38. −9
39. undefined
40. 14

Section 1.5 (Exercises on page 76.)

1. 25
2. 81
3. 64
4. 1
5. −1
6. −1
7. 1
8. 9
9. 4
10. 64
11. −27
12. 1
13. 0
14. 1
15. 49
16. 11

17. undefined
18. -64
19. 0
20. 81
21. 1
22. 20
23. 0
24. -25
25. 12
26. -27
27. -1
28. 7
29. -4
30. -5
31. 15
32. -7
33. 1
34. -7
35. 12
36. 6
37. -20
38. 0
39. 2
40. 0

Section 1.6 (Exercises on page 90.)

1. $88 = 2 \cdot 2 \cdot 2 \cdot 11$
2. $70 = 2 \cdot 5 \cdot 7$
3. $65 = 5 \cdot 13$
4. $50 = 2 \cdot 5 \cdot 5$
5. $98 = 2 \cdot 7 \cdot 7$
6. $100 = 2 \cdot 2 \cdot 5 \cdot 5$
7. $108 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$
8. $40 = 2 \cdot 2 \cdot 2 \cdot 5$
9. $135 = 3 \cdot 3 \cdot 3 \cdot 5$
10. $78 = 2 \cdot 3 \cdot 13$
11. GCF(50,70) = 10
12. GCF(70,98) = 14
13. GCF(40,78) = 2
14. GCF(40,108) = 4
15. GCF(88,98) = 2
16. GCF(50,65) = 5
17. GCF(108,135) = 27
18. GCF(65,70) = 5
19. GCF(100,135) = 5
20. GCF(88,100) = 4
21. LCM(50,70) = 350
22. LCM(70,98) = 490
23. LCM(40,78) = 1560

24. LCM(40,108) $= 1080$
25. LCM(88,98) $= 4312$
26. LCM(50,65) $= 650$
27. LCM(108,135) $= 540$
28. LCM(65,70) $= 910$
29. LCM(100,135) $= 2700$
30. LCM(88,100) $= 2200$

Section 1.7 (Exercises on page 122.)

1. $\frac{2}{7}$
2. $-\frac{3}{5}$
3. $-\frac{5}{9}$
4. $\frac{11}{13}$
5. $-\frac{2}{3}$
6. $\frac{21}{22}$
7. $-\frac{5}{6}$
8. $\frac{3}{4}$
9. $-\frac{3}{11}$
10. $-\frac{7}{10}$
11. $\frac{8}{20}$
12. $-\frac{2}{8}$
13. $-\frac{9}{21}$
14. $\frac{30}{60}$
15. $-\frac{12}{18}$
16. $\frac{20}{32}$
17. $-\frac{36}{45}$
18. $\frac{42}{48}$
19. $-\frac{7}{42}$
20. $\frac{16}{72}$
21. $\frac{2}{3}$
22. $-\frac{1}{5}$
23. $-\frac{2}{5}$
24. $\frac{3}{7}$
25. $-\frac{4}{9}$
26. $\frac{5}{8}$
27. $-\frac{3}{5}$
28. $\frac{1}{3}$

29. $\frac{2}{5}$

30. $-\frac{13}{16}$

31. $3\frac{2}{3}$

32. $-7\frac{1}{2}$

33. $14\frac{1}{4}$

34. $-4\frac{3}{8}$

35. $20\frac{2}{3}$

36. $-14\frac{1}{2}$

37. $4\frac{10}{11}$

38. $-11\frac{5}{9}$

39. $27\frac{1}{3}$

40. $-14\frac{3}{5}$

41. $\frac{23}{7}$

42. $-\frac{11}{5}$

43. $\frac{23}{4}$

44. $-\frac{21}{11}$

45. $\frac{35}{3}$

46. $-\frac{47}{8}$

47. $\frac{93}{10}$

48. $-\frac{25}{6}$

49. $\frac{77}{9}$

50. $-\frac{59}{24}$

51. $-\frac{3}{5} \quad \leq \quad \frac{1}{2}$

52. $\frac{4}{9} \quad \leq \quad \frac{11}{24}$

53. $\frac{8}{13} \quad > \quad \frac{3}{5}$

54. $-\frac{5}{7} \quad > \quad -\frac{11}{15}$

55. $\frac{31}{8} \quad > \quad \frac{52}{19}$

56. $-\frac{100}{3} \quad \leq \quad -\frac{65}{2}$

57. $\frac{15}{17} \quad > \quad -\frac{110}{133}$

58. $\frac{11}{5} \quad \leq \quad \frac{9}{4}$

59. $-\frac{5}{6} \quad \leq \quad -\frac{4}{5}$

60. $6 \quad > \quad \frac{53}{9}$

Section 1.8 (Exercises on page 144.)

1. $\frac{9}{8}$ or $1\frac{1}{8}$
2. $\frac{2}{3}$
3. $\frac{1}{2}$
4. $-\frac{1}{4}$
5. $\frac{98}{15}$ or $6\frac{8}{15}$
6. $\frac{7}{4}$ or $1\frac{3}{4}$
7. $\frac{41}{12}$ or $3\frac{5}{12}$
8. $-\frac{17}{6}$ or $-2\frac{5}{6}$
9. $-\frac{11}{9}$ or $-1\frac{2}{9}$
10. $\frac{29}{4}$ or $7\frac{1}{4}$
11. $-\frac{25}{12}$ or $-2\frac{1}{12}$
12. $-\frac{65}{63}$ or $-1\frac{2}{63}$
13. $\frac{1}{2}$
14. $\frac{113}{12}$ or $9\frac{5}{12}$
15. $\frac{41}{12}$ or $3\frac{5}{12}$
16. $-\frac{41}{10}$ or $-4\frac{1}{10}$
17. $-\frac{11}{6}$ or $-1\frac{5}{6}$
18. $\frac{7}{2}$ or $3\frac{1}{2}$
19. $\frac{58}{35}$ or $1\frac{23}{35}$
20. $-\frac{7}{8}$
21. $\frac{41}{24}$ or $1\frac{17}{24}$
22. $-\frac{11}{5}$ or $-2\frac{1}{5}$
23. $-\frac{1}{8}$
24. $\frac{7}{10}$
25. $-\frac{71}{9}$ or $-7\frac{8}{9}$
26. $-\frac{3}{5}$
27. $\frac{41}{42}$
28. $-\frac{17}{2}$ or $-8\frac{1}{2}$
29. $\frac{25}{7}$ or $3\frac{4}{7}$
30. $-\frac{3}{16}$

Section 1.9 (Exercises on page 159.)

1. $\frac{5}{3}$ or $1\frac{2}{3}$
2. $-\frac{11}{7}$ or $-1\frac{4}{7}$

3. $-\frac{1}{9}$
4. $\frac{5}{11}$
5. $-\frac{9}{28}$
6. $\frac{35}{22}$ or $1\frac{13}{22}$
7. none
8. $\frac{1}{84}$
9. $-\frac{17}{12}$ or $-1\frac{5}{12}$
10. $\frac{7}{43}$
11. $\frac{1}{36}$
12. $\frac{2}{15}$
13. $-\frac{27}{64}$
14. $\frac{7}{10}$
15. $-\frac{3}{2}$ or $-1\frac{1}{2}$
16. $-\frac{25}{2}$ or $-12\frac{1}{2}$
17. $\frac{3}{2}$ or $1\frac{1}{2}$
18. 1
19. $-\frac{5}{28}$
20. $-\frac{3}{32}$
21. $\frac{51}{28}$ or $1\frac{23}{28}$
22. $\frac{22}{7}$ or $3\frac{1}{7}$
23. $\frac{2}{9}$
24. 0
25. undefined
26. $-\frac{17}{88}$
27. $\frac{25}{9}$ or $2\frac{7}{9}$
28. -1
29. $\frac{16}{81}$
30. $\frac{4}{9}$
31. 10
32. $-\frac{16}{11}$ or $-1\frac{5}{11}$
33. $-\frac{3}{17}$
34. $\frac{9}{2}$ or $4\frac{1}{2}$
35. -21
36. $\frac{24}{35}$
37. $\frac{4}{9}$
38. $-\frac{1}{4}$
39. -4
40. 1
41. $-\frac{1}{2}$

42. 1
43. undefined
44. $\frac{8}{7}$ or $1\frac{1}{7}$
45. 0
46. $-\frac{2}{9}$
47. $\frac{27}{14}$ or $1\frac{13}{14}$
48. $\frac{7}{10}$
49. $-\frac{1}{6}$
50. $\frac{1}{21}$

Section 1.10 (Exercises on page 186.)

1. 0.5
2. $0.\overline{7}$
3. 0.375
4. 0.09
5. $0.5\overline{3}$
6. $0.\overline{90}$
7. 0.28
8. 0.8125
9. 4.75
10. $0.2\overline{27}$
11. $\frac{37,630,073}{10,000,000}$ or $3\frac{7,630,073}{10,000,000}$
12. $\frac{142,178}{10,000}$ or $14\frac{2,178}{10,000}$
13. $-\frac{900,035}{100,000}$ or $-9\frac{35}{100,000}$
14. $\frac{5,124,992}{1,000,000}$ or $5\frac{124,992}{1,000,000}$
15. $\frac{1,562}{10}$ or $156\frac{2}{10}$
16. $-\frac{5,588}{100}$ or $-55\frac{88}{100}$
17. $\frac{7}{1,000,000}$
18. $\frac{11,011}{1,000}$ or $11\frac{11}{1,000}$
19. $-\frac{187,002}{10,000}$ or $-18\frac{7,002}{10,000}$
20. $\frac{306}{10,000}$
21. 22.08
22. -73.10
23. 1,350.01
24. -3.14
25. 57.00
26. -18.56
27. $1.5\overline{6} \quad \geq \quad 1.56$
28. $-23.\overline{52} \quad < \quad -23.5\overline{2}$
29. $8.67 \quad \geq \quad -8.67$
30. $4.\overline{3} \quad \geq \quad 4.\overline{30}$
31. $-2.9192 \quad < \quad -2.\overline{91}$

32. $\frac{3}{5} \quad \geq \quad 0.6$
33. $\frac{4}{9} \quad < \quad 0.\overline{45}$
34. $\frac{9}{14} \quad < \quad 0.643$
35. $\frac{16}{5} \quad \geq \quad 3.1$
36. $-\frac{1}{6} \quad \geq \quad -0.167$
37. $32.5 \;<\; 32.54343 \;<\; 32.5\overline{43} \;<\; 32.5\overline{4}$
38. $-18.\overline{7} \;<\; -18.7\overline{1} \;<\; -18.7101 \;<\; -18.71$
39. $23.\overline{23} \;<\; 23.\overline{32} \;<\; 32.\overline{23} \;<\; 32.\overline{32}$
40. $-0.1 \;<\; 0.0\overline{1} \;<\; -0.\overline{01} \;<\; -0.0101$
41. $\frac{6}{25}$
42. $\frac{33}{100}$
43. $\frac{1}{20}$
44. $\frac{13}{10}$ or $1\frac{3}{10}$
45. $\frac{19}{100}$
46. $\frac{3}{4}$
47. $\frac{4}{5}$
48. $\frac{1}{100}$
49. $\frac{0}{1}$ or 0
50. $\frac{2}{1}$ or 2
51. 0.24
52. 0.33
53. 0.05
54. 1.3
55. 0.19
56. 0.75
57. 0.8
58. 0.01
59. 0
60. 2
61. 52%
62. 3.1%
63. 18%
64. 250%
65. 999%
66. 0.5%
67. 70%
68. 44%
69. 22.5%

70. 100%
71. $33.\overline{3}\%$
72. 50%
73. 25%
74. 20%
75. $16.\overline{6}\%$
76. 75%
77. $55.\overline{5}\%$
78. 8%
79. 70%
80. 34%

Section 1.11 (Exercises on page 205.)

1. 765.3223
2. 37.4944
3. 4.64
4. 771.327
5. -14.177
6. -4.325
7. -272.832
8. 208.486
9. 13.46
10. 269.795
11. 256.851
12. 198.418
13. -29.5097
14. -73.14
15. -10.22
16. 337.41
17. 21.835
18. -155.76
19. -12.65
20. 584.693
21. 124.845
22. 0.7955
23. -241.92
24. 3.7
25. -2.0832
26. 9.792
27. 263.11
28. -0.8194
29. -28.52
30. -0.0924
31. $1,737.2$
32. -4.72
33. $-4,252.834$
34. $2,400.23$
35. -339.61
36. $20,700,340$

37. -201.9
38. 1.56
39. $320,100$
40. -200.5
41. 32.6
42. 2.342
43. -1.54
44. -200.4
45. 0.314
46. 13.25
47. -3.22
48. -2.005
49. 3.002
50. -51.8
51. 1.3
52. 3.36
53. 2.7
54. 4.2
55. 69
56. 17.76
57. 73.8
58. 25.74
59. 17.5
60. 19

Section 1.12 (Exercises on page 215.)

1. 7
2. no real number
3. -7
4. 5
5. -5
6. -5
7. 8
8. 4
9. no real number
10. -2
11. no real number
12. -2
13. -6
14. -6
15. 22
16. 16
17. 24
18. -27
19. no real number
20. -42

Section 1.13 (Exercises on page 226.)

1. -3
2. 8
3. -2.9
4. 4.1
5. $-\frac{1}{6}$
6. $\frac{2}{17}$
7. $-\frac{1}{2}$
8. $\frac{1}{18}$
9. -4
10. $\frac{15}{13}$ or $1\frac{2}{13}$
11. $-\frac{3}{2}$ or $-1\frac{1}{2}$
12. $\frac{1}{45}$
13. Associative Property of Addition
14. Additive Inverse
15. Distributive Property
16. Multiplicative Identity
17. Commutative Property of Multiplication
18. Additive Inverse
19. Associative Property of Multiplication
20. Additive Identity
21. Commutative Property of Addition
22. Distributive Property
23. Multiplicative Inverse
24. Commutative Property of Multiplication
25. Associative Property of Addition
26. Commutative Property of Addition
27. Multiplicative Inverse
28. Associative Property of Multiplication
29. Multiplicative Identity
30. Additive Identity
31. 5.15
32. 511
33. 7
34. 270
35. 301
36. 250

Chapter 2

Section 2.1 (Exercises on page 239.)

1. $3x^5 - 4x^4 + x^2$; this expression has three terms.
   The first term is: $3x^5$, and this is a **variable term** with coefficient $= 3$.
   The second term is: $-4x^4$, and this is a **variable term** with coefficient $= -4$.
   The third term is: x^2, and this is a **variable term** with coefficient $= 1$.

2. $xy - 2yz$; this expression has two terms.
   The first term is: xy, and this is a **variable term** with coefficient $= 1$.
   The second term is: $-2yz$, and this is a **variable term** with coefficient $= -2$.

3. $5 - x$; this expression has two terms.
   The first term is: 5, and this is a **constant term**.
   The second term is: $-x$, and this is a **variable term** with coefficient $= -1$.

4. $13x^2 - 11x + 1$; this expression has three terms.
   The first term is: $13x^2$, and this is a **variable term** with coefficient $= 13$.
   The second term is: $-11x$, and this is a **variable term** with coefficient $= -11$.
   The third term is: 1, and this is a **constant term**.

5. $x + y + xy$; this expression has three terms.
   The first term is: $3x^5$, and this is a **variable term** with coefficient $= 3$.
   The second term is: $-4x^4$, and this is a **variable term** with coefficient $= -4$.
   The third term is: x^2, and this is a **variable term** with coefficient $= 1$.

6. $8 + 3x^2y - 2xy^2$; this expression has three terms.
   The first term is: 8, and this is a **constant term**.
   The second term is: $3x^2y$, and this is a **variable term** with coefficient $= 3$.
   The third term is: $-2xy^2$, and this is a **variable term** with coefficient $= -2$.

7. -6
8. -2
9. -1
10. -8
11. $-1|$
12. 11
13. -3
14. 7
15. 13
16. -1
17. $\dfrac{2}{3}$
18. -5
19. 6
20. 1
21. $11x - 4$
22. $-m + 4n$
23. $2x - y + 2$
24. $-x - 2y$
25. $-8x + 17$
26. $-2x^2 + x - 10$
27. 3
28. $11a$
29. $4x + 3$
30. $-4xy - 6x + 2y$
31. $7x - 14$
32. $-4x - 3$
33. $5m + n$
34. $x + 33$

Section 2.2 (Exercises on page 257.)

1. No. Move all terms containing a variable to one side of the equation. Answer should be $x = 0$
2. $x = \dfrac{8}{3}$ or $2\dfrac{2}{3}$
3. $x = 20$
4. $x = 3$
5. $x = -23$
6. $x = -14$
7. All real numbers
8. $x = 7$
9. $x = -4.6$
10. $x = -4$
11. No solution
12. $x = -\dfrac{4}{7}$
13. $x = \dfrac{35}{76}$
14. $x = \dfrac{92}{33}$ or $2\dfrac{26}{33}$
15. No solution

16. $x = \frac{47}{7}$ or $6\frac{5}{7}$

17. $x = \frac{51}{7}$ or $7\frac{2}{7}$

18. $x = \frac{21}{17}$ or $1\frac{4}{17}$

19. $x = \frac{3}{5}$

20. $x = \frac{13}{4}$ or $3\frac{1}{4}$

21. $x = \frac{38}{23}$ or $1\frac{15}{23}$

22. $x = \frac{53}{29}$ or $1\frac{24}{29}$

23. $x = \frac{68}{7}$ or $9\frac{5}{7}$

24. $x = \frac{29}{15}$ or $1\frac{14}{15}$

25. $x = -\frac{1}{3}$

26. All real numbers

27. $x = -\frac{24}{5}$ or $-4\frac{4}{5}$

28. $x = -\frac{7}{5}$ or $-1\frac{2}{5}$

29. No solution

30. $x = -\frac{13}{8}$ or $-1\frac{5}{8}$

31. $x = -1$

32. $x = 0.2$

Section 2.3 (Exercises on page 268.)

1. $5 - 2x = 3(x - 5)$; $\quad x = 4$

2. $2x + 7 = x - 6$; $\quad x = -13$

3. $5 \cdot x = 2(x - 3)$; $\quad x = -2$

4. $2(x - 3) = 3 - x$; $\quad x = 3$

5. $x + (2x - 3) = 15$;
   First piece $= 6$ feet;
   Second piece $= 9$ feet.

6. $5x + x = 12$;
   First piece $= 10$ feet;
   Second piece $= 2$ feet.

7. $(2x - 1) + x + (3x + 1) = 12$;
   First piece $= 3$ feet;
   Second piece $= 2$ feet;
   Third piece $= 7$ feet.

8. $x + (2x - 1) + (x + 2)$;
   First piece $= 4$ feet;
   Second piece $= 7$ feet;
   Third piece $= 6$ feet.

9. $x + (x + 7,000) = 57,000$;
   Waitress earns $\$25,000$;
   Cook earns $\$32,000$.

10. $x + 2x = 81,000$;
    Secretary earns $\$27,000$;
    Owner earns $\$54,000$.

11. $x + (x + 1) = 297$;
    Page numbers are 148 and 149.

12. $x + (x + 5) = 29$;
    First CD cost \$12; Second CD cost \$17.

13. $x + (x + 9) = 241$;
    First game $= 116$;
    Second game $= 125$.

14. $(x + 4) + x = 86$;
    Front nine score $= 45$;
    Back nine score $= 41$.

Section 2.4 (Exercises on page 283.)

1. 8 ounces for \$1.29.
2. 64 ounces for \$4.69.
3. 12 for \$0.85.
4. 8 ounces for \$3.99.
5. 5 cups
6. \$51.43
7. 25 minutes
8. $1\dfrac{7}{8}$ cups of flour
9. 7 inches
10. 35 gallons
11. 14 pounds
12. $\$1,174.90$
13. 0.8365 cm
14. 4048 yards
15. 1.24 miles
16. 55.8 mi/hr
17. 446.4 mi/hr
18. 99 pounds
19. $150,000\ mm^2$
20. $0.77\ yd^3$

Section 2.5 (Exercises on page 295.)

1. $[-3, \infty)$
2. $(2, 8]$
3. $(-\infty, 7)$
4. $\left[3, \frac{11}{2}\right]$
5. $\left[\frac{3}{4}, \infty\right)$
6. $(-\infty, 14]$
7. $[-5, \infty)$
8. $\left[-\frac{2}{5}, \infty\right)$
9. $[42, \infty)$
10. $(-\infty, 8]$
11. $\left(-\infty, \frac{19}{4}\right)$
12. $\left[-\frac{7}{4}, \infty\right)$
13. $(-\infty, 6)$
14. $\left(-\infty, \frac{5}{4}\right]$
15. $\left(-\infty, \frac{71}{150}\right]$
16. $\left(-\frac{16}{13}, \infty\right)$
17. $(-\infty, 42]$
18. $(-\infty, 20]$
19. $(-\infty, -6]$
20. $\left(-\infty, \frac{76}{11}\right)$
21. $[3, 4]$
22. $\left[\frac{8}{15}, \frac{23}{15}\right)$
23. $\left[\frac{4}{3}, \frac{25}{6}\right)$
24. $\left(-\frac{5}{3}, \frac{25}{6}\right]$
25. $\left[-\frac{18}{5}, \frac{38}{5}\right)$

26. $\left[\frac{23}{12}, \frac{37}{8}\right)$

27. They would need to drive more than 2800 miles.

28. The average weekly sales would need to be less than $\$8,333.33$.

29. (a) when $3 < h \leq 4$.
    (b) when $h > 9$
    (c) when $h \leq 6$

Chapter 3

Section 3.1 (Exercises on page 309.)

1. solution
2. not a solution
3. not a solution
4. solution
5. (a) $(-4, -\frac{29}{2})$
   (b) $(2, \frac{1}{2})$
   (c) $(0, -\frac{9}{2})$
   (d) $(\frac{9}{5}, 0)$
6. (a) $\left(\frac{11}{7}, -1\right)$
   (b) $(2, -2)$
   (c) $\left(\frac{2}{7}, 2\right)$
   (d) $\left(1, \frac{1}{3}\right)$
7. (a) $\left(0, -\frac{7}{5}\right)$
   (b) $(2, 1)$
   (c) $\left(\frac{1}{3}, -1\right)$
   (d) $\left(\frac{3}{2}, \frac{2}{5}\right)$
8. (a) $(4, 0)$
   (b) $(-4, 3)$
   (c) $(-12, 6)$
   (d) $\left(\frac{8}{3}, \frac{1}{2}\right)$
9.

x	y
1	$\frac{7}{3}$
$\frac{1}{4}$	$\frac{1}{3}$
-2	$-\frac{17}{3}$

10.

x	y
0	$\frac{2}{3}$
-3	$-\frac{19}{3}$
$\frac{1}{2}$	$\frac{11}{6}$

11.

x	y
-5	-2
-8	-4
-1	$\frac{2}{3}$
7	6

12.

x	y
0	-4
18	2
10	$-\frac{2}{3}$
5	$-\frac{7}{3}$

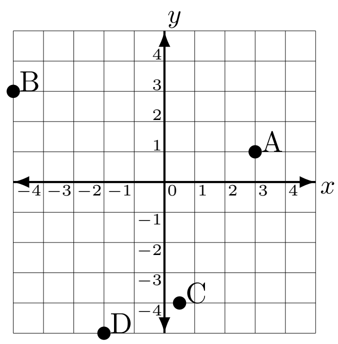
13.

x	y
$\frac{2}{3}$	-3
-3	$\frac{-17}{2}$
4	2
1	$-\frac{5}{2}$

14.

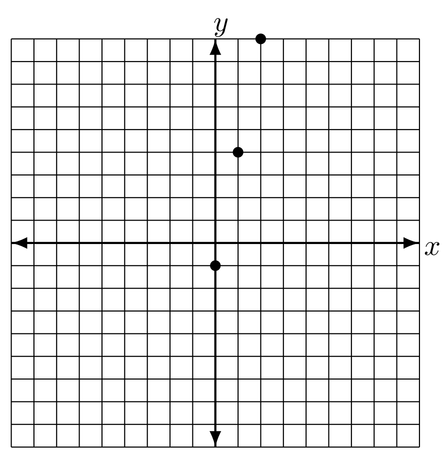
x	y
$-\frac{3}{2}$	-2
0	$-\frac{7}{5}$
6	1
-4	-3

15.

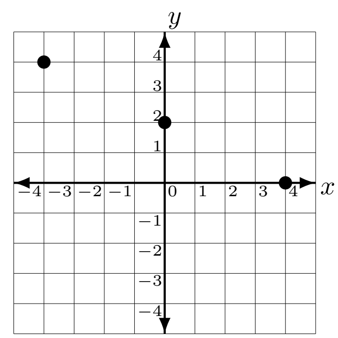


For exercises #16–#25 there are many solutions. Given below is one example for each problem.

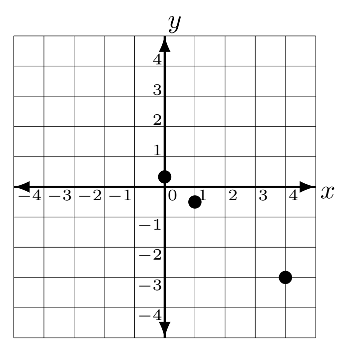
16. $(0, -1)$, $(1, 4)$, $(2, 9)$



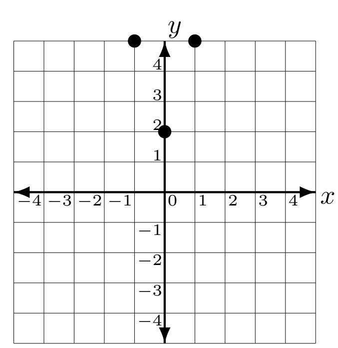
17. $(0, 2)$, $(4, 0)$, $(-4, 4)$



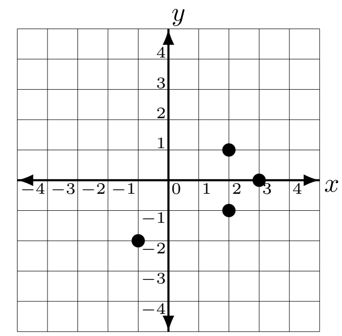
18. $\left(0, \frac{1}{3}\right)$, $\left(1, -\frac{1}{2}\right)$, $(4, -3)$



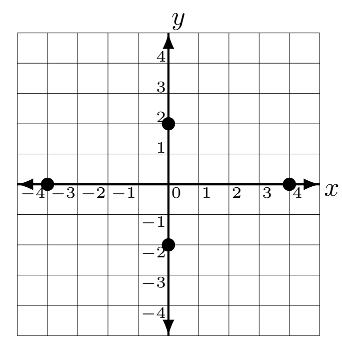
19. $(0, 2)$, $(-1, 5)$, $(1, 5)$



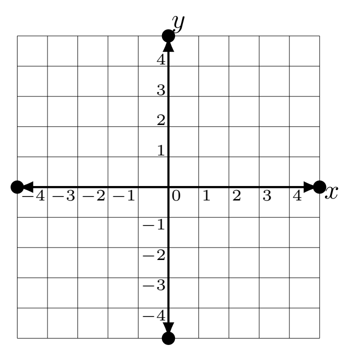
20. $(3,0),\ (2,1),\ (2,-1),\ (-1,-2)$



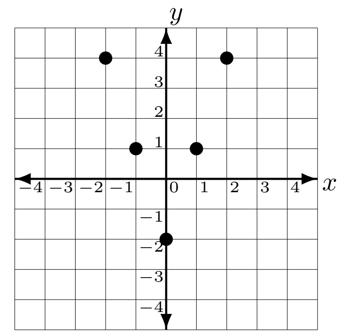
21. $(0,2),\ (0,-2),\ (-4,0),\ (4,0)$



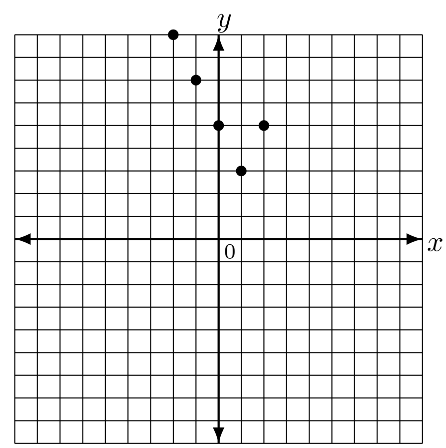
22. $(0,5),\ (0,-5),\ (-5,0),\ (5,0)$



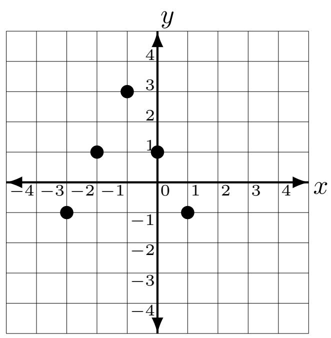
23. $(0,-2),\ (1,1),\ (-1,1),\ (2,4),\ (-2,4)$



24. $(0,5),\ (-1,7),\ (1,3),\ (2,5),\ (-2,9)$

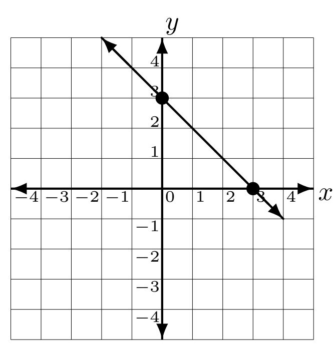


25. $(-1,3),\ (0,1),\ (-2,1),\ (1,-1),\ (-3,-1)$

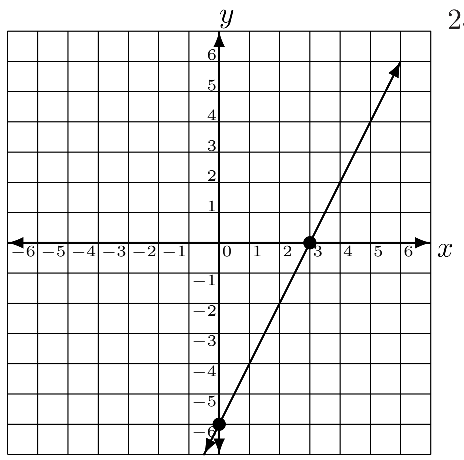


Section 3.2 (Exercises on page 321.)

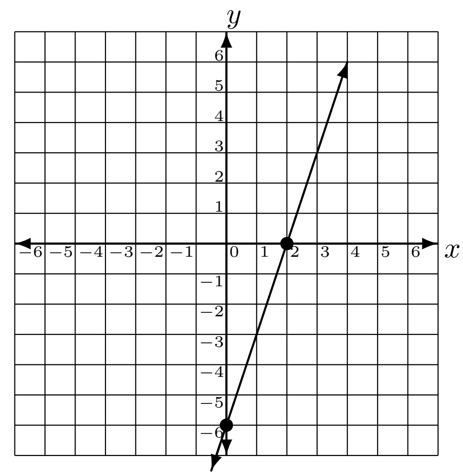
1. $(-2,0),\quad (0,1)$
2. $(2,0),\quad (0,3)$
3. $(4,0),\quad (0,-2)$
4. $(-4,0),\quad (0,-1)$
5. $(-2,0)$, no y-intercept
6. $(0,1)$, no x-intercept
7. $(2,0),\quad (0,-4)$
8. $(2,0),\quad (0,-6)$
9. $(2,0),\quad (0,4)$
10. $(2,0),\quad (0,-1)$
11. $(2,0),\quad \left(0,\frac{4}{5}\right)$
12. $\left(\frac{4}{3},0\right),\quad (0,-1)$
13. $\left(\frac{1}{3},0\right),\quad \left(0,-\frac{1}{5}\right)$
14. $(5,0),\quad \left(0,-\frac{10}{3}\right)$
15. $(3,0),\quad \left(0,-\frac{12}{5}\right)$
16. $\left(\frac{1}{2},0\right),\quad \left(0,-\frac{3}{5}\right)$
17. $\left(\frac{1}{2},0\right),\quad (0,-2)$
18. $\left(-\frac{2}{7},0\right),\quad \left(0,\frac{2}{3}\right)$
19. $\left(\frac{5}{6},0\right),\quad \left(0,-\frac{5}{3}\right)$
20. $\left(\frac{11}{9},0\right),\quad \left(0,\frac{11}{2}\right)$
21.



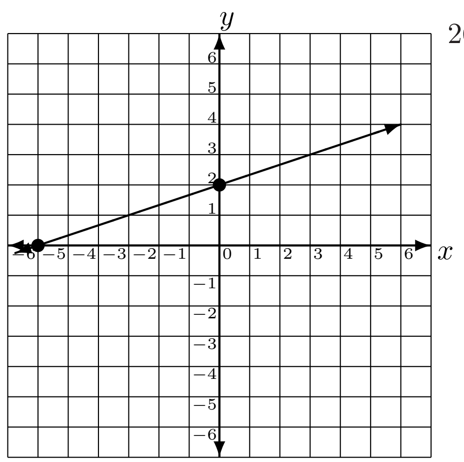
22.



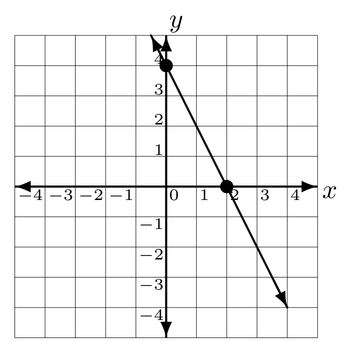
25.



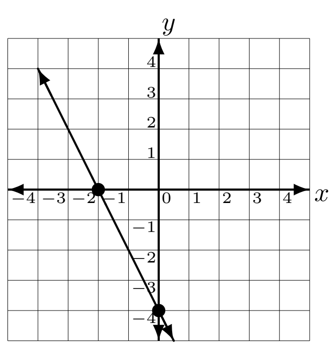
23.



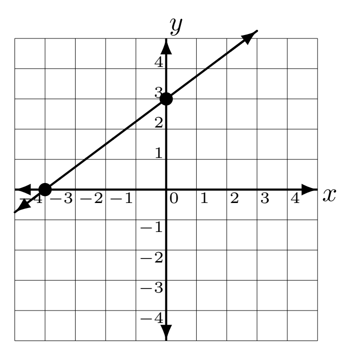
26.



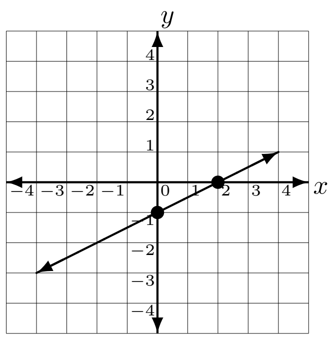
24.



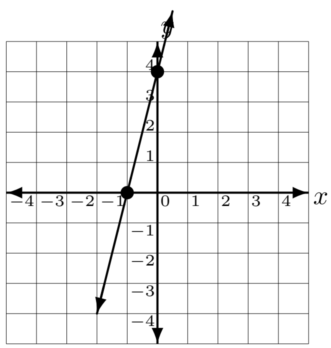
27.



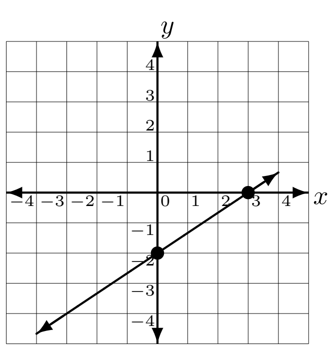
28.


y
x
−4 −3 −2 −1 0 1 2 3 4
4 3 2 1 −1 −2 −3 −4

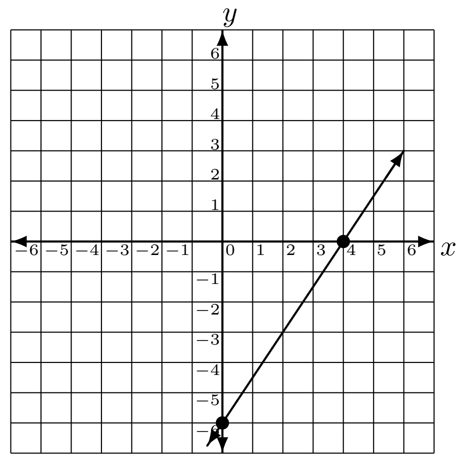
29.


y
x
−4 −3 −2 −1 0 1 2 3 4
4 3 2 1 −1 −2 −3 −4

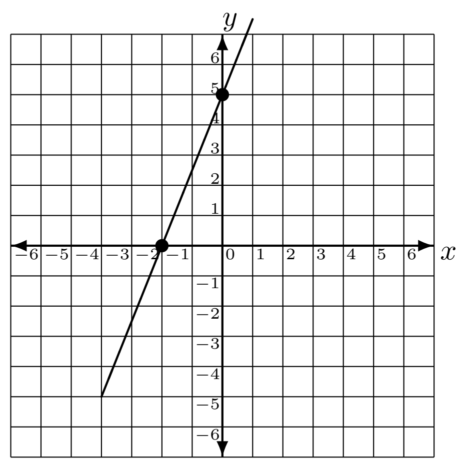
30.


y
x
−4 −3 −2 −1 0 1 2 3 4
4 3 2 1 −1 −2 −3 −4

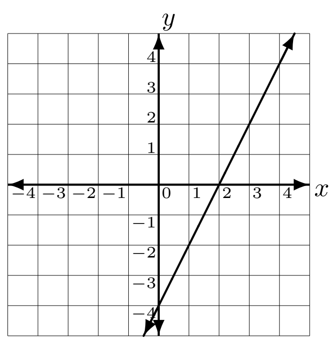
31.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

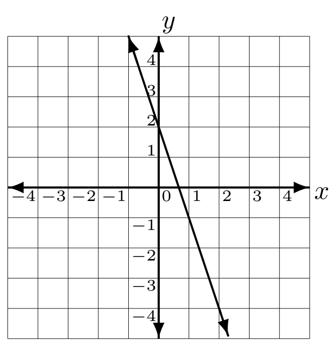
32.


y
x
−6 −5 −4 −3 −2 −1 0 1 2 3 4 5 6
6 5 4 3 2 1 −1 −2 −3 −4 −5 −6

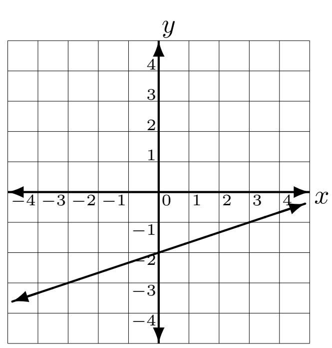
33.



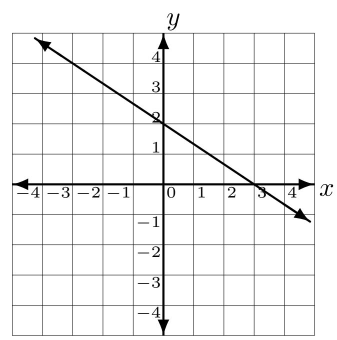
34.



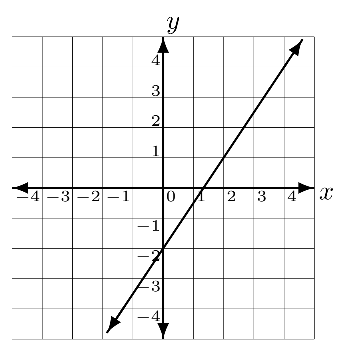
35.



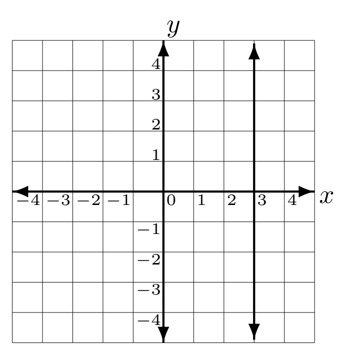
36.



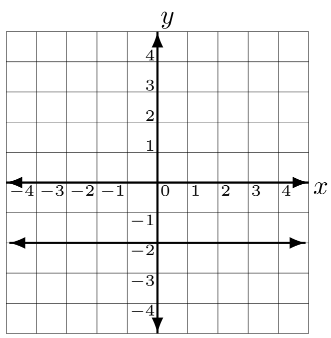
37.



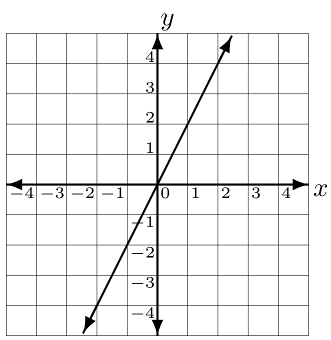
38.



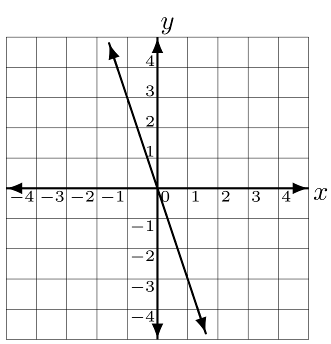
39.



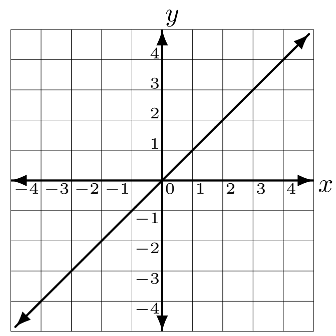
40.



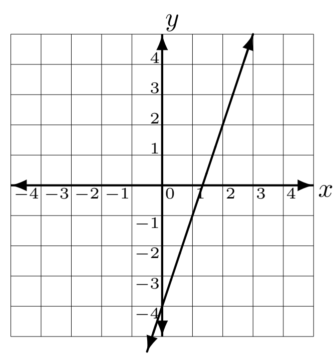
41.



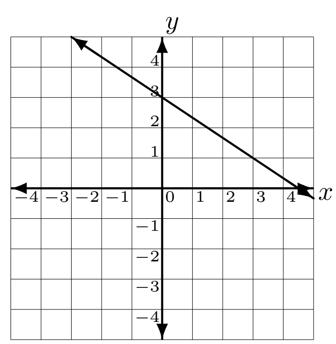
42.



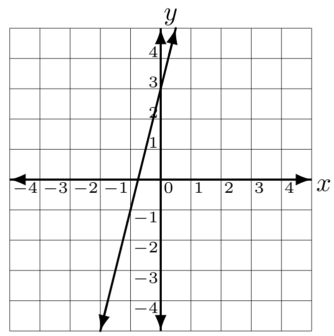
43.



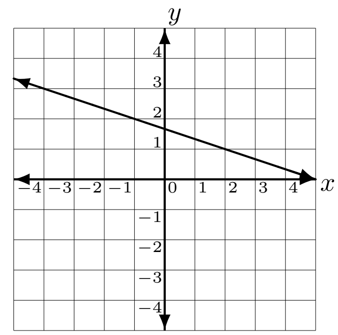
44.



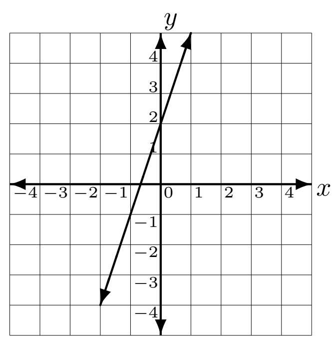
45.



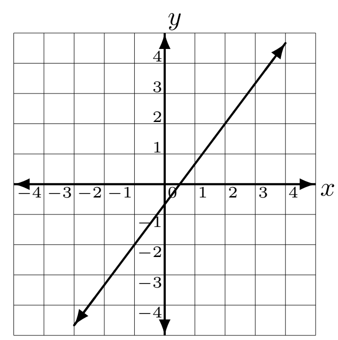
47.



46.



48.



Section 3.3 (Exercises on page 328.)

1. (a) cost of item x

   (b) sales tax amount S

   (c) $S(12) = 0.78$. The sales tax amount for an item costing \$12 is \$0.78.

   (d) $D(20) = \$1.30$

2. (a) time t hours

   (b) distance D in miles

   (c) $D(2) = 110$ miles

   (d) $D(3) = 165$. In 3 hours, the car travels 165 miles.

3. (a) number of hours h spent mowing

   (b) area of lawn A remaining to be mowed.

   (c) $A(2) = 900$. After mowing for 2 hours, 900 square yds remains to be mowed.

(d) $A(4) = 300$ sq yds

4. (a) number of hours t on the job
   (b) the charges C
   (c) $C(2) = 140$. For 2 hours on the job, the charge is \$140.
   (d) $C(3) = \$185$.

5. (a) dollars d charged per space.
   (b) number N of spaces rented.
   (c) $N(10) = 160$. The manager can rent 160 spaces if she charges \$10 per rental space.
   (d) $N(20) = 120$ spaces.

6. (a) number x of baseball caps
   (b) total cost C
   (c) $C(400) = 5080$. The total cost for 400 baseball caps is \$5080.
   (d) $C(350) = \$4480$.

7. (a) the number of ornaments x
   (b) the profit P
   (c) $P(200) = 1800$. A profit of \$1800 is made from the product and sale of 200 ornaments.
   (d) $P(350) = \$4050$.

8. (a) x microwaves
   (b) production cost C
   (c) $C(150) = 3900$. The production cost for 150 microwaves is \$3900.
   (d) $C(75) = \$3450$

9. (a) x computers
   (b) revenue R
   (c) $R(8) = 12000$. The revenue from the sale of 8 computers is $\$12,000$.
   (d) $R(10) = \$15,000$

10. (a) number of years t since reservoir constructed
    (b) water level W measured in feet.
    (c) $W(4) = 50$. Four years after reservoir constructed, the water level is 50 feet.
    (d) $W(7) = 65$ feet

11. (a) width w measured in meters
    (b) perimeter P measured in meters
    (c) $P(12) = 52$. The perimeter of the garden with width 12 meters is 52 meters.
    (d) $P(20) = 68$ meters.

12. (a) minutes m
    (b) monthly cost A in dollars
    (c) $A(100) = 14.25$. The monthly cost for 100 minutes is \$14.25.
    (d) $A(250) = \$27.75$

13. (a) time t in seconds
    (b) height H of the rocket

(c) $H(2) = 116$. The height of the rocket at 2 seconds is 116 feet.
(d) $H(3) = 76$ feet

14. (a) width w measured in feet
(b) area A measured in square feet
(c) $A(4) = 16$. When the width is 4 feet the area is 16 square feet.
(d) $A(2) = 12$ square feet

15. (a) length s of the side in meters
(b) area A of the square in square meters
(c) $A(10) = 100$. The area of a square with side 10 meters is 100 square meters.
(d) $A(13) = 169$ sq meters

16. (a) -3
(b) 6
(c) -12

17. (a) 12
(b) 120
(c) 180

18. (a) 50
(b) 250
(c) -50

19. (a) 8
(b) -4
(c) 2

20. (a) -2
(b) -14
(c) 6

21. (a) 155
(b) 11
(c) 35

22. (a) 3
(b) -1
(c) 13

23. (a) -7
(b) -1
(c) -5

24. (a) 56
(b) 31
(c) 40

25. (a) 1
(b) 9
(c) 2

26. (a) 4
(b) 10
(c) 6

27. (a) 3
(b) $\frac{5}{2}$
(c) 1

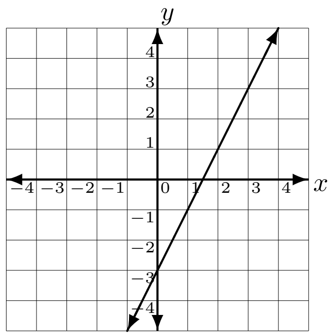
28. (a) 3
(b) $\frac{7}{2}$
(c) 5

29. (a) 5
(b) 0
(c) -4

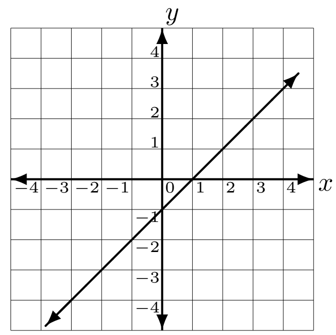
30. (a) -5
(b) 0
(c) 4

Section 3.4 (Exercises on on page 344.)

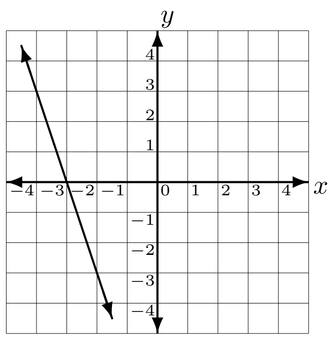
1.



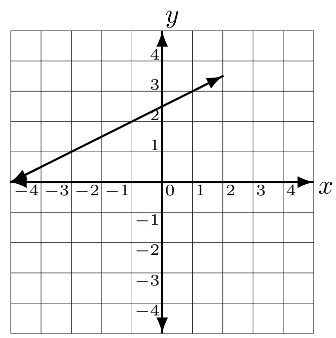
4.



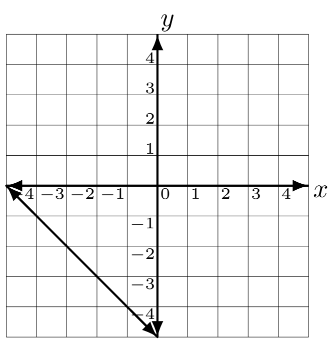
2.



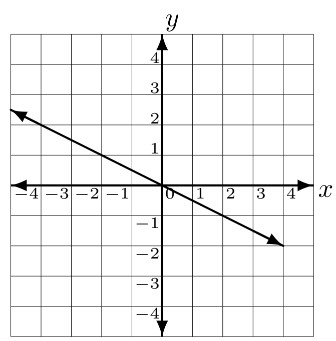
5.



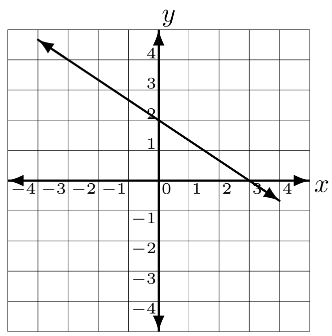
3.



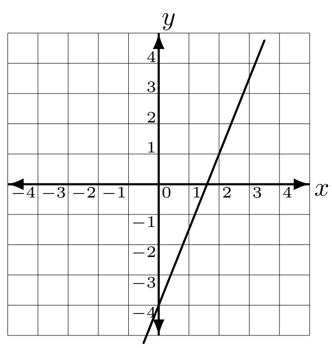
6.



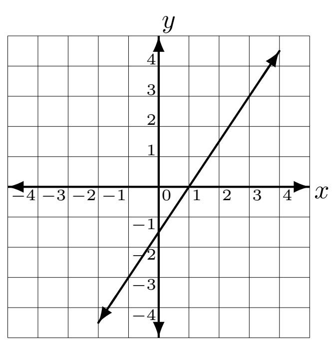
7.



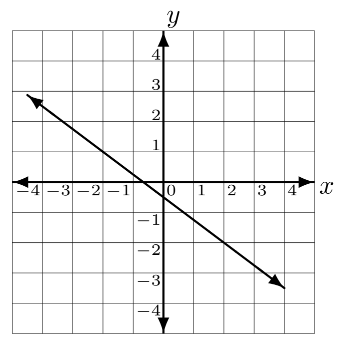
8.



9.



10.



11. $m = 5$

12. $m = \dfrac{9}{11}$

13. slope undefined

14. $m = 1$

15. $m = 0$

16. $m = -\dfrac{7}{2}$

17. $m = 7$

18. $m = -\dfrac{1}{2}$

19. $m = 12$

20. $m = -4$

21. $m = -\dfrac{1}{18}$

22. $m = -\dfrac{1}{3}$

23. $m = -\dfrac{3}{2}, \ \ (0, 5)$

24. $m = \dfrac{7}{3}, \ \ (0, 4)$

25. $m = \dfrac{7}{3}, \ \ (0, -1)$

26. $m = -\dfrac{7}{2}, \ \ (0, 0)$

27. $m = -\frac{3}{2}$, $\left(0, \frac{1}{12}\right)$
28. $m = \frac{4}{5}$, $\left(0, -\frac{6}{5}\right)$
29. $m = \frac{3}{5}$, $\left(0, \frac{4}{5}\right)$
30. $m = \frac{7}{9}$, $\left(0, -\frac{1}{9}\right)$
31. $m = -\frac{5}{2}$ $\left(0, \frac{13}{2}\right)$
32. $m = -\frac{3}{4}$ $\left(0, -\frac{9}{4}\right)$
33. $m = \frac{4}{3}$ $(0, -3)$
34. $m = -\frac{2}{5}$ $\left(0, \frac{6}{5}\right)$
35. $m = -2$
36. $m = \frac{3}{4}$
37. $m = -\frac{2}{3}$
38. $m = 0$
39. $m = \frac{2}{3}$
40. $m = -\frac{1}{2}$
41. $\frac{4}{3}$
42. slope undefined
43. perpendicular
44. neither
45. parallel
46. perpendicular
47. perpendicular
48. neither
49. perpendicular
50. parallel
51. parallel
52. neither

Section 3.5 (Exercises on page 355.)

1. $y = -2x + 2$
2. $y = \frac{3}{4}x - 3$
3. $y = -\frac{2}{3}x + 4$
4. $y = -3$
5. $y = -x - 2$
6. $y = \frac{2}{3}x + \frac{4}{3}$
7. $y = -2x - 7$
8. $y = \frac{4}{3}x - 2$
9. $x = -2$
10. $y = -2x + 1$

11. $y = 4x - 14$
12. $y = -\frac{1}{3}x - 10$
13. $y = 6x - 10$
14. $y = -8x - 13$
15. $y = \frac{1}{2}x - \frac{17}{2}$
16. $y = \frac{2}{3}x + \frac{43}{3}$
17. $y = \frac{2}{5}x - 3$
18. $y = -\frac{3}{7}x + 8$
19. $y = 2x - 4$
20. $y = 3x - 16$
21. $y = 8x + 11$
22. $y = \frac{4}{3}x + \frac{1}{3}$
23. $y = \frac{2}{3}x$
24. $x = 0$
25. $y = 4$
26. $y = \frac{1}{3}$
27. $x = -\frac{7}{3}$
28. $y = 2$
29. $x = 1$
30. $y = 5$
31. $x = -5$
32. $y = \frac{1}{5}x + 3$
33. $y = -\frac{4}{11}x + 8$
34. $y = -x + 19$
35. $y = x - 14$
36. $y = \frac{2}{3}x + \frac{25}{3}$
37. $y = -\frac{3}{2}x - 9$
38. $y = \frac{3}{4}x - \frac{9}{2}$
39. $y = \frac{3}{2}x - \frac{11}{2}$
40. $y = -\frac{1}{2}x + 9$
41. $y = \frac{1}{2}x - 1$
42. $y = \frac{3}{4}x + \frac{3}{2}$
43. $y = \frac{5}{3}x + \frac{14}{3}$
44. $y = \frac{5}{2}x - 4$
45. $y = -\frac{1}{4}x - 1$
46. $y = \frac{1}{2}x - \frac{5}{2}$
47. $y = \frac{5}{3}x + 10$

Index