MATH 10022

Core Mathematics II

http://www.math.kent.edu/ebooks/10022/CMII.pdf

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Contents

Chapter 1

Functions

1.1 Functions

A **function** f is a rule that assigns to every element x contained in a set A, *exactly one* element y in a set B. Another way to to think of a function is the phrase "for every x, there is only one y." The symbol $f(x)$, read "f of x," is called the **value of the function at** x and is usually equated with the variable y. In other words, we write $y = f(x)$. The set A is called the **domain** of the function. The domain of a function is the set of all values of x for which the function is defined. If x is any element in the domain, then x is called the **independent variable**. The domain can also be thought of as the set of all input values. The **range** of the function is the set of all possible values of $f(x)$, as x varies throughout the domain. Hence, the range of a function is the set of all y values assumed by the function. If y represents an output of the function f from an input x, then y is called the **dependent variable**.

A function may be defined by a set of ordered pairs, a table, an arrow diagram, a graph, or an equation. Because functions play an important role in mathematics, it is important to recognize when a particular relationship represents a function.

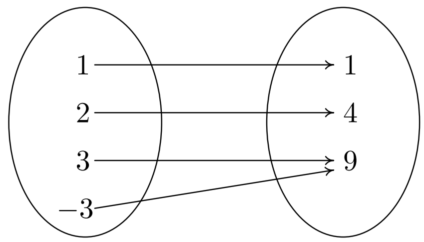
Example 1. *Determine which of the following are examples of functions. For each function, determine the domain and range.*

(a) $\{(1,2),\ (3,6),\ (6,8),\ (9,2),\ (12,5)\}$

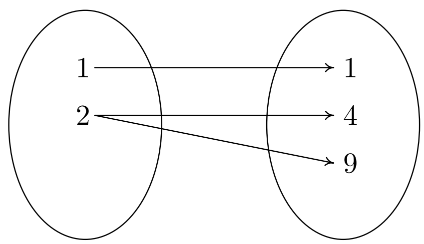
(b)

x	1	2	-5	2	-4
y	-1	4	6	7	9

(c)



(d)



(e) $x + y^2 = 9$ *where* x *is the input.*

Solution 1. *(a) Remember that ordered pairs are written as* (x, y). *Since every element in the first position is paired with exactly one element in the second position, this relationship defines a function. For this example,*

$$\begin{aligned} \textit{Domain} &= \{1, 3, 6, 9, 12\} \\ \textit{Range} &= \{2, 5, 6, 8\} \end{aligned}$$

(b) This relationship is not a function, since 2 is paired with both 4 and 7.

(c) This relationship defines a function, since every element in the first set is paired with exactly one element in the second set.

$$\begin{aligned} \textit{Domain} &= \{-3, 1, 2, 3\} \\ \textit{Range} &= \{1, 4, 9\} \end{aligned}$$

(d) *This relationship is not a function, since 2 is paired with both 4 and 9.*

(e) *This relationship is also not a function. Consider when* $x = 5$. *We have* $5 + y^2 = 9$ *or* $y^2 = 4$. *Thus,* $y = \pm 2$. *Therefore, the number 5 is paired with both 2 and* -2, *which violates the definition for a function.*

Note that in part *(e)* of the previous example, we were able to show that the relationship was not a function by using an example. However, you can not prove that a relationship is a function by showing it works for one example.

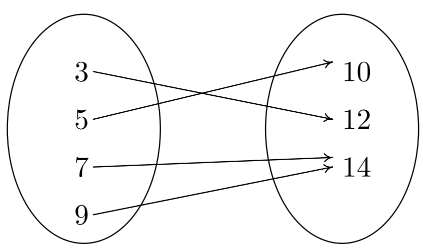
Practice 1. *Determine which of the following are examples of functions. For each function, determine the domain and range.* (Answer on page 10)

(a) $\{(3,2),\ (5,-8),\ (7,6),\ (5,4),\ (9,11)\}$

(b)

x	3	6	9	12	15
y	-5	8	12	8	9

(c)

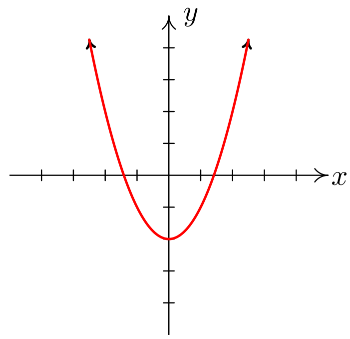


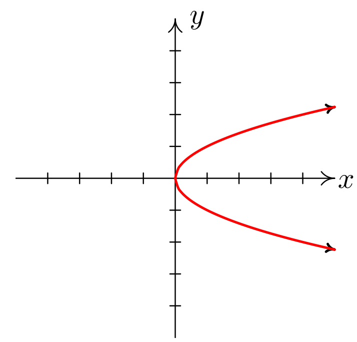
We mentioned that a function can also be defined by a graph. The **graph of a function** is a set of points (x, y) in the xy-plane such that $y = f(x)$. However, which set of points are graphs of functions? The following test gives us the answer.

The Vertical Line Test: A set of points in the xy-plane is the graph of a function if and only if no vertical line intersects the set of points more than once.

If every vertical line intersects a curve only once at (x, y), then exactly one value is defined by $y = f(x)$. However, if a vertical line intersects a curve twice at (x, y_1) and (x, y_2), then the value of x is paired with two different output values. Hence, the definition for a function is violated.

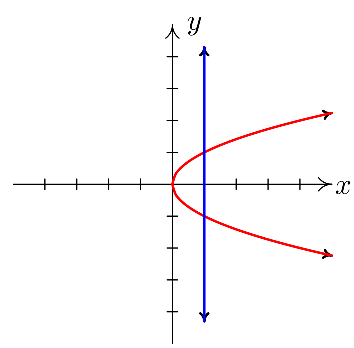
Example 2. *Determine if each of following curves is the graph of a function.*





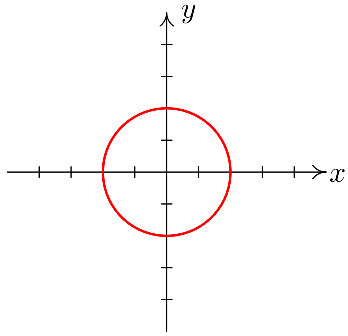
Solution 2. *(a) Since every vertical line would intersect the curve only once, the curve is the graph of a function.*

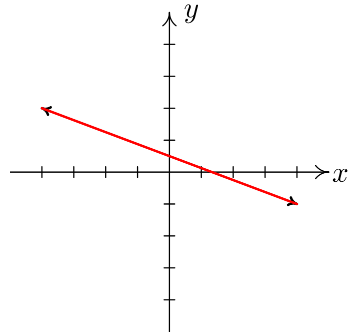
(b) Notice that the following vertical line intersects the curve twice.



Therefore, the curve is not the graph of a function.

Practice 2. *Determine if each of the following curves is the graph of a function. (Answer on page 10)*





In our definition of a function $y = f(x)$, the independent variable x serves as a placeholder for all input values. Therefore, to evaluate a function at a number, we substitute the number for the placeholder.

Example 3. *Consider the function* $f(x) = 3x^2 - 2x - 8$. *Find*

(a) $f(0)$

(b) $f(-1)$

(c) $f(2)$

(d) $f\left(\frac{1}{2}\right)$

Solution 3. *(a)* $f(0) = 3(0)^2 - 2(0) - 8 = -8$

(b) $f(-1) = 3(-1)^2 - 2(-1) - 8 = 3 + 2 - 8 = -3$

(c) $f(2) = 3(2)^2 - 2(2) - 8 = 12 - 4 - 8 = 0$

(d) $f\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) - 8 = \frac{3}{4} - 1 - 8 = -\frac{33}{4}$

Practice 3. *Consider the function* $f(x) = x^2 - 5x + 3$. *Find each of the following. (Answers on page 10)*

(a) $f(-1)$

(b) $f(3)$

(c) $f(-2)$

(d) $f\left(\frac{1}{3}\right)$

ANSWERS TO PRACTICE PROBLEMS

1. (a) not a function
   (b) function;
   Domain= $\{3, 6, 9, 12, 15\}$;
   Range = $\{-5, 8, 9, 12\}$
   (c) function;
   Domain= $\{3, 5, 7, 9\}$;
   Range = $\{10, 12, 14\}$
2. (a) not a function
   (b) function
3. (a) 9
   (b) -3
   (c) 17
   (d) $\frac{13}{9}$

SECTION 1.1 EXERCISES

(Answers are found on page 227.)

Determine which of the following are examples of functions. For each function, determine the domain and range.

1. $\{(7, -2),\ (5, 6),\ (3, -8),\ (4, -8),\ (10, -2)\}$

2. $\{(3, 1),\ (3, 2),\ (3, 3),\ (3, 4),\ (3, 5)\}$

3. $\{(1, 7),\ (-1, 7),\ (2, 7),\ (-2, 7)\}$

4. $\{(4, 64),\ (-3, -27),\ (-5, -125),\ (1, 1),\ (-1, 1)\}$

5.

x	3	2	-1	7	1
y	-10	2	3	4	8

6.

x	1	2	3	4	5
y	3	3	3	3	3

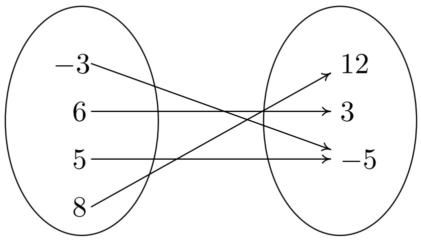
7.

x	-2	-1	0	1	2
y	4	1	0	1	4

8.

x	7	8	3	8
y	4	3	-6	1

9.



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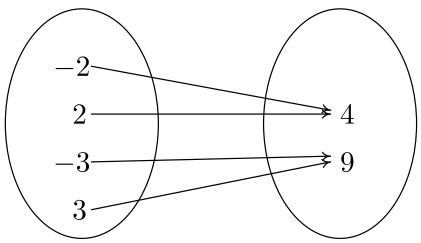

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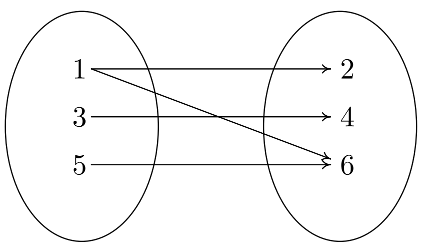

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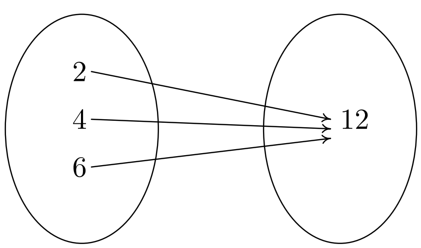


Determine which of the following are examples of functions.

13. $y = 8x - 2$

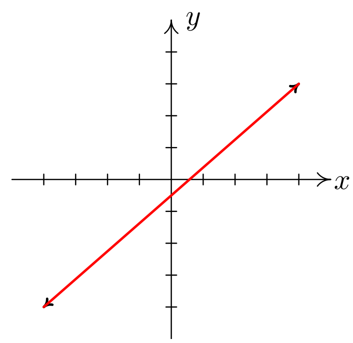
14. $x^2 + y^2 = 4$

15. $3x^2 + 7y = 11$ where x is the input.

16. $3x + 7y^2 = 11$ where x is the input.

Determine if each of the following curves is a graph of a function.

17.



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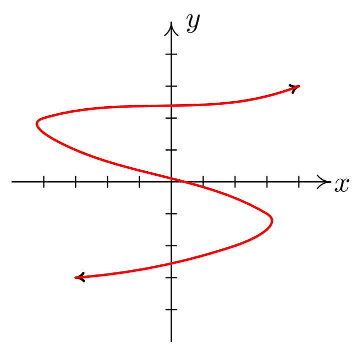

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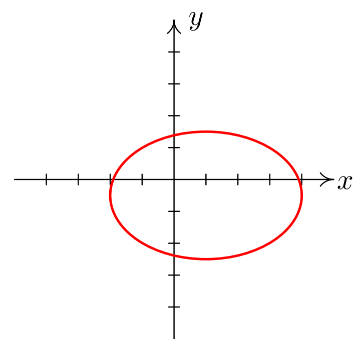

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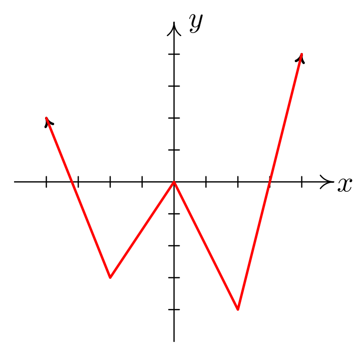

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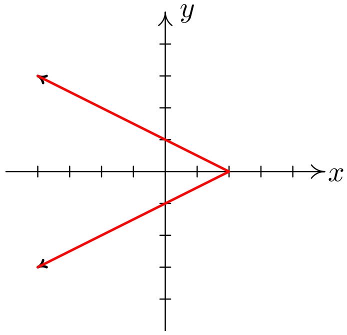

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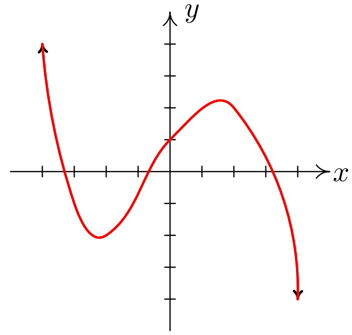

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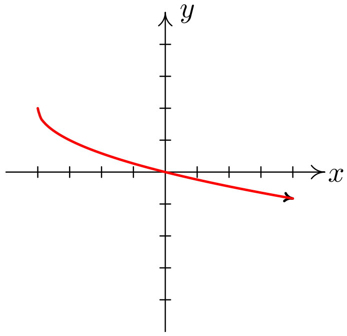

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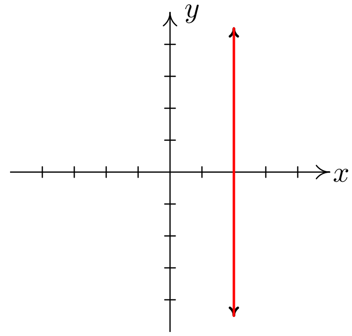


Evaluate the following functions.

25. For $f(x) = 3x - 7$, find

    (a) $f(0)$
    (b) $f(-5)$
    (c) $f(3)$
    (d) $f(-8)$

26. For $f(x) = x^2 - 3x - 4$, find

    (a) $f(0)$
    (b) $f(2)$
    (c) $f(3)$
    (d) $f\left(\frac{1}{2}\right)$

27. For $g(x) = x^3 - x^2$, find

    (a) $g\left(\frac{1}{2}\right)$
    (b) $g\left(-\frac{1}{2}\right)$
    (c) $g(1)$
    (d) $g(-2)$

28. For $R(x) = \dfrac{4x+2}{2x-1}$, find

    (a) $R(0)$
    (b) $R(1)$
    (c) $R(3)$
    (d) $R(-1)$

29. For $s(t) = 2t^2 - 7t$, find

    (a) $s(-2)$
    (b) $s(-1)$
    (c) $s(1)$
    (d) $s(2)$

30. For $h(w) = -13$, find

    (a) $h(0)$
    (b) $h(44)$
    (c) $h\left(-\frac{1}{5}\right)$
    (d) $h(91)$

31. For $f(x) = 3|x-5| + 7$, find

    (a) $f(0)$
    (b) $f(2)$
    (c) $f(4)$
    (d) $f(-3)$

32. For $P(x) = x + \dfrac{1}{x}$, find

    (a) $P(1)$
    (b) $P(-1)$
    (c) $P(2)$
    (d) $P(-4)$

33. For $f(x) = -4(x+1)^2 - 3$, find

    (a) $f(0)$
    (b) $f(-1)$
    (c) $f(-3)$
    (d) $f(2)$

34. For $f(x) = 5(x-1)^2 + 3x$, find

(a) $f(0)$
(b) $f(2)$
(c) $f(-2)$
(d) $f(1)$

35. For $f(x) = 3\sqrt{x-1} + 4$, find

(a) $f(1)$
(b) $f(5)$
(c) $f(10)$
(d) $f(17)$

Chapter 2

Linear Equation and Applications

2.1 Solving Linear Equations

In this chapter, we will discuss applications of linear equations. Before we work any applications, we need to review solving a linear equation from Core Mathematics I. Recall that a **linear equation in one variable** is an equation that can be written in the form

$$ax + b = c$$

for real numbers a, b, and c, with $a \neq 0$. They are also called first-degree equations, because the variable is raised to the first power. A **solution** of an equation in one variable is a number that makes the equation true when each occurrence of the variable in the equation is replaced by the number.

In order to solve a linear equation, we use the Addition Property of Equality and the Multiplication Property of Equality that were discussed in Core Mathematics I. These two properties are stated again for convenience. They are very useful for solving all kinds of equations (not just linear ones).

Addition Property of Equality:

$$\text{If} \quad a = b \quad \text{then} \quad a + c = b + c.$$

Multiplication Property of Equality: Let $c \neq 0$

$$\text{If} \quad a = b \quad \text{then} \quad ac = bc.$$

Remember to perform the same operation on **both** sides of the equation.

The Addition Property allows us to subtract the same quantity from both sides by adding the opposite of the quantity. The Multiplication Property allows us to divide each side by the same nonzero quantity by multiplying both sides by its reciprocal. (Click here to review solving linear equations from Core Math I.)

Example 1. *Solve for x:* $9(x+5)-2(2x-1)=3(x+8)-6$

Solution 1. *We first use the distribute property, and then isolate the variable.*

$$\begin{aligned}9(x+5)-2(2x-1)&=3(x+8)-6\\9x+45-4x+2&=3x+24-6\\5x+47&=3x+18\\2x+47&=18\\2x&=-29\\x&=-\frac{29}{2}\end{aligned}$$

Practice 1. *Solve for x:* $3(5x+1)-(7x+2)=2(x+6)$. *(Answer on page 21)*

Next, we consider a linear equation containing fractions. How do we deal with the fractions? We can choose to leave the fractions in the problem or to rewrite the equation without fractions. Remember that by the multiplication property of equality we can multiply both sides of our equation by the same nonzero constant without changing the solution. If we choose to eliminate the fractions, which constant do we choose? Consider

$$\frac{1}{2}\left(x-\frac{1}{3}\right)=\frac{x}{4}+7.$$

If we multiply both sides by two, the fraction on the right hand side of the equation would not be eliminated. If we multiply by four, the $\frac{1}{3}$ will still remain inside the parentheses. Therefore, it might be easier to clear all parentheses using the distributive property *before* multiplying every term by the least common denominator. When we use the distributive property first, we get

$$\frac{1}{2}x - \frac{1}{6} = \frac{x}{4} + 7.$$

Now we see that if we multiply by twelve, all the fractions in the equation will be eliminated. Recall that in this example, twelve is the least common multiple; that is, the smallest natural number that is a multiple of all of the denominators involved. (Click here to review least common multiple from Core Math I) Therefore, we can eliminate fractions by multiplying each term by the least common multiple of all the denominators. The next example illustrates this procedure.

It is important to remember when clearing fractions that all terms need to be multiplied by the least common denominator, not just the fractions.

Example 2. *Solve for x:* $\dfrac{1}{2}\left(x - \dfrac{1}{3}\right) = \dfrac{x}{4} + 7$

Solution 2. *We first distribute, and then eliminate the fractions by multiplying every term by* 12.

$$\begin{aligned}
\frac{1}{2}\left(x - \frac{1}{3}\right) &= \frac{x}{4} + 7 \\
\frac{1}{2}x - \frac{1}{6} &= \frac{x}{4} + 7 \\
12\left(\frac{1}{2}x\right) - 12\left(\frac{1}{6}\right) &= 12\left(\frac{x}{4}\right) + 12(7) \\
6x - 2 &= 3x + 84 \\
3x - 2 &= 84 \\
3x &= 86 \\
x &= \frac{86}{3}
\end{aligned}$$

Therefore, the solution is $x = \dfrac{86}{3}$.

Practice 2. *Solve* $\frac{2}{3}\left(2x - \frac{1}{4}\right) = \frac{3x}{5} + 2$ *(Answer on page 21)*

Suppose we apply our properties and the variable is eliminated. What is the solution? The next example illustrates one such case.

Example 3. *Solve for x:* $5(3x - 2) + \frac{1}{2}(4x + 6) = 10x + 7(x + 1)$

Solution 3. *We first distribute, and then isolate the variable.*

$$5(3x - 2) + \frac{1}{2}(4x + 6) = 10x + 7(x + 1)$$
$$15x - 10 + 2x + 3 = 10x + 7x + 7$$
$$17x - 7 = 17x + 7$$
$$-7 \neq 7$$

Therefore, there is no solution to this equation.

When an equation has no solution this means that there is no value of x for which the expression will ever yield a true statement. Does this mean that whenever the variable is eliminated from the equation, there will be no solution? The next example illustrates that this is not the case.

Example 4. *Solve for x:* $\frac{1}{4}(x - 1) + \frac{1}{3}\left(3x + \frac{9}{4}\right) = \frac{1}{2}(4x + 1) - \frac{3}{4}x$

Solution 4. *We first distribute, and then eliminate the fractions* *by multiplying every term by 4.*

$$\frac{1}{4}(x-1)+\frac{1}{3}\left(3x+\frac{9}{4}\right)=\frac{1}{2}(4x+1)-\frac{3}{4}x$$

$$\frac{1}{4}x-\frac{1}{4}+x+\frac{3}{4}=2x+\frac{1}{2}-\frac{3}{4}x$$

$$4\left(\frac{1}{4}x\right)-4\left(\frac{1}{4}\right)+4(x)+4\left(\frac{3}{4}\right)=4(2x)+4\left(\frac{1}{2}\right)-4\left(\frac{3}{4}x\right)$$

$$x-1+4x+3=8x+2-3x$$

$$5x+2=5x+2$$

$$2=2$$

Therefore, the answer is all real numbers.

When solving a linear equation, if the variable is eliminated and the resulting statement is true then the answer is all real numbers. This means that the original equation is an **identity**. An identity is a statement that is true regardless of what number is replaced in the variable. A linear equation that has one solution is called a **conditional** equation. If the variable is eliminated and the resulting statement is false, then the equation has no solution.

Practice 3. *Solve:* $6(4x-3)=\frac{1}{2}(6x+4)+4\left(\frac{21}{4}x-5\right)$ *(Answer below)*

Practice 4. *Solve:* $2-3\left(4x+\frac{1}{5}\right)=4x-8\left(2x+\frac{1}{3}\right)$ *(Answer below)*

ANSWERS TO PRACTICE PROBLEMS

1. $x=\frac{11}{6}$
2. $x=\frac{65}{22}$
3. all real numbers
4. no solution

SECTION 2.1 EXERCISES

(Answers are found on page 229.)

Solve each equation.

1. $7(5x+1)=3(x+2)$

2. $3(7x-9)=5(x-1)+6$

3. $7(x-6)=2(x+4)+5x$

4. $2x-9=2(x+3)+5x$

5. $8x-7(6x-7)+2=3$

6. $3x-2(x+5)=7x-9$

7. $4(x+2)-2(2x-1)=10$

8. $4(2x-3)+8(x+1)=2x-1$

9. $5(3x-4)+6x=4x-9$

10. $4x+3(2x-9)+6(x-1)=0$

11. $9(x+1)+6(x+3)-4(x+2)=0$

12. $3(4x-9)-7(5x+1)=2(x+3)-6$

13. $3(x+5)-8(x+3)=-5(x+2)$

14. $-\frac{2}{7}x+2x=\frac{1}{2}x+\frac{17}{2}$

15. $\dfrac{1}{2}(2x-3)+5=\dfrac{1}{3}(3x+4)$

16. $\dfrac{3}{5}x-\dfrac{1}{10}x=x-\dfrac{5}{2}$

17. $\dfrac{1}{9}(x+18)+\dfrac{1}{3}(2x+3)=x+3$

18. $\frac{1}{3}\left(6x-7\right)=\frac{1}{2}(4x-5)+\frac{1}{6}$

19. $-\dfrac{1}{4}(x-12)+\dfrac{1}{2}(x+2)=x+4$

20. $-\dfrac{5}{6}x-\left(x-\dfrac{1}{2}\right)=\dfrac{1}{4}(x+1)$

21. $\dfrac{2}{3}x-\left(x+\dfrac{1}{4}\right)=\dfrac{1}{12}(x+4)$

22. $\dfrac{1}{3}(x+3)+\dfrac{1}{6}(x-6)=x+3$

23. $\dfrac{1}{2}(x+2)+\dfrac{3}{4}(x+4)=x+5$

24. $\dfrac{1}{4}(8x-1)+\dfrac{9}{4}=\dfrac{1}{2}(4x+5)-\dfrac{1}{2}$

(Click here for additional linear equation exercises from Core Math I)

2.2 Applications of Linear Equations: Geometry

We now consider how linear equations are used to solve application problems. The key to solving any application problem begins with the translation. Can you think of any words that represent addition, subtraction, multiplication or division? Below are some of the key words that are used to represent the different operations.

- Addition – sum, added to, increased by, more than
- Subtraction – difference, minus, decreased by, subtracted from
- Multiplication – product, times, double (2x), tripled (3x)
- Division – quotient, ratio, divided by

Although there are no step by step directions that apply to every application problem, stated below are some suggested steps.

1. Read the problem carefully and explain what your variable represents.
2. Create and label any pictures, charts, or diagrams that will ease your problem solving.
3. Translate the problem into an equation. Be sure that you can read the problem back.
4. Use the addition and multiplication properties to solve the equation.
5. Check that your answer makes sense in the problem and that you have answered the question being asked. If the problem asks for two answers, make sure you give both.

Example 1. *The sum of twice a number and three is seven more than the number. Find the number.*

Solution 1. *Let $x =$ the number.*

We translate the expression to obtain

$$2x + 3 = 7 + x.$$

Next, we have

$$\begin{aligned} 2x + 3 &= 7 + x \\ x + 3 &= 7 \\ x &= 4 \end{aligned}$$

Therefore, our number must be 4.

Practice 1. *The sum of three times a number and four is the same as the number increased by two. Find the number.* (Answer on page 31.)

When translating, the order the words appear in the sentence is sometimes important. To illustrate this, consider the difference between the following expressions:

- The sum of four times a number and 5 is translated $4x + 5$.
- Four times the sum of a number and 5 is translated $4(x + 5)$.

Example 2. *Three times the difference between twice the number and four is five times the number increased by six. Find the number.*

Solution 2. *Letting $x =$ the number, we translate the expression as*

$$3(2x - 4) = 5x + 6.$$

Thus,

$$\begin{aligned} 3(2x - 4) &= 5x + 6 \\ 6x - 12 &= 5x + 6 && \textit{distributive property} \\ 6x - 12 - 5x &= 5x + 6 - 5x && \textit{subtracting } 5x \\ x - 12 &= 6 \\ x - 12 + 12 &= 6 + 12 && \textit{adding } 12 \\ x &= 18 \end{aligned}$$

Hence, the number is 18.

CAUTION: Remember when working with differences, the order is important. The expressions a minus b or the difference of a and b are both translated $a - b$. However, a less than b is translated $b - a$.

Practice 2. *Six less than four times a number is the same as twice the sum of the number and three. Find the number.* *(Answer on page 31.)*

Two integers which differ by 1, such as page numbers in a book, are called **consecutive integers**. Examples are 4 and 5, 9 and 10, 15 and 16, etc. In an application problem, if we let x be the first integer, how would we represent the second integer? If we consider our example of 4 and 5, we see that we need to add 1 to the first integer to obtain the second integer. Hence, if x is the first integer, then $x+1$ is the second. For consecutive even or consecutive odd integers, the numbers now differ by two; for example, 7 and 9, or 4 and 6. How do we set up the variables for this problem? The next example answers this question.

Example 3. *Find two consecutive* ***even*** *integers such that the smaller added to three times the larger gives a sum of 62.*

Solution 3. *Since we are dealing with consecutive even integers we will let*

$$\begin{aligned} x &= \textit{first integer} \\ x+2 &= \textit{second integer} \end{aligned}$$

Translating the expression yields

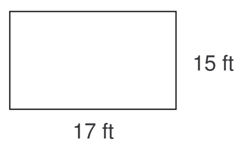
$$\begin{aligned} x + 3\,(x+2) &= 62 && \\ x + 3x + 6 &= 62 && \textit{distributive property} \\ 4x + 6 &= 62 && \\ 4x + 6 - 6 &= 62 - 6 && \textit{subtract } 6 \\ 4x &= 56 && \\ \frac{4x}{4} &= \frac{56}{4} && \textit{divide by } 4 \\ x &= 14 && \end{aligned}$$

Since the problem asks for two consecutive integers the answer is 14 *and* 16*.*

Practice 3. *Find two consecutive positive integers such that the sum of three times the smaller and five times the larger is 85.* (Answers on page 31.)

We now turn our attention to the use of formulas to solve an application problem. For carpeting a floor, painting a wall, and building a deck, it is sometimes beneficial to know the square units of the items being worked on. For other jobs, knowing the perimeter is useful. For those not familiar with these formulas, they can be found in Appendix A on page 259.

Example 4. *Mary wants to carpet her living room which is 15 feet wide by 17 feet long. What is the minimum square footage of carpeting Mary must purchase?*



Solution 4. *To solve we use the area formula for a rectangle with* $w = 15$ *and* $l = 17$*. Thus,*

$$\begin{aligned} \textit{Area in square feet} &= l \cdot w \\ &= (17 \textit{ ft})(15 \textit{ ft}) \\ &= 255 \textit{ square feet} \end{aligned}$$

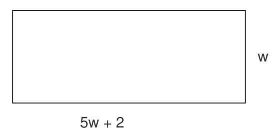
Practice 4. *Jennifer wants to spread lawn fertilizer on her back yard which is 105 feet wide by 75 feet long. How many square feet does she need to cover? If each bag of fertilizer covers* $1,000$ *square feet, how many bags must Jennifer purchase?* (Answers on page 31.)

We now consider an example where the formula is being used without substituting in a number for the length and width.

Example 5. *The length of a rectangle is 2 inches more than five times the width. If the perimeter is 106 inches, find the length and width of the rectangle.*

Solution 5. *First, since both dimensions are in terms of the width we have*

$$\begin{aligned} w &= \textit{the width of the rectangle} \\ 5w+2 &= \textit{the length of the rectangle} \end{aligned}$$



Next, we know the perimeter of any rectangle is given by $P = 2w + 2l$. Since the perimeter of this rectangle is 106 inches we have

$$\begin{aligned} P &= 2w + 2l \\ 106 &= 2w + 2(5w + 2) \\ 106 &= 2w + 10w + 4 \\ 106 &= 12w + 4 \\ 102 &= 12w \\ 8.5 &= w \end{aligned}$$

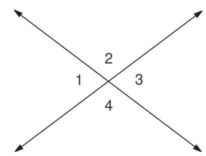
Since the problem asks for both the width and length, we find that

$$\begin{aligned} \textit{width} &= 8.5 \textit{ feet} \\ \textit{length} &= 5(8.5) + 2 = 44.5 \textit{ feet} \end{aligned}$$

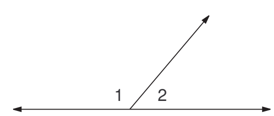
Practice 5. *The amount of fencing required to enclose a small rectangular garden is 34 feet. If the length is 1.5 times the width, find the length of the garden.* *(Answer on page 31.)*

We now turn our attention to some angle properties from geometry. Three of these definitions are stated below.

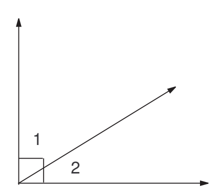
- **Vertical Angles** are opposite angles formed by intersecting lines. Vertical angles always have the same measurement. In the following diagram 1 and 3 are vertical angles. Likewise, 2 and 4 are vertical angles.



- **Supplementary Angles** are two angles whose sum is 180°. If A and B are supplementary angles, then we say that A is the **supplement** of B. Since 180° is the measure of a straight line, supplementary angles form a straight line. In the following diagram, 1 and 2 are supplementary angles. Likewise, in the previous diagram (with respect to vertical angles) 1 and 2 are supplementary; 2 and 3 are supplementary; 3 and 4 are supplementary; and 1 and 4 are supplementary.

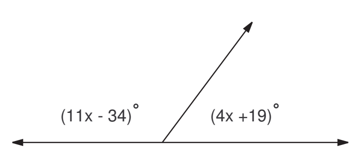


- **Complementary Angles** are two angles whose sum is 90°. If A and B are complementary angles, we say that A is the **complement** of B, and vice versa. In the following diagram, 1 and 2 are complementary angles.



Let us first consider some examples to illustrate the above definitions. Additional angle definitions can be found in Appendix A on page 259.

Example 6. *Find the measure of each marked angle.*



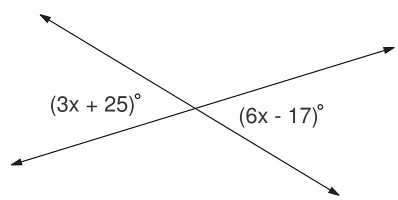
Solution 6. *First, we need to recognize that these two angles are supplementary angles. So their sum is* $180°$*.*

$$\begin{aligned} 11x - 34 + 4x + 19 &= 180 \\ 15x - 15 &= 180 \\ 15x &= 195 \\ x &= 13 \end{aligned}$$

Now that we have the value of x*, we substitute this into both expressions to yield the measure of each angle; namely,*

$$\begin{aligned} 11(13) - 34 &= 109° \\ 4(13) + 19 &= 71° \end{aligned}$$

Practice 6. *Find the measure of each marked angle.* *(Answers on page 31.)*



Example 7. *Find the measure of the angle whose complement is five times its measure.*

Solution 7. *Recalling the definition stated earlier, we have the following variable expressions:*

$$\begin{aligned} x &= \textit{the angle} \\ 90 - x &= \textit{the complement of the angle} \end{aligned}$$

Since the complement is five times the angle $(5x)$*, we have the* *following:*

$$\begin{aligned}
5x &= 90 - x \\
5x + x &= 90 - x + x \\
6x &= 90 \\
\frac{6x}{6} &= \frac{90}{6} \\
x &= 15
\end{aligned}$$

Therefore, the measure of the angle we are looking for is $15°$.

Practice 7. *Find the measure of the angle whose supplement is 8 times its measure. (Answer below.)*

ANSWERS TO PRACTICE PROBLEMS

1. -1
2. 6
3. $10,\ 11$
4. 7875 square feet; 8 bags
5. 10.2 feet
6. $67°$ and $67°$
7. $20°$

SECTION 2.2 EXERCISES

(Answers are found on page 229.)

1. Twice the sum of a number and three is eighteen. What is the number?

2. The sum of twice a number and three is nineteen. What is the number?

3. Twelve is two more than twice a number. What is the number?

4. If seven is added to twice a number the sum is one hundred one. What is the number?

5. Seven times a number equals that number added to sixty. What is the number?

6. The product of a number and two hundred two equals the sum of that number and four hundred two. What is the number?

7. Twelve times a number equals the sum of twice that number and fifty. What is the number?

8. The product of eight and a number equals twice the sum of that number and thirty-six. What is the number?

9. The sum of seventy-two and a number equals the number subtracted from ninety. Find the number.

10. If the sum of a number and 8 is tripled the result is the sum of twenty-four and three times the number. Find the number.

11. The product of a number and three is the half the sum of the number and three hundred. Find the number.

12. If the sum of a number and five is tripled, the result is one less than twice the number. Find the number.

13. If five times the smaller of two consecutive integers is added to three times the larger, the result is 59. Find both integers.

14. If four is added to twice a number and this sum is multiplied by three, the result is the same as if the number is multiplied by two and four is added to the product. What is the number?

15. The sum of three numbers is 81. The second number is twice the first number, and the third number is three less than four times the first. Find the three numbers.

16. A number divided by twelve added to fifteen equals fifteen subtracted from one-fourth of the number. Find the number.

17. Find two consecutive odd integers whose sum is one hundred eighty.

18. Find two consecutive even integers whose sum is twenty-six.

19. Find two consecutive odd integers whose sum is ten thousand thirty-two.

20. Find three consecutive even integers whose sum is ten thousand thirty-two.

21. Find three consecutive integers whose sum is ten thousand thirty-two.

22. If twice the sum of two consecutive even integers is increased by five, the result is 97. Find both integers.

23. Find three consecutive integers such that the sum of three times the middle integer and five times the smallest integer is the same as seven times the largest integer.

24. A 17-foot wire is to be cut so that one piece is 2 feet longer than twice the shorter piece. Find the length of both pieces.

25. Adult tickets for a show cost \$5.50 while children's tickets cost \$2.50. If there were twice as many adults as children and the total receipts for the show were $\$1,026$, how many adults were at the show?

26. Kayla, Savanna, and Cheyenne are the pitchers on the local girls softball team. Kayla pitched twelve more games than Savanna and Cheyenne pitched four more than twice the number of games Savanna pitched. If the team played a total of 28 games, how many games did each girl pitch?

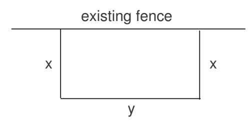
27. Roger, Will, and Jacob sold magazine subscriptions to raise money for playground equipment. Will sold nine more subscriptions than Roger. Jacob sold three less than twice as many subscriptions as Roger. If together they sold a total of 66 magazine subscriptions, how many subscriptions did each boy sell?

28. As an employee at the local market, Charlotte's duties include stocking shelves, cashier, and training new employees. Yesterday during an eight hour workday, Charlotte trained new employees half as long as she was cashier and stocked shelves three-quarters of an hour less than she was cashier. How many hours did Charlotte spend on each duty?

29. Angela needs to build a rectangular storage unit. She wants the length of the rectangle to be three feet more than twice the width. The perimeter of the rectangle is 36 feet. Find the length and width of the storage unit.

30. The perimeter of a rectangular garden is 126 feet. If the length of the garden is seven feet more than three times the width, find the dimensions of the garden.

31. A small pasture is to be fenced off with 100 yards of new fencing along an existing fence. The existing fence will serve as one side of a rectangular enclosure. (See diagram that follows). What integer dimensions produce the maximum area that can be enclosed by the new fencing?



32. In problem #31, suppose that the 100 yards of fencing needs to be used to enclose the entire area. What integer dimensions produce the maximum enclosed area?

33. Mark decides to paint his living room walls Winter Snow. According to the can, each gallon of paint will cover 300 square feet. If his rectangular living room is 20 feet long and 15 feet wide with 10 foot ceilings, how many gallons of paint will Mark need to buy to complete his project if

    (a) only one coat is needed to complete the project?

    (b) two coats are needed to complete the project?

34. A thirty-nine inch wire is cut into four pieces. The second piece is six inches less than twice the first piece. The third piece is as long as the first piece. The fourth piece is as long as the second piece. Find the length of each piece.

35. A thirty-six foot long board is cut into two pieces. The length of one of the pieces is four more than three times the length of the other piece. Find the length of the two pieces.

36. A 200-yard bolt of cloth is cut into two sections of unequal length. One section is ten yards less than twice the length of the other section. Find the length of the sections.

37. A 50 inch board is cut into three pieces so that the second piece is four times as long as the first piece and the third piece is five inches longer than the second piece. Find the length of the third piece.

38. Michael recorded two songs during his 45-minute recording session. He took twice as long to record the second song compared to the first song. How long did it take for him to record the first song?

39. Emma recorded three songs during her 180-minute recording session. She took thirty minutes more to record the second song than the first, and three times as long to record the third song as the first. How long did she take to record each song?

40. Suppose that the total number of tickets sold at last Thursday's matinee at Regal Cinema was 1050. If there were half as many children's tickets sold as adult tickets, how many adult tickets were sold for the matinee shows last Thursday?

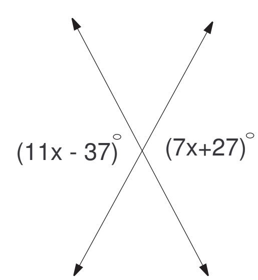
41. Suppose that the total number of tickets sold at last Friday's evening shows at Regal Cinema was 2008. If the number of adult tickets sold was fifty-eight more than twelve times the number of children's tickets sold, how many of each type ticket was sold at last Friday's evening shows?

42. The length of a rectangular yard is 2 less than five times the width. If 44 yards of fencing are required to fence all four sides of the yard, find the length of the yard.

43. The perimeter of Sally's garden is 60 feet. What is the length of her garden if its width is 10 feet?

Find the measure of each marked angle.

44.



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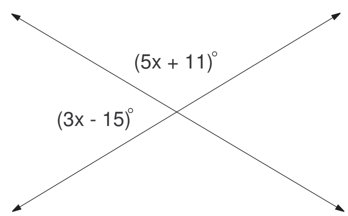

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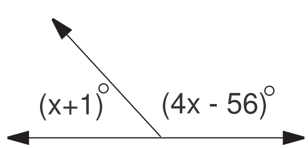

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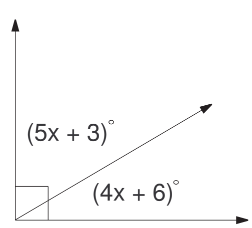

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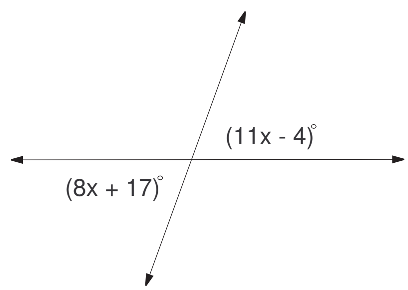

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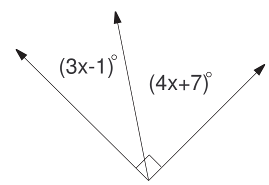


50. Find the measure of an angle such that the sum of the measures of its complement and its supplement is 100°.

51. The supplement of an angle measures 15° more than four times its complement. Find the measure of the angle.

2.3 Applications of Linear Equations: Percents

Percent problems are probably the most commonly used type of problems in every day life – sales tax, discounts, depreciation, etc. In Core Mathematics I we discussed transforming a percent into a decimal and a decimal into a percent. (Click here to review percents and decimals from Core Math I) When working with percents, there are typically three types of problems:

1. Given the whole and the percent, find the part;
2. Given the whole and the part, find the percent;
3. Given the part and the percent, find the whole.

Given the whole and the percent

If we are given the whole and the percent, the part can be found by multiplying the percent by the whole. The word "of" is a key to where the multiplication takes place. Consider the following example.

Example 1. *Suppose that* 75% *of 400 fourth graders in a school district have passed their proficiency exams. How many fourth graders have passed their proficiency exam?*

> **Solution 1.** *Recall that* $75\% = 0.75$. *Therefore,*
>
> $$75\% \text{ of } 400 = 0.75 \times 400 = 300.$$
>
> *Therefore, 300 fourth graders have passed their proficiency exams.*

Practice 1. *If* $42\frac{1}{2}\%$ *of the 80 faculty at the local high school have master's degrees, how many faculty have master's degrees?* (Answer on page 43).

Given the whole and the part

Once again, we will recall that the word "of" is a key to where the multiplication takes place.

Example 2. *If* \$450 *of a loan has been repaid, what percent of the* \$2000 *loan has been repaid?*

Solution 2. *Let x be the percent. Then we have that*

$$450 \textit{ is } x \textit{ percent of } 2000.$$

Translating and solving we get

$$\begin{aligned} 450 &= x \cdot 2000 \\ \frac{450}{2000} &= \frac{x \cdot 2000}{2000} \\ 0.225 &= x \end{aligned}$$

Therefore, \$450 *is* 22.5% *of* \$2000.

Practice 2. *A skirt originally selling for* \$45 *is on sale for* \$27.90. *What percent of the original price is the sale price? What is the discount rate?* *(Answer on page 43).*

Given the percent and the part

Example 3. *Joe decides to save* 15% *of his weekly salary. If he saves* \$75 *every week, what is Joe's weekly salary?*

Solution 3. *Let x = Joe's weekly salary. Then*

$$15\% \textit{ of his weekly salary is } \$75.$$

Translating we get

$$\begin{aligned} .15x &= 75 \\ \frac{.15x}{.15} &= \frac{75}{.15} \\ x &= 500. \end{aligned}$$

Thus, Joe's weekly salary is \$500.

Practice 3. *The Miller family decides that they can spend* \$120 *of their monthly income on entertainment. What is the the Miller family's monthly income if they spend* 2.5% *of their monthly income on entertainment?* *(Answer on page 43).*

When solving application problems, it sometimes helps to write your own formula as the next example illustrates.

Example 4. *Marilyn is paid* \$315 *per week, plus a* 5% *commission on sales. What is her total earnings if her sales were* \$625.

Solution 4. *From the problem we know that*

$$\begin{aligned} \textit{total earnings} &= \$315 + (5\% \textit{ of her sales}) \\ &= \$315 + (5\% \textit{ of } \$625) \\ &= 315 + (.05 \times 625) \\ &= 315 + 31.25 \\ &= 346.25 \end{aligned}$$

Hence, her total earnings are \$346.25.

Practice 4. *If income tax is* \$3,750 *plus* 28% *of taxable income over* \$28,000*, how much is the income tax on a taxable income of* \$35,000. *(Answer on page 43).*

Sometimes we deal with discounts and markups in the same problem.

Example 5. *Kim paid* \$330 *for a dresser to sell at her antique shop. She wants to price it so that she can offer a* 10% *discount and still make a* 20% *profit off the price she paid for it. What should be the marked price of the dresser at the antique shop?*

Solution 5. *Let x = the price the dresser should be marked.*

$$\textit{Profit Kim wants to make} = .20(330) = \$66.$$

Price willing to sell the dresser: $\$330 + \$66 = \$396$.

NOTE: $\$396$ *is the price after Kim offers a* 10% *discount.*

$$\begin{aligned} 396 &= \textit{marked price} - \textit{discount} \\ 396 &= x - .10x \\ 396 &= .90x \\ 440 &= x \end{aligned}$$

Thus, Kim should mark the dresser at $\$440$.

Practice 5. *Buy 4 Less marks down all remaining winter coats at a discount of* 40% *off the original price. After several weeks, in a last attempt to reduce inventory, the coats are further discounted* 50% *off the last marked price. Your friend believes that this means the coats are now* 90% *off the original price. Is your friend correct? If not, what percent off are the coats from the original price?* *(Answers on page 43).*

Now let us turn our attention to some interest problems. If you have car loans, savings accounts, or any investments, you should be concerned about the type of interest you are paying or that you earn on the investment. There are two common types of interest – simple and compound. **Simple interest** pays interest only on the original amount of money you deposited or invested, called the **principal**. **Compound interest** pays interest on the interest in the account as well. Before we discuss a formula, lets look at the following example.

Example 6. *Suppose you invest* $\$10,000$ *in a bank account that earns an annual interest rate of* 3% *simple interest.*

- *(a) How much interest would you earn in one year?*
- *(b) How much interest would you earn in three years?*
- *(c) How much interest would you earn in a half a year?*

Solution 6.

(a) Because we are dealing with simple interest, the account only pays interest once per year and only on the original amount deposited. Recall that $3\% = 0.03$. *Therefore, the amount of interest is* $0.03 \times 10,000 = \$300$.

(b) Once again, the account only pays interest once per year and only on the original amount deposited. Hence, we know that the account pays \$300 *in interest per year, so for 3 years we would have* $3 \times 300 = \$900$*.*

(c) We know that we earn \$300 *per year so in half a year we would earn half as much; namely,* $\frac{1}{2} \times 300 = \$150$*.*

Now, the original amount of \$10,000 is called the principal and we will denote this by p. The interest rate is r and the time is t. Using the above example as motivation, what would you consider a valid formula for the interest I earned in terms of p, r, and t? Well, lets summarize our results:

$$\begin{aligned} \text{1 year interest} &= 10,000 \times 0.03 \times 1 = 300 \\ \text{3 years interest} &= 10,000 \times 0.03 \times 3 = 900 \\ \text{half year interest} &= 10,000 \times 0.03 \times \frac{1}{2} = 150 \end{aligned}$$

Do you see the pattern? Principal times rate times time. We can state this formula as follows.

Formula for simple interest is given by

$$I = prt$$

where I is the interest, p is the principal, r is the annual interest rate, and t is the time in years.

Example 7. *Mary deposits* \$2000 *into an account earning* 4.5% *simple interest for 2 years. When the 2 years are over, how much money does Mary have in the account?*

Solution 7. *Since we are earning simple interest we will use the above formula with* $p = 2000$*,* $r = 0.045$ *and* $t = 2$*. Thus,*

$$\begin{aligned} I &= prt \\ &= 2000 \cdot 0.045 \cdot 2 \\ &= 180. \end{aligned}$$

Therefore, the amount of money in the account at the end of two years is $\$2000 + \$180 = \$2180$*.*

Practice 6. *Savanna deposits* $\$15,000$ *into an account paying simple interest. At the end of two years, the amount in the account is* $\$16,875$*. What is the annual percentage rate on this account?* (Answers on page 43).

How much more can one earn with compound interest rather than simple interest? Lets return to our previous example to find out.

Example 8. *Mary deposits* $\$2000$ *into an account earning* 4.5% *interest compounded annually.*

(a) Find the interest and account balance after 1 year.

(b) Find the account balance after 2 years.

(c) How much more money do you have after 2 years if interest is compounded annually than you did in Example 7 when interest was simple interest?

Solution 8. *(a) We use the formula to get*

$$\begin{aligned} I &= prt \\ &= 2000 \times 0.045 \times 1 \\ &= 90. \end{aligned}$$

So there is $\$90$ *of interest in the account and a balance of* $\$2090$ *after 1 year.*

(b) For the second year, we earn interest on the interest. The amount of interest we earn in the second year is

$$I = 2090 \times 0.045 \times 1 = 94.05$$

The account therefore has $\$2090 + \$94.05 = \$2184.05$ *after 2 years.*

(c) From example 7 we found that the amount in the account after two years of simple interest is $\$2,180$*. Hence, the difference of* $\$2184.05 - \$2180 = \$4.05$ *is the additional amount received for compound rather than simple interest.*

Practice 7. *The Baker Family invests* $30,000 *for their son's education into an account paying* 6.75% *compounded annually. How much will the account contain after 3 years? How much more interest will the account have earned than if their account only paid* 6.75% *simple interest? (Answer below).*

ANSWERS TO PRACTICE PROBLEMS

1. 34
2. 62%, 38%
3. $4800
4. $5710
5. The final price is 70% off the original.
6. 6.25%
7. $36,494.30; $419.30 more

SECTION 2.3 EXERCISES

(Answers are found on page 231.)

1. Which is better: A 10% discount followed by a 10% markup, or a 10% markup followed by a 10% discount? Explain your reasoning.

2. Which is better: A discount of 20% followed by the addition of sales tax at a rate of 6%, or the addition of sales tax at a rate of 6% followed by a discount of 20%? Explain your reasoning.

3. Mary deposited $100 in an account paying 5% annual simple interest. At the end of one year how much interest will Mary's account earn?

4. William deposited $1000 in an account paying $4\frac{1}{2}$% annual simple interest. At the end of one year how much interest will William's account earn?

5. Lester deposited $100 in an account paying $4\frac{1}{4}$% annual simple interest. At the end of two years how much will Lester earn in interest on his account?

6. Linda deposited $1000 in an account paying $3\frac{1}{2}$% annual simple interest. At the end of four years how much will Linda earn in interest on her account?

7. Maurita A. Coss, a financial planner, invested money at 4% annual simple interest one year ago today. If she earned $32 in simple interest in one year, how much money at 4% annual simple interest did she invest one year ago today?

8. A chemist needs a 50% solution of sulfuric acid having a volume of 72 liters. How much pure acid is in the 72 liters of 50% solution?

9. How much pure acid is in 48 liters of a 40% solution of sulfuric acid?

10. Mackenzie's collection of twenty-four CDs is 75% country. How many country CDs does Mackenzie have in her collection?

11. How much orange juice is in twelve cups of a 75% solution of orange juice?

12. A math class has the names of 25 students on the roster. What is the percent of students present on a particular day when only one of the students is absent?

13. A geography class has the names of 30 students on the roster. If 18 of the students in the class are female, what percentage of the class is male?

14. The Calculus class is 25% male. If there are 18 women in the class, how many student are in this class?

15. An introductory psychology class is 40% male. If there are 300 students in the class, how many are women?

16. For General Chemistry, 20% of the students earned an A- or an A for their final grade. If there are 35 names on the roster, how many students earned at least an A- in the class?

17. Tanya spent one and a half hours completing her Math 10022 homework today and six hours working on her other assignments. What percent of her total homework time today did she spend on her mathematics homework?

18. If there are 45 ounces of gold in 180 ounces of a gold alloy, what is the percent of gold in the alloy?

19. How many ounces of pure copper are there in 300 ounces of an alloy that is 40% copper?

20. Suppose that this year's sale at Nordstrom included a \$120 wool scarf from Italy that before the sale sold for \$150. What is the percent of sale discount on the scarf?

21. Aaron invested $\$14,000$ of his $\$20,000$ inheritance in an account yielding 3% in annual simple interest. He invested the remainder of his inheritance in an account earning 2% in annual simple interest. How much did he earn in simple interest from both accounts?

22. Andrew invested \$300 at 3% simple interest. On the same day he also invested \$200 at 4% simple interest. Find his total yearly earnings from the interest on both accounts at the end of one year.

23. Amy was driving to a city 180 miles from her starting point. After she had traveled 45 miles, what percent of her proposed journey had she completed?

24. Jennifer has 80 coins, all nickels and dimes. If she has 30 dimes, what percentage of her coins are nickels?

25. A plumber charges \$80 in time and materials to unstop a drain. If the parts cost \$20, what percentage of his bill was for his labor?

26. An auto mechanic charges \$80 in time and materials to repair a rim leak on a tire. If the parts cost \$10, what percentage of his bill was for his labor?

27. Kim is a real estate agent who makes a 6% commission of every property she sells. How much commission did she make on the $\$100,000$ piece of land she recently sold to Embassy Suites for their new hotel?

28. Suppose the Beldon family decides to give 8% of their monthly income to charity. If their income the past month was $\$150,000$, how much money would they have given to charity this past month?

29. Suppose that Tom is a recent graduate of Kent State University and decides to put \$20 a week in a savings account. If his weekly take-home pay is \$800, what is the percentage of his take-home pay that he puts into savings?

30. You are offered two different real estate positions. Job 1 pays \$250 per week plus a 6% commission on sales. Job 2 pays 9% commission only. If sales average about $\$20,000$ per week, which job would pay more?

31. To find the cost of a \$20 item discounted 25%, Amy multiplied by $\frac{1}{4}$ and Jack multiplied by $\frac{3}{4}$. Both found the right answer. Explain how this could be.

32. Candidates A, B, and C were in an election in which 1000 votes were cast. Candidate A received 35% of the votes while Candidate B received 52 more votes than Candidate C. Which candidate won the election and by how many votes?

33. The Miller family spends twenty percent of their monthly salary on food. Twice the amount spent on food is spent on the house payment, and one half of the amount spent on food is spent for utilities. If the total amount spent on these three amounts is $\$1,050$, what is the Miller family's monthly salary?

34. The size of the smallest angle of a triangle is 30% of the size of the largest angle. The size of the third angles is $20°$ more than the smallest angle. Find the size of each angle. (Note that the sum of the angles of a triangle is $180°$.)

35. College tuition at Local University in 2005 is up 9% that of tuition in 2004. If the tuition in 2005 is $\$7,050$, what was the tuition in 2004?

36. Carl decided to plant a rectangular shaped garden. If the width is 80% of the length and the perimeter is 45 feet, what is the area of the garden?

37. Sally bought textbooks at the bookstore for \$244.33, which included 6% sales tax. What did the books cost?

38. John sells a decorative wreath at his floral shop for \$34.95. If the wreath has a markup of 65%, what was John's cost of the wreath? (Round to two decimal places)

39. A pair of slippers originally costs \$35.50. This week the slippers are advertised at 35% off. What is the sale price of the slippers (before tax)? (Round to two decimal places)

40. A new 10 piece cookware set sells for \$348. This weekend it is on sale for 33% off. What is the sale price of the cookware (before tax)? (Round answer to two decimal places.)

41. Kelly bought a bike and a year later sold it for 30% less than what she paid for it. If she sold the bike for \$126, what did she pay for it?

42. In a basketball game, 17 free throws went in the basket and 3 missed. What percent were missed?

43. A car was to be sold for a 12% discount which amounted to \$1800. How much would the car sell for after the discount?

44. Sarah paid $\$36,648.12$, including tax, for her new car. The sales tax rate is 7.25%. What was the actual price of the car (before tax)? (Round answer to two decimal places.)

45. If the federal income tax is $\$3,750$ plus 28% of taxable income over $\$65,000$, what is the federal income tax on a taxable income of $\$87,000$?

46. The university basketball team won 91 games, which was 70% of the games played. How many games did the team play?

47. At the local market, bananas originally sold for 45 cents a pound. Today, the cost is 52 cents a pound. What is the percent increase in the cost of bananas per pound? (Round answer to two decimal places.)

48. A laptop that was originally purchased for $\$1,850$ now has an estimated value of $\$1,258$. What is the percent decrease in the value of the laptop?

49. Brenda purchased clothing totaling \$140 before taxes. If the sales tax rate is 6.75%, what was her total cost for the clothing?

50. Suppose you invest $\$25,500$ in a bank account that earns an annual interest rate of $2\frac{1}{2}\%$ simple interest.

    (a) How much interest would you earn in one year?

    (b) How much interest would you earn in three years?

    (c) How much interest would you earn in a half a year?

51. If $\$12,000$ is deposited into an account earning 4% interest compounded annually,

    (a) Find the interest and account balance after 1 year.

    (b) Find the account balance after 2 years.

    (c) How much additional money do you have after 2 years if interest is compounded annually than you would have if interest was calculated using simple interest?

52. Beverly deposits \$18,250 into an account paying simple interest. At the end of three years, the amount in the account is \$21,808.75. What is the annual percentage rate on this account?

2.4 Applications of Linear Equations: Proportions

Much of our use of mathematics involves comparison and change. A **ratio** is a quotient of two quantities. The ratio of the number a to the number b is written

$$a \text{ to } b, \quad \frac{a}{b}, \quad \text{or} \quad a : b.$$

Fractions are ratios of integers but not all ratios are fractions. For example, the ratio of the diagonal of a square to one of its sides is never a ratio of integers. Indeed, a ratio may compare any kind of quantity or magnitude to any other kind such as dollars to ounces in Example 1.

One application of ratios is in unit pricing. This helps one to determine which size of a product offered at different prices represents the best bargain.

Example 1. *The local Bargain Basement offers the following prices for a jar of Fluff Marshmallow Creme:*

Size	*Price*
15 oz	\$1.35
32 oz	\$2.76
48 oz	\$4.56

Which size represents the best buy?

Solution 1. *To find the best buy, we compare the price for each jar to the number of ounces per jar. This ratio gives the price per ounce. The best deal, of course, will be the item with the lowest price per ounce. The following chart gives those ratios:*

Size	*cost per ounce*
15 oz	$\frac{\$1.35}{15} = 0.09$
32 oz	$\frac{\$2.76}{32} = 0.08625$
48 oz	$\frac{\$4.56}{48} = 0.095$

Comparing the price per ounce, we determine that the 32 ounce jar is the best buy.

The above example illustrates that buying the largest size does not always result in the best buy.

Practice 1. *The local grocery store had the following prices on coffee:*

Size	*Price*
13 oz	\$2.75
34 oz	\$7.34
60 oz	\$13.52

Which size represents the best buy? (Answer on page 52).

A **proportion** is a statement that says that two ratios are equal. Ratios and proportions are not limited to the area of mathematics. In map making, proportions are used to represent the distance between two cities. Likewise, for scale models we use proportions. Even in cooking, the idea of a proportion is used whenever we increase or decrease the ingredients to accommodate the number of servings we want. If three of the numbers in a proportion are known, then we can always solve for the fourth number by multiplying both sides of the equation by the least common multiple of the denominators.

Example 2. *A new commercial states that three out of four dentists recommend the new Flash toothpaste. If 411 dentists recommend the toothpaste, how many dentists were interviewed?*

Solution 2. *Let x = the number of dentists interviewed. Then*

$$\begin{aligned}
\frac{3}{4} &= \frac{411}{x} \\
4x\left(\frac{3}{4}\right) &= 4x\left(\frac{411}{x}\right) && \textit{multiply by LCM of denominators} \\
3x &= 4(411) \\
3x &= 1644 \\
x &= 548
\end{aligned}$$

Hence, 548 dentists were interviewed.

Note that in the above example, there are four correct ways to set up the proportion. They are $\frac{3}{4} = \frac{411}{x}$, $\frac{4}{3} = \frac{x}{411}$, $\frac{3}{411} = \frac{4}{x}$, and $\frac{411}{3} = \frac{x}{4}$. After multiplying by the LCM of the denominators in each proportion, we obtain $3x = 1644$.

Practice 2. *Cheyenne scored 75 goals during her soccer practice. If her success-to-failure rate is* $5:4$*, how many times did she attempt to goal?* *(Answer on page 52.)*

Example 3. *You have just invented a new recipe to enter in the local cooking contest. As invented, the recipe calls for 2 cups of onions for three servings. Contest rules state that the recipe must make four servings. How many cups of onions must be used for the recipe to make four servings?*

Solution 3. *Let* x = *the number of cups of onions for four servings.*

$$\begin{aligned} \frac{x}{4} &= \frac{2}{3} \\ 12\left(\frac{x}{4}\right) &= 12\left(\frac{2}{3}\right) \quad \textit{multiply by LCM of denominators} \\ 3x &= 8 \\ x &= \frac{8}{3} \\ x &= 2\frac{2}{3} \end{aligned}$$

Therefore, for four servings the recipe would require $2\frac{2}{3}$ *cups of onions.*

Practice 3. *A recipe that makes 3 dozen peanut butter cookies requires* $1\frac{1}{4}$ *cups of flour. How much flour would you require for 5 dozen cookies?* *(Answer on page 52).*

Example 4. *The driver at pump 2 filled her tank with 12.5 gallons of gasoline for* \$33*. How many gallons of gasoline did the driver at pump 6 get if his total was* \$42.90*?*

Solution 4. *Let $x =$ the number of gallons of gasoline for driver at pump 6.*

$$\frac{x}{42.90} = \frac{12.5}{33}$$
$$33x = 536.25$$
$$x = 16.25$$

Therefore, the driver at pump 6 got 16.25 gallons of gasoline.

Practice 4. *Akron, OH is located approximately 60 miles west of the Pennsylvania border. An Ohio map represents this distance as 3 inches. On the same map, Youngstown, OH is represented by approximately $\frac{11}{20}$ of an inch from the Pennsylvania border. How far is Youngstown, OH from the Pennsylvania border?* (Answer below).

ANSWERS TO PRACTICE PROBLEMS

1. The 13 ounce can is the best buy
2. 135 goals attempted
3. $2\frac{1}{12}$ cups of flour
4. 11 miles

SECTION 2.4 EXERCISES

(Answers are found on page 232.)

1. Find the best buy for each item.

   (a) Salad Dressing: 8 ounces for \$0.98, 16 ounces for \$1.34, or 32 ounces for \$2.74

   (b) Chocolate Chips: 6 ounces for \$0.94, 12 ounces for \$1.50, or 24 ounces for \$2.83

   (c) Strawberry Jam: 11 ounces for \$1.19, 24 ounces for \$2.64, or 32 ounces for \$3.68.

2. If a light flashes every 6 seconds, how many times will it flash in three-fourths of an hour?

3. If a man 6 feet tall casts a shadow 4 feet long, how long a shadow will be cast by a cell phone tower that is 75 feet tall?

4. If 500 sheets of paper makes a pile two inches high, how many sheets of paper are there in a pile a foot high?

5. If the scale of a map is $\frac{1}{3}$ inch equals one mile, how long is a straight road that is one and one-half inches on the map?

6. If a cricket chirps 75 times in a minute, how many times does the cricket chirp in 8 seconds?

7. If the human heart pumps 5 quarts of blood through the human body in a minute, how many gallons of blood will it pump in 1 hour?

8. If John walks a mile in fifteen minutes, how far can he walk in three-quarters of an hour?

9. If Grace pays \$3 for two pens, how many pens can she buy with \$60?

10. A recipe for 36 cookies requires $1\dfrac{3}{4}$ cups of chocolate chips. How many cups are required for 90 cookies?

11. If 7 gallons of premium gasoline costs \$17.43, how much would it cost to fill up with 25 gallons of premium gasoline? (Round to two decimal places)

12. A holiday recipe calls for $1\dfrac{2}{3}$ cups of flour. If this recipe feeds 4 people, how many cups of flour would you need in the altered recipe in order to feed 10 people?

13. Cheyenne scored 20 goals during this soccer season. If she scored 5 goals for every 9 attempts, how many goals did she miss?

14. A recipe that makes 4 dozen oatmeal raisin cookies calls from $1\dfrac{1}{4}$ cups of flour.

    (a) How much flour would you need if you doubled the recipe?

    (b) How much flour would you need for half of the recipe?

    (c) How much flour would you need to make 5 dozen cookies?

15. Your architect has drawn up the blueprints for your new house. In the drawing $\frac{1}{2}$ inch represents 3 feet. The length of the house is 42 feet. How many inches would this represent on the blueprint?

16. A local business produces color markers. The shop manager has found that out of every 1200 markers produced, 3 will be defective. How many markers were produced on this machine, if 7 markers were found to be defective?

17. You invented a new recipe to enter into the local cooking contest. As invented, the recipe calls for $1\frac{1}{2}$ cups of chopped onions for four servings. The contest rules state that the recipe must make six servings. How many cups of onions must be used in order for the recipe to make six servings?

18. If $\frac{3}{4}$ inch on a map represents an actual distance of 20 miles, what is the distance, in miles, represented by $4\frac{1}{8}$ inches on the same map?

19. The official ratio of width to length for the US flag is 10 to 19. If a flag is 45 inches wide, what is the length of the flag?

20. East Elementary has 1500 students. The student to teacher ratio is 20 to 1. How many teachers are employed at East Elementary?

21. East Elementary has 1500 students. The ratio of female students to male students is 16 to 9. How many female students attend East Elementary?

22. Wildlife officials want to estimate the number of fish in Bluegill Lake. As a result, they tag a total of 150 fish from the lake. What is the estimated number of fish in Bluegill Lake if six months later officials catch 250 fish and 75 are tagged?

23. The directions on a bag of cement are to mix 1 part cement, 2 parts sand, and 3 part gravel. If you have 14 pounds of sand, how many pounds of cement and how many pounds of gravel should be mixed with this sand?

24. A metal tube measuring 3 feet long weighs 28 pounds. What is the weight of a similar tube that measures $2\frac{1}{4}$ feet?

25. If there are 7.62 cm in 3 inches, how many centimeters are there in 8 inches?

26. A family purchases 2 gallons of milks every 3 weeks. At this rate, how many gallons of milk can this family expect to purchase in a year?

27. According to Physics, the weight of an object on the moon is proportional to its weight on Earth. If a 144 pound woman weighs 24 pounds on the moon, what would a 84 pound boy weigh on the moon?

28. The property tax for a home with an assessed value of $\$145,000$ is $\$1,373.88$. What is the property tax for a house with an assessed value of $\$124,000$

Chapter 3

Systems of Linear Equations

3.1 Graphing Method

In Core Mathematics I, we graphed a linear equation in two variables. (Click here to review graphing linear equations in two variables from Core Math I.) In this section we consider the solution to a system of linear equations. A **system of linear equations** consists of two or more linear equations. A **solution** of a system of linear equations is an ordered pair of numbers that is a solution to every equation in the system.

Example 1. *Determine if* $(-3,2)$ *is a solution of the following system:*

$$\begin{cases} 2x+5y = 4 \\ 4x-3y = -18 \end{cases}$$

Solution 1. *In order for* $(-3,2)$ *to be a solution for the system, it must be a solution to each of the linear equations. To check, we substitute* $x=-3$ *and* $y=2$ *into each equation. Checking the first equation, we find*

$$\begin{aligned} 2(-3)+5(2) &\stackrel{?}{=} 4 \\ -6+10 &\stackrel{?}{=} 4 \\ 4 &= 4\checkmark \end{aligned}$$

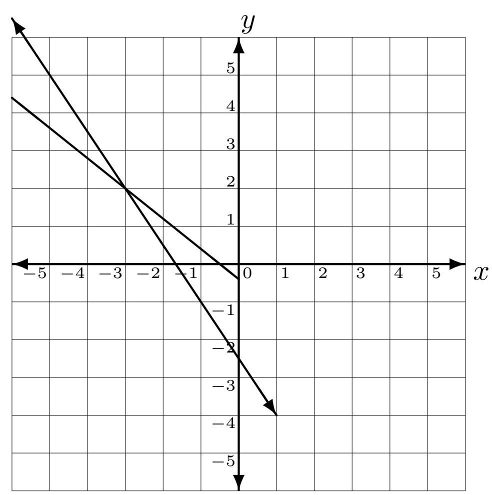
Since we ended up with a true statement, $(-3, 2)$ is a solution of the first linear equation. However, before it can be a solution of the system, it must also be a solution of the second equation. Checking the second equation, we find

$$\begin{aligned} 4(-3) - 3(2) &\stackrel{?}{=} -18 \\ -12 - 6 &\stackrel{?}{=} -18 \\ -18 &= -18\checkmark \end{aligned}$$

Thus, $(-3, 2)$ is a solution of the given system.

Practice 1. *Determine if $(-1, 4)$ is a solution of* $\begin{cases} 3x + 2y = 5 \\ 4x + 5y = 12 \end{cases}$ *(Answer on page 64)*

The set of all ordered pairs that are solutions of a system forms the **solution set** of the system. The goal is to find the solution set of a given system. In Example 1, $(-3, 2)$ was a solution of the given system. If we graph each equation in the system on a common set of axes, we have the following graph.



Notice that $(-3, 2)$ is a point on both graphs of the linear equations. In other words, $(-3, 2)$ is a point of intersection of the two graphs of linear equations involved in the system. Therefore, one method for solving a system of linear equations is by graphing and identifying the points of intersection.

Steps for the Graphing Method:

1. Graph each linear equation on a common set of axes.
2. Locate the points of intersection.
3. Check each proposed point of intersection to confirm that it is a solution to each linear equation in the system.

Example 2. *Solve the following system by graphing.*

$$\begin{cases} x + y = 1 \\ -x + y = 3 \end{cases}$$

Solution 2. *We begin by graphing each of the linear equations on a common set of axes. Both equations can be graphed by using the $x-$intercept and $y-$intercept. We will plot one additional point for each line as well. Therefore, we have*

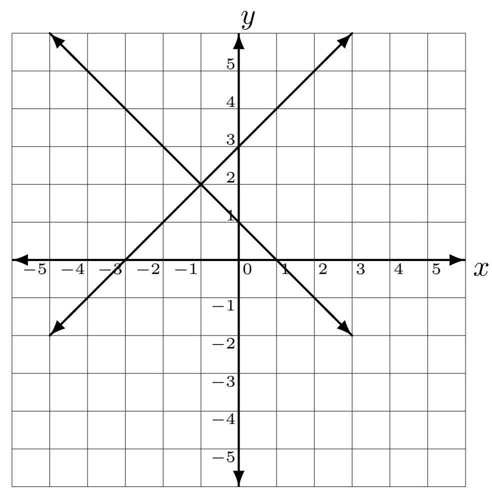
$x + y = 1$

x	y
1	0
0	1
-1	2

$-x + y = 3$

x	y
-3	0
0	3
1	4

Graphing these two lines by plotting the above points, we obtain



It appears that the two lines intersect at $(-1, 2)$. In order to conclude that $(-1, 2)$ is our solution, we must check to see if it is a solution to each equation. Therefore, substituting $x = -1$ and $y = 2$ in both equations we get,

first equation:	*second equation:*
$(-1) + 2 \stackrel{?}{=} 1$	$-(-1) + 2 \stackrel{?}{=} 3$
$1 = 1\checkmark$	$3 = 3\checkmark$

Hence, $(-1, 2)$ is the solution of the system.

Practice 2. *Solve the following system by graphing.* $\begin{cases} 3x - y = 7 \\ x - y = 5 \end{cases}$ *(Answer on page 64)*

Example 3. *Solve the following system by graphing.* $\begin{cases} 2x + y = 5 \\ x + 3y = 5 \end{cases}$

Solution 3. *We will also graph each of these equation by plotting points. We notice that*

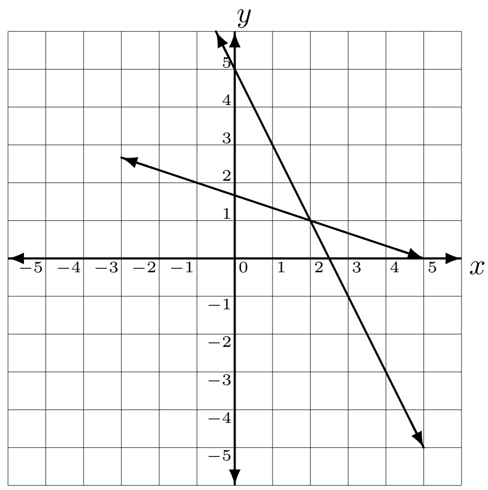
$2x + y = 5$

x	y
0	5
1	3
3	-1

$x + 3y = 5$

x	y
-1	2
2	1
5	0

Graphing each of these lines on a common set of axes, we obtain



It appears that the graphs intersect at $(2, 1)$*. Checking this into each equation, we see that it results in a true statement for each since*

first equation:

$$2(2) + 1 \stackrel{?}{=} 5$$
$$5 = 5\checkmark$$

second equation:

$$2 + 3(1) \stackrel{?}{=} 5$$
$$5 = 5\checkmark$$

Therefore, $(2, 1)$ *is the solution of the given system.*

Practice 3. *Solve the following system by graphing.* $\begin{cases} 3x - y = -6 \\ -x + 2y = -3 \end{cases}$ *(Answer on page 64)*

In the previous two examples we solved a system of linear equations that had exactly one solution. However, if we consider all the possible outcomes that we can have when graphing two lines, then we see that two lines can either intersect in exactly one point, can never intersect, or intersect everywhere. If the lines never intersect, then they must be parallel. Recall from Core Mathematics I that two lines are parallel if they have exactly the same slope. If the lines intersect everywhere then the lines are equivalent, and we would have an infinite number of solutions. The next two examples illustrate each of these cases.

Example 4. *Solve the following system by graphing.* $\begin{cases} x - 2y = 4 \\ -2x + 4y = 0 \end{cases}$

Solution 4. *Again, plotting points we find*

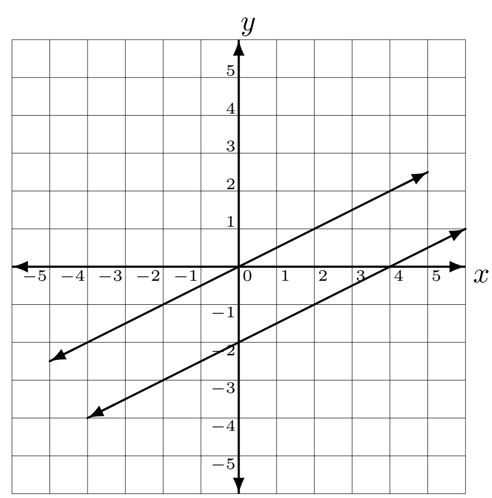
$x - 2y = 4$

x	y
0	-2
2	-1
4	0

$-2x + 4y = 0$

x	y
0	0
2	1
4	2

Therefore, plotting these points we obtain the following graphs



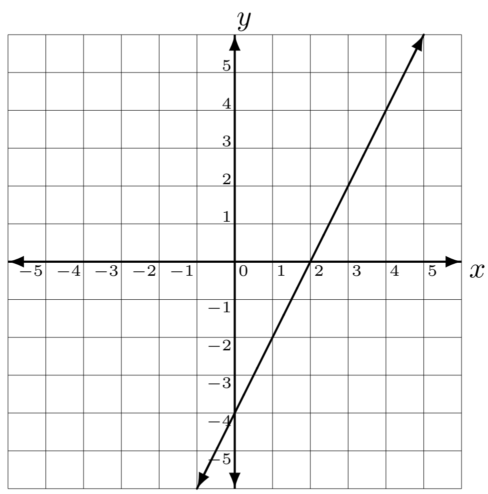
The two lines appear to be parallel, but graphs can be deceiving. Remember that parallel lines have identical slopes. So, in order *to guarantee these lines are parallel, we determine the slope of each line by writing both lines in slope-intercept form.*

$$\begin{aligned} x - 2y &= 4 \\ -2y &= -x + 4 \\ y &= \frac{1}{2}x - 2 \end{aligned} \qquad\qquad \begin{aligned} -2x + 4y &= 0 \\ 4y &= 2x \\ y &= \frac{1}{2}x \end{aligned}$$

Hence, the lines really are parallel since they have the same slope of $m = \frac{1}{2}$. *Therefore, the system has no solution.*

Example 5. *Solve the following system by graphing.* $\begin{cases} 2x - y = 4 \\ -4x + 2y = -8 \end{cases}$

Solution 5. *The intercepts of the first line are* $(2,0)$ *and* $(0,-4)$, *and the intercepts for the second line are also* $(2,0)$ *and* $(0,-4)$. *Since the intercepts are equal, the two lines must be the same. Therefore, both lines are given by the following graph*



The lines intersect infinitely many times, so the system has an infinite number of solutions.

Practice 4. *Solve the following system by graphing.* $\begin{cases} 3x - y = 4 \\ -9x + 3y = -12 \end{cases}$

(Answer on page 64)

Practice 5. *Solve the following system by graphing.* $\begin{cases} 3x - 2y = 6 \\ -6x + 4y = 5 \end{cases}$ *(Answer below)*

ANSWERS TO PRACTICE PROBLEMS

1. Not a solution.
2. $(1, -4)$
3. $(-3, -3)$
4. infinite number of solutions
5. no solution

SECTION 3.1 EXERCISES

(Answers are found on page 233.)

Solve each system by graphing. Plot at least two points for each line. Remember to check your solution in both equations.

1. $\begin{cases} x - 3y = 10 \\ x + y = 2 \end{cases}$

2. $\begin{cases} x - y = 6 \\ 3x - y = 10 \end{cases}$

3. $\begin{cases} 3x - 6y = 9 \\ 2x - 4y = 10 \end{cases}$

4. $\begin{cases} x - y = 1 \\ x + 2y = 7 \end{cases}$

5. $\begin{cases} 3x - 4y = 12 \\ x + y = -3 \end{cases}$

6. $\begin{cases} 2x + y = 5 \\ 3x + 9y = 15 \end{cases}$

7. $\begin{cases} x + 5y = -9 \\ 4x - 3y = -13 \end{cases}$

8. $\begin{cases} x + 4y = 1 \\ x - 4y = 1 \end{cases}$

9. $\begin{cases} 5x + 5y = 45 \\ 3x + 2y = 18 \end{cases}$

10. $\begin{cases} 3x - 5y = 4 \\ -6x + 10y = -8 \end{cases}$

11. $\begin{cases} 4x - y = 6 \\ 2x + 3y = 10 \end{cases}$

12. $\begin{cases} x + 2y = 12 \\ -3x + 5y = -25 \end{cases}$

13. $\begin{cases} x + y = 7 \\ -2x + 3y = 16 \end{cases}$

14. $\begin{cases} 3x + 4y = 26 \\ 4x + 3y = 30 \end{cases}$

15. $\begin{cases} 3x - 2y = 12 \\ 4x + y = 5 \end{cases}$

16. $\begin{cases} 2x - y = 4 \\ 4x + y = 2 \end{cases}$

17. $\begin{cases} 4x - y = 6 \\ -4x + y = 8 \end{cases}$

18. $\begin{cases} 3x + 2y = 27 \\ x - y = 4 \end{cases}$

19. $\begin{cases} 2x - y = 2 \\ 4x + y = 10 \end{cases}$

20. $\begin{cases} x + y = 4 \\ x - y = 2 \end{cases}$

21. $\begin{cases} x + y = 0 \\ 3x - 2y = 10 \end{cases}$

22. $\begin{cases} 2x + y = -6 \\ 4x + 2y = 8 \end{cases}$

23. $\begin{cases} -3x + y = -2 \\ 2x + y = 8 \end{cases}$

24. $\begin{cases} 4x + y = 3 \\ 3x + 5y = 15 \end{cases}$

25. $\begin{cases} 3x - 2y = 20 \\ 2x + 7y = 5 \end{cases}$

26. $\begin{cases} 5x + y = 2 \\ 10x + 2y = 4 \end{cases}$

27. $\begin{cases} 5x - 4y = 8 \\ -5x - 4y = 8 \end{cases}$

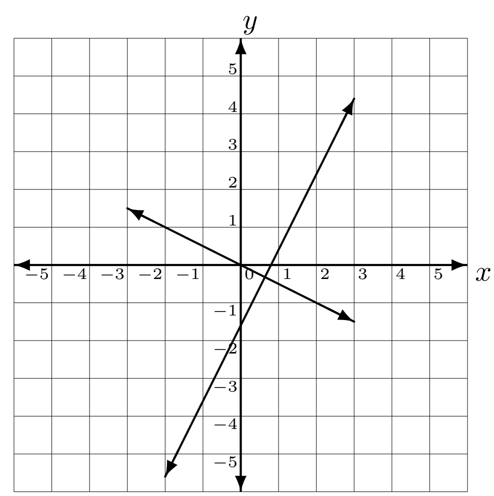
28. $\begin{cases} 2x + y = -2 \\ 2x + 3y = 6 \end{cases}$

3.2 Substitution Method

The graphing method for solving a system of linear equations has the advantage that it is visual and that the solution corresponds to the intersection points. However, a major weakness is that graphs can be deceiving. Some lines that look parallel are not, or sometimes two lines clearly intersect but in a hard to guess location. Consider the following system

$$\begin{cases} 6x - 3y = 5 \\ \ \ x + 2y = 0 \end{cases}$$

The graphs of these two equations are given below.



Can you determine the solution of this system? The solution is $\left(\frac{2}{3}, -\frac{1}{3}\right)$, but it is probably hard to determine from the graph.

In this section we will discuss an algebraic way to solve a system of linear equations. The method to be discussed in this section is the **Substitution Method**. The Substitution Method is useful when one equation can be solved easily for one of the variables.

Steps for the Substitution Method:

1. Choose one of the equations and solve for one variable in terms of the other variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the equation from Step 2. (There will be one equation with one variable).
4. Substitute the solution from Step 3 into either of the original equations. This will give the value of the other variable.

Remember that a system of linear equations is not completely solved until values for both x and y are found. To avoid this mistake, write all answers as an ordered pair.

Example 1. *Solve the following system using the Substitution Method.*

$$\begin{cases} 6x - 3y = 5 \\ x + 2y = 0 \end{cases}$$

Solution 1. *The second equation can be easily solved for x giving us*

$$\begin{aligned} x + 2y &= 0 \\ x &= -2y \end{aligned}$$

We now substitute this into the x variable in the first equation. This gives us

$$\begin{aligned} 6(-2y) - 3y &= 5 \\ -12y - 3y &= 5 \\ -15y &= 5 \\ y &= -\frac{1}{3} \end{aligned}$$

Next, we need to find the value of our x variable by substituting this into one of the equations. Since we already know that $x = -2y$, substituting into this equation gives us

$$\begin{aligned} x &= -2\left(-\frac{1}{3}\right) \\ x &= \frac{2}{3} \end{aligned}$$

Hence, the solution of the given system is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Practice 1. *Solve the following system using the Substitution Method.*

$$\begin{cases} 4x + 3y = -3 \\ x - 3y = -2 \end{cases}$$

(Answer on page 74)

Example 2. *Solve the following system using the Substitution Method.*

$$\begin{cases} 3x + 2y = -8 \\ 2x + y = 5 \end{cases}$$

Solution 2. *Notice that the second equation can be solved easily for y, giving us*

$$\begin{aligned} 2x + y &= 5 \\ y &= -2x + 5 \end{aligned}$$

This is what we now substitute into the y variable in our first equation. This gives us:

$$\begin{aligned} 3x + 2(-2x + 5) &= -8 \\ 3x - 4x + 10 &= -8 \\ -x + 10 &= -8 \\ -x &= -18 \\ x &= 18 \end{aligned}$$

Next, we need to find the value of our y *variable by substituting* $x = 18$ *into one of the equations. Since we already know that* $y = -2x + 5$*, substituting in this equation gives us:*

$$\begin{aligned} y &= -2(18) + 5 \\ y &= -36 + 5 \\ y &= -31 \end{aligned}$$

Thus, $(18, -31)$ *is the solution of the given system.*

Practice 2. *Solve the following system using the Substitution Method.*

$$\begin{cases} 2x + y = 4 \\ 4x + 3y = 10 \end{cases}$$

(Answer on page 74)

Example 3. *Solve the following system using the Substitution Method.*

$$\begin{cases} x - y = -3 \\ 4x + 3y = -5 \end{cases}$$

Solution 3. *Notice that the first equation can be solved quickly for either* x *or* y*. We will solve for* x*.*

$$\begin{aligned} x - y &= -3 \\ x &= y - 3 \end{aligned}$$

We now substitute this into the x *variable in our second equation.*

$$\begin{aligned} 4(y - 3) + 3y &= -5 \\ 4y - 12 + 3y &= -5 \\ 7y - 12 &= -5 \\ 7y &= 7 \\ y &= 1 \end{aligned}$$

We now substitute $y = 1$ into one of our equations in order to find the value of x. Since we already know that $x = y - 3$, substituting $y = 1$ into this equation yields

$$\begin{aligned} x &= 1 - 3 \\ x &= -2 \end{aligned}$$

So, the solution of the given system is $(-2, 1)$.

Practice 3. *Solve the following system using the Substitution Method.*

$$\begin{cases} -x + y = 9 \\ 3x - 2y = 7 \end{cases}$$

(Answer on page 74)

We have seen that the substitution method can be used on those systems where one of the linear equation's coefficients is one. However, this is not the only situation that the Substitution Method can be easily used. In fact, the Substitution Method can be used to solve any system of linear equations.

Example 4. *Solve the following system using the Substitution Method.*

$$\begin{cases} 6x + 3y = 12 \\ 3x + 2y = 5 \end{cases}$$

Solution 4. *Notice that the first equation can be solved for y without fractions, giving us*

$$\begin{aligned} 6x + 3y &= 12 \\ 3y &= -6x + 12 \\ y &= -2x + 4 \end{aligned}$$

Next, we substitute this expression into our y variable in our second equation.

$$\begin{aligned} 3x + 2(-2x + 4) &= 5 \\ 3x - 4x + 8 &= 5 \\ -x + 8 &= 5 \\ -x &= -3 \\ x &= 3 \end{aligned}$$

We now substitute $x = 3$ into one of our equations in order to find the value of y. Since we already know that $y = -2x + 4$, substituting $x = 3$ into this equation yields

$$\begin{aligned} y &= -2(3) + 4 \\ y &= -6 + 4 \\ y &= -2 \end{aligned}$$

Therefore, $(3, -2)$ is the solution of the given system.

Practice 4. *Solve the following system using the Substitution Method.*

$$\begin{cases} 5x + 10y = 20 \\ 2x + 6y = 10 \end{cases}$$

(Answer on page 74)

All of our previous examples were **consistent systems**. A system is consistent if there is exactly one solution. However, this is not always the situation. Recall from Section 3.1 that two lines may not intersect or they can represent the same line. A system is **inconsistent** if there is no solution. This happens when the two equations represent parallel lines. On the other hand, a system is **dependent** if there is an infinite number of ordered pairs as solutions. This occurs when the two equations represent the same line.

Example 5. *Solve the following system using the Substitution Method.*

$$\begin{cases} 2x - y = 3 \\ -6x + 3y = 9 \end{cases}$$

Solution 5. *Notice that the first equation can be solved easily for y.*

$$\begin{aligned} 2x - y &= 3 \\ -y &= -2x + 3 \\ y &= 2x - 3 \end{aligned}$$

We now substitute this into the y variable in our second equation.

$$\begin{aligned} -6x + 3(2x - 3) &= 9 \\ -6x + 6x - 9 &= 9 \\ -9 &= 9 \end{aligned}$$

Since this is a false statement, the system is inconsistent. Therefore, there is no solution.

Example 6. *Solve the following system using the Substitution Method.*

$$\begin{cases} -x + 2y = -3 \\ 3x - 6y = 9 \end{cases}$$

Solution 6. *Notice the first equation can easily be solved for x. Therefore, we have*

$$\begin{aligned} -x + 2y &= -3 \\ -x &= -2y - 3 \\ x &= 2y + 3 \end{aligned}$$

Next, we substitute this expression into the x variable in the second equation.

$$\begin{aligned} 3(2y + 3) - 6y &= 9 \\ 6y + 9 - 6y &= 9 \\ 9 &= 9 \end{aligned}$$

Since we arrived at a true statement, the two equations in the system are equivalent, or the exact same line. Hence, any solution of one of them is a solution of the other. Therefore, there are an infinite number of solutions.

Practice 5. *Solve the following system using the Substitution Method.*

$$\begin{cases} 2x + 4y = 6 \\ 5x + 10y = 16 \end{cases}$$

(Answer on page 74)

Practice 6. *Solve the following system using the Substitution Method.*

$$\begin{cases} x - 3y = 6 \\ -4x + 12y = -24 \end{cases}$$

(Answer on page 74)

Finally, we will consider an example with fractional coefficients.

Example 7. *Solve the following system using the Substitution Method.*

$$\begin{cases} \frac{1}{4}x + \frac{1}{2}y = 3 \\ 3x - 5y = 14 \end{cases}$$

> **Solution 7.** *For this example, it is easiest to solve for the x variable in the first equation. Isolating the x term on one side and multiplying by four we obtain,*
>
> $$\begin{aligned} \frac{1}{4}x + \frac{1}{2}y &= 3 \\ \frac{1}{4}x &= 3 - \frac{1}{2}y \\ x &= 12 - 2y \end{aligned}$$
>
> *Next, we substitute this into the x variable in our second equation. As a result, we obtain*
>
> $$\begin{aligned} 3(12 - 2y) - 5y &= 14 \\ 36 - 6y - 5y &= 14 \\ 36 - 11y &= 14 \\ -11y &= -22 \\ y &= 2 \end{aligned}$$
>
> *To finish the problem, we substitute $y = 2$ into one of our equations to obtain the value of x. Since we know that $x = 12 - 2y$, substituting into this equation yields*
>
> $$\begin{aligned} x &= 12 - 2(2) \\ x &= 12 - 4 \\ x &= 8 \end{aligned}$$
>
> *Hence, the solution to the given system is $(8, 2)$.*

Practice 7. *Solve the following system using the Substitution Method.*

$$\begin{cases} \frac{1}{3}x - \frac{1}{6}y = \ \ 4 \\ 2x - 5y = 48 \end{cases}$$

(Answer below)

ANSWERS TO PRACTICE PROBLEMS

1. $\left(-1, \frac{1}{3}\right)$
2. $(1, 2)$
3. $(25, 34)$
4. $(2, 1)$
5. No solution
6. Infinite number of solutions
7. $(9, -6)$

SECTION 3.2 EXERCISES

(Answers are found on page 233.)

Solve each system using the Substitution Method.

1. $\begin{cases} 2x + 7y = -12 \\ x = 3 - 2y \end{cases}$

2. $\begin{cases} -2x - 4y = 0 \\ 3x - \ \ y = 7 \end{cases}$

3. $\begin{cases} -2x + \ \ y = 1 \\ x - 3y = 5 \end{cases}$

4. $\begin{cases} -2x + 4y = \ \ 7 \\ x - 2y = -7 \end{cases}$

5. $\begin{cases} -4x + 8y = 0 \\ 2x - 4y = 0 \end{cases}$

6. $\begin{cases} -x + \ \ y = 5 \\ 2x - 2y = 9 \end{cases}$

7. $\begin{cases} -2x + y = -3 \\ 2x - y = \ \ 3 \end{cases}$

8. $\begin{cases} 2x + 3y = 5 \\ x - 4y = 6 \end{cases}$

9. $\begin{cases} 4x + y = 10 \\ 3x + 2y = 5 \end{cases}$

10. $\begin{cases} 4x - 3y = 11 \\ 8x + 4y = 12 \end{cases}$

11. $\begin{cases} 4x - 2y = 6 \\ -6x + 3y = 5 \end{cases}$

12. $\begin{cases} -5x + 2y = 3 \\ 4x + y = 2 \end{cases}$

13. $\begin{cases} 7x - y = 38 \\ 2x + 5y = -5 \end{cases}$

14. $\begin{cases} 8x - 3y = -4 \\ 7x - y = 3 \end{cases}$

15. $\begin{cases} x - 5y = 7 \\ 2x - y = 14 \end{cases}$

16. $\begin{cases} 4x + 2y = 5 \\ 2x + y = -4 \end{cases}$

17. $\begin{cases} 6x + 3y = 12 \\ 9x + 6y = 15 \end{cases}$

18. $\begin{cases} x + 7y = 3 \\ \frac{5}{2}x + 6y = -4 \end{cases}$

19. $\begin{cases} x + \frac{1}{2}y = -5 \\ x - \frac{1}{2}y = 1 \end{cases}$

20. $\begin{cases} \frac{1}{2}x + 5y = -4 \\ 7x - 3y = 17 \end{cases}$

21. $\begin{cases} 2x + 5y = -20 \\ x + \frac{5}{2}y = -10 \end{cases}$

22. $\begin{cases} 3x - 4y = -12 \\ x + y = -3 \end{cases}$

23. $\begin{cases} x + 4y = -9 \\ 4x - 13y = -13 \end{cases}$

24. $\begin{cases} 2x - y = 6 \\ 3x + y = 6 \end{cases}$

25. $\begin{cases} 4x - 2y = 8 \\ 3x + 2y = -6 \end{cases}$

26. $\begin{cases} 7x - y = 8 \\ x + 2y = 11 \end{cases}$

27. $$\begin{cases} 3x - y = 4 \\ x + 3y = 5 \end{cases}$$

28. $$\begin{cases} x - 4y = 12 \\ 3x + y = -5 \end{cases}$$

3.3 Elimination Method

Although the substitution method can be used for solving any system of linear equations, it works the easiest when one equation can be easily solved for one of the variables without fractions. However, if this is impossible, the Elimination Method can be used.

Steps for the Elimination Method:

1. Align the variables.
2. Multiply one or both equations by the appropriate constant so that the coefficients of one of the variables are opposites of each other.
3. Add the two equations together. (If you have done everything correctly, this will produce one equation with at most one variable).
4. Solve the equation from Step 3.
5. Substitute the solution from Step 4 into either of the original equations. This will give the value of the other variable.

As with the substitution method, remember that a system of linear equations is not completely solved until values for both x and y are found. To avoid this mistake, write all answers as ordered pairs.

Example 1. *Solve the following system using the Elimination Method.*

$$\begin{cases} 6x - 5y = 25 \\ 4x + 15y = 13 \end{cases}$$

Solution 1. *To eliminate the y variable, we multiply the first equation by 3 and leave the second equation alone,*

$$\begin{aligned} 3(6x - \quad 5y &= 25) \\ 4x + \quad 15y &= 13 \end{aligned}$$

We now have

$$\begin{aligned} 18x - \ 15y &= 75 \\ 4x + \ 15y &= 13 \end{aligned}$$

Adding these two equations together, and solving for x, we get

$$\begin{aligned} 22x &= 88 \\ x &= 4 \end{aligned}$$

Now, we must find the value for y by substituting $x = 4$ into one of the two original equations. Substituting into the first equation gives

$$\begin{aligned} 6(4) - 5y &= 25 \\ 24 - 5y &= 25 \\ -5y &= 1 \\ y &= -\frac{1}{5} \end{aligned}$$

Hence, the solution of the given system is $\left(4, -\frac{1}{5}\right)$.

Practice 1. *Solve the following system using the Elimination Method.*

$$\begin{cases} 5x - y = 13 \\ 2x + y = \ \ 8 \end{cases}$$

(Answer on page 83)

Notice in the first example, we only had to multiply one equation by a constant in order to eliminate one of the variables. However, sometimes it is necessary to multiply both equations by a constant in order to eliminate a variable. This is illustrated in our next example.

Example 2. *Solve the following system using the Elimination Method.*

$$\begin{cases} 8x + 9y = 13 \\ 6x - 5y = 45 \end{cases}$$

Solution 2. *To eliminate the x variable we will multiply the first equation by -3 and the second equation by 4.*

$$\begin{aligned} -3(8x + \ 9y &= 13) \\ 4(6x - \ 5y &= 45) \end{aligned}$$

This gives us

$$\begin{aligned} -24x - \ 27y &= -39 \\ 24x - \ 20y &= 180 \end{aligned}$$

Adding these two equations together and solving for y yields

$$\begin{aligned} -47y &= 141 \\ y &= -3 \end{aligned}$$

Next, we need to find the value for x by substituting $y = -3$ into either of the two original equations. If we substitute $y = -3$ into the first equation, we get

$$\begin{aligned} 8x + 9(-3) &= 13 \\ 8x - 27 &= 13 \\ 8x &= 40 \\ x &= 5 \end{aligned}$$

Therefore, $(5, -3)$ is the solution of the given equation.

Practice 2. *Solve the following system using the Elimination Method.*

$$\begin{cases} 4x - 5y = 35 \\ 3x - 4y = 24 \end{cases}$$

(Answer on page 83)

In Section 3.2 we discussed an inconsistent system and a dependent system. Recall that a system is **inconsistent** if there is no solution. This happens when the two equations represent parallel lines. On the other hand, a system is **dependent** if there is an infinite number of ordered pairs as solutions. This occurs when the two equations represent the same line.

Example 3. *Solve the following system using the Elimination Method.*

$$\begin{cases} -6x + 9y = 12 \\ \ \ 2x - 3y = -4 \end{cases}$$

Solution 3. *In order to eliminate the x variable, we multiply the second equation by 3.*

$$\begin{aligned} -6x +\ & 9y = 12 \\ 3(2x -\ & 3y = -4) \end{aligned}$$

This gives us

$$\begin{aligned} -6x +\ & 9y = \ \ 12 \\ 6x -\ & 9y = -12 \end{aligned}$$

When we add these two equations together, we get

$$0 = 0$$

Since this is a true statement, we know that the system is dependent. Therefore, there are an infinite number of solutions.

Example 4. *Solve the following system using the Elimination Method.*

$$\begin{cases} -5x + \ \ 4y = 1 \\ 15x - 12y = 4 \end{cases}$$

Solution 4. *If we multiply the first equation by* 3

$$\begin{aligned} 3(-5x + \ \ 4y &= 1) \\ 15x - \ 12y &= 4 \end{aligned}$$

we will be able to eliminate the x variable. This gives us

$$\begin{aligned} -15x + \ 12y &= 3 \\ 15x - \ 12y &= 4 \end{aligned}$$

Adding these two equations together results in

$$0 = 7$$

Since this is a false statement, we know that the system is inconsistent. Therefore, there is no solution.

Practice 3. *Solve the following system using the Elimination Method.*

$$\begin{cases} 3x - \ 5y = \ \ 2 \\ -6x + 10y = -4 \end{cases}$$

(Answer on page 83)

Practice 4. *Solve the following system using the Elimination Method.*

$$\begin{cases} 4x - 6y = 5 \\ -2x + 3y = 7 \end{cases}$$

(Answer on page 83)

Next, we turn our attention to fractional coefficients.

Example 5. *Solve the following system using the Elimination Method.*

$$\begin{cases} 3x + \frac{7}{2}y = \frac{3}{4} \\ -\frac{1}{2}x + \frac{5}{3}y = -\frac{5}{4} \end{cases}$$

Solution 5. *We will solve this problem by eliminating the x variable. In order to eliminate the x variable and also eliminate all fractions, we will multiply the first equation by 4 and the second equation by 24.*

$$\begin{aligned} 4\left(3x + \tfrac{7}{2}y = \tfrac{3}{4}\right) \\ 24\left(-\tfrac{1}{2}x + \tfrac{5}{3}y = -\tfrac{5}{4}\right) \end{aligned}$$

This gives us

$$\begin{aligned} 12x + 14y &= 3 \\ -12x + 40y &= -30 \end{aligned}$$

Adding these two equations together gives us

$$\begin{aligned} 54y &= -27 \\ y &= -\frac{1}{2} \end{aligned}$$

To find the $x-$variable, we substitute $y = -\frac{1}{2}$ into either of the equations. Substituting into the first equation gives us

$$\begin{aligned} 3x + \frac{7}{2}\left(-\frac{1}{2}\right) &= \frac{3}{4} \\ 3x - \frac{7}{4} &= \frac{3}{4} \\ 3x &= \frac{5}{2} \\ x &= \frac{5}{6} \end{aligned}$$

So, $\left(\frac{5}{6}, -\frac{1}{2}\right)$ is the solution of the given system.

Practice 5. *Solve the following system using the Elimination Method.*

$$\begin{cases} \frac{1}{3}x + \frac{1}{6}y = 1 \\ \frac{1}{2}x - \frac{1}{4}y = 0 \end{cases}$$

(Answer on page 83)

ANSWERS TO PRACTICE PROBLEMS

1. $(3,2)$
2. $(20,9)$
3. infinite number of solutions
4. no solution
5. $\left(\frac{3}{2},3\right)$

SECTION 3.3 EXERCISES

(Answers are found on page 234.)

Solve each system using the Elimination Method.

1. $\begin{cases} -2x+3y=-1 \\ x-2y=3 \end{cases}$

2. $\begin{cases} 4x-3y=11 \\ 8x+4y=12 \end{cases}$

3. $\begin{cases} 2x-5y=4 \\ 3x-2y=4 \end{cases}$

4. $\begin{cases} 7x-y=8 \\ 2x-2y=-3 \end{cases}$

5. $\begin{cases} 3x+5y=6 \\ 5x-3y=-7 \end{cases}$

6. $\begin{cases} 9x+2y=16 \\ 9x+2y=4 \end{cases}$

7. $\begin{cases} 3x-4y=-2 \\ 4x-6y=-1 \end{cases}$

8. $\begin{cases} 9x+4y=-3 \\ 6x+7=-6y \end{cases}$

9. $\begin{cases} 3x+3y=-1 \\ 2x-2y=5 \end{cases}$

10. $\begin{cases} x-y=2 \\ 2x-y=-3 \end{cases}$

11. $\begin{cases} 4x+y=24 \\ x+2y=2 \end{cases}$

12. $\begin{cases} \frac{1}{2}x-5y=6 \\ 3x-2y=8 \end{cases}$

13. $\begin{cases} x+8y=-18 \\ \frac{1}{2}x+3y=6 \end{cases}$

14. $\begin{cases} 4x=7y-6 \\ 9y+12x=12 \end{cases}$

15. $\begin{cases} 4x + 6y - 6 = 0 \\ 3y - 3 = -2x \end{cases}$

16. $\begin{cases} 3x + \frac{1}{3}y = 13 \\ 4x - \frac{3}{2}y = -6 \end{cases}$

17. $\begin{cases} 0.2x + 0.6y = 1 \\ 0.5x + 0.6y = -0.2 \end{cases}$

18. $\begin{cases} \frac{1}{2}x + \frac{1}{6}y = \frac{1}{2} \\ \frac{1}{10}x - \frac{3}{10}y = \frac{3}{10} \end{cases}$

19. $\begin{cases} 2x - y = 10 \\ 3x + 4y = -7 \end{cases}$

20. $\begin{cases} 3x + 4y = -10 \\ 2x - 3y = -1 \end{cases}$

21. $\begin{cases} 6x - 5y = 25 \\ 4x + 15y = 13 \end{cases}$

22. $\begin{cases} 3x + 5y = 1 \\ 2x + 3y = 0 \end{cases}$

23. $\begin{cases} 2x - 3y = 5 \\ x + 4y = 3 \end{cases}$

24. $\begin{cases} 4x - 2y = -1 \\ 6x - 4y = -3 \end{cases}$

25. $\begin{cases} 2x + 3y = 2 \\ 4x - 5y = -1 \end{cases}$

Solve the following systems of equations using either the Substitution Method or the Elimination Method.

26. $\begin{cases} 4x - 5y = 35 \\ 3x - 4y = 24 \end{cases}$

27. $\begin{cases} 4x - 3y = 5 \\ 3x + 4y = -2 \end{cases}$

28. $\begin{cases} 3x - 2y = 14 \\ y = 2x - 8 \end{cases}$

29. $\begin{cases} x + 3y = -16 \\ y = 6x + 1 \end{cases}$

30. $\begin{cases} x = 4y - 14 \\ x + y = 1 \end{cases}$

31. $\begin{cases} 3x + 4y = 2 \\ 6x + 8y = 1 \end{cases}$

32. $\begin{cases} 4x - 3y = 5 \\ 12x - 9y = 15 \end{cases}$

33. $\begin{cases} 5x - 4y = 8 \\ -5x - 4y = 8 \end{cases}$

34. $\begin{cases} 2x + y = -2 \\ 2x + 3y = 6 \end{cases}$

35. $\begin{cases} -3x + 8y = -7 \\ 2x - 7y = 3 \end{cases}$

36. $\begin{cases} 9x + 2y = -5 \\ 4x - 3y = 10 \end{cases}$

37. $\begin{cases} \frac{5}{2}x - \frac{1}{3}y = 1 \\ 4x - \frac{2}{3}y = -1 \end{cases}$

38. $\begin{cases} 4x - 2y = 21 \\ 3x + y = -10 \end{cases}$

39. $\begin{cases} -x + 8y = 15 \\ 2x + 3y = -11 \end{cases}$

40. $\begin{cases} 6x + 2y = -3 \\ -3x + y = \frac{5}{2} \end{cases}$

41. $\begin{cases} 3x + 4y = -4 \\ -x + 8y = 6 \end{cases}$

42. $\begin{cases} 8x - 2y = -3 \\ 4x + 6y = 3 \end{cases}$

43. $\begin{cases} \frac{1}{2}x - \frac{1}{5}y = 1 \\ \frac{1}{4}x + 3y = 2 \end{cases}$

44. $\begin{cases} 5x + 2y = 1 \\ -3x + 5y = 0 \end{cases}$

45. $\begin{cases} 7x - y = 3 \\ 4x + 5y = -4 \end{cases}$

46. $\begin{cases} 8x + 3y = 5 \\ -2x + 3y = 6 \end{cases}$

47. $\begin{cases} 3x - 5y = 1 \\ 6x + 7y = -2 \end{cases}$

3.4 Applications of Systems of Linear Equations

In this section we will cover a few applications of linear systems. The typical problem will involve describing a situation with two linear equations and then solving the system using your favorite method.

Example 1. *The sum of two numbers is 53 and their difference is 15. Find both numbers.*

> **Solution 1.** *If we let x be one of the numbers and y represent the other number, we have the following system to solve*
>
> $$\begin{cases} x + y = 53 \\ x - y = 15 \end{cases}$$
>
> *Notice that the system can be solved using the elimination method without the need to multiply either equation by a constant. Thus, adding the two equations together and solving for x, we get*
>
> $$\begin{aligned} 2x &= 68 \\ x &= 34 \end{aligned}$$
>
> *Substituting $x = 34$ into the first equation gives us*
>
> $$\begin{aligned} 34 + y &= 53 \\ y &= 19 \end{aligned}$$
>
> *Therefore, the two numbers are 19 and 34.*

Practice 1. *The sum of two numbers is 79 and their difference is 33. Find both numbers.* *(Answer on page 93)*

Example 2. *Suzie purchased only apples and oranges at the local market. Each apple costs 25 cents and each orange costs 30 cents. If Suzie purchased 15 pieces of fruit for* \$4.05*, determine the number of apples and the number of oranges she purchased.*

Solution 2. *Let*

$$\begin{aligned} x &= \text{ the number of apples purchased} \\ \text{and} \quad y &= \text{ the number of oranges purchased.} \end{aligned}$$

Since Suzie purchased a total of 15 pieces of fruit, The first equation of the system is $x + y = 15$*. Since the apple cost is* $0.25x$ *and the orange cost is* $0.30y$*, the second equation of the system is* $0.25x + 0.30y = 4.05$*. Thus, the system to be solved is*

$$\begin{cases} x + \quad y = \ 15 \\ 0.25x + 0.30y = 4.05 \end{cases}$$

Solving the first equation for x*, we get* $x = 15-y$*. If we substitute this into the second equation, we obtain*

$$\begin{aligned} 0.25(15-y) + 0.30y &= 4.05 \\ 3.75 - 0.25y + 0.30y &= 4.05 \\ 3.75 + 0.05y &= 4.05 \\ 0.05y &= 0.30 \\ y &= 6 \end{aligned}$$

Substituting $y = 6$ *into* $x = 15 - y$ *results in* $x = 15 - 6 = 9$*. Therefore, Suzie purchased 9 apples and 6 oranges.*

Practice 2. *Peter purchased Matchbox cars and Star Wars figures at a garage sale for a total of* $5.50*. Each Star Wars figure cost 50 cents and each Matchbox car cost 25 cents. If Peter purchased a total of 17 items, how many Star Wars figures and how many Matchbox cars did he buy?* *(Answer on page 93)*

Next, let us consider an example involving simple interest. Recall the following simple interest formula from Section 2.3.

Formula for simple interest is given by

$$I = prt$$

where I is the interest, p is the principal, r is the annual interest rate, and t is the time in years.

The word "simple" means that the interest is paid only once, so no compounding occurs like in a savings account where interest might be paid monthly.

Example 3. *Erin took an early retirement incentive of* \$500,000*. Her plan is to invest this money and live off the interest. Some of the money will be the principal for a savings account that pays* 3% *simple interest. The remainder of the money will serve as the principal for a stock fund that pays on average* 10% *simple interest. If the goal is to achieve* \$35,000 *in interest per year for living expenses, how much principal should be invested at* 3% *and* 10% *respectively?*

Solution 3. *We know that there are two different principals which sum to* \$500,000*. So if we let*

$$\begin{aligned} x &= \text{ the principal invested at } 3\% \\ \text{and} \quad y &= \text{ the principal invested at } 10\% \end{aligned}$$

the first equation of the system is $x + y = 500,000$.

Next, we calculate the interest on each account. The interest for one year from the principal invested at 3% *would be* $(x)(0.03)(1)$ *or just* $0.03x$*. Similarly, the interest for one year from the principal invested at* 10% *would be* $(y)(0.10)(1)$ *or just* $0.10y$*. Since the interests must sum to* \$35,000*, the second equation of the system is* $0.03x + 0.10y = 35,000$*. Therefore, we need to solve the system*

$$\begin{cases} x + \quad y = 500,000 \\ 0.03x + 0.10y = \ \ 35,000 \end{cases}$$

Since both x *and* y *are easy to solve for in our first equation, we will use the Substitution Method to solve this system. If we solve for* x *in the first equation, we obtain*

$$\begin{aligned} x + y &= 500,000 \\ x &= 500,000 - y \end{aligned}$$

Substituting this into the x *variable in our second equation and* *solving for y (by rounding to two decimal places), we obtain*

$$\begin{aligned} 0.03(500,000 - y) + 0.10y &= 35,000 \\ 15,000 - 0.03y + 0.10y &= 35,000 \\ 15,000 + 0.07y &= 35,000 \\ 0.07y &= 20,000 \\ y &= 285,714.29 \end{aligned}$$

We find the value of x by substituting this value into the y variable in $x = 500,000 - y$ to get $x = 500,000 - 285714.29 = 214,285.71$. Therefore, Erin needs to invest \$214,285.71 in the account paying 3% interest, and invest \$285,714.29 in the account paying 10% interest.

Practice 3. *Suppose instead that Erin wants to invest a portion of her \$500,000 incentive bonus in a stock account earning 6% interest and the rest in a bond fund earning 9% interest. If her goal is to achieve \$35,000 in interest per year for living expenses, how much principal should be invested at 6% and 9% respectively?* (Answer on page 93)

Next, we consider some examples involving distance. Here the key formula is $d = r \cdot t$, where d is the distance an object travels at an average rate r over a time period t. The units for d and t must agree with the units of r. For example, if r is measured in miles per hour then d must be measured in miles and t in hours. If r is given in feet per minute and t in hours then we would need to convert t to minutes or r to feet per hour.

Example 4. *Roberta and Gene recently completed a two day 1050 mile trip to Daytona. On the first day of the trip they averaged 60 miles per hour. On the second day of the trip, they averaged 65 miles per hour. If the trip took a total of 17 hours to complete, determine their driving time on each day of the trip.*

Solution 4. *If we let*

$$\begin{aligned} x &= \text{ the driving time for day one of the trip} \\ \text{and} \quad y &= \text{ the driving time for day two of the trip} \end{aligned}$$

then the first equation of the system is $x+y=17$ *since their trip took a total of 17 hours.*

The distances from each day sum to 1050. Using $d = rt$, *the distance traveled on day one is* $60x$ *and the distance traveled on day two is* $65y$. *So, the second equation of the system would be* $60x + 65y = 1050$. *Thus, we need to solve*

$$\begin{cases} x + \quad y = \quad 17 \\ 60x + 65y = 1050 \end{cases}$$

Solving the first equation for y, *we find that* $y = 17 - x$. *Substituting this into the second equation, we get*

$$\begin{aligned} 60x + 65(17 - x) &= 1050 \\ 60x + 1105 - 65x &= 1050 \\ 1105 - 5x &= 1050 \\ -5x &= -55 \\ x &= 11 \end{aligned}$$

To solve for y, *the driving time on day two, we substitute* $x = 11$ *into* $y = 17 - x = 17 - 11 = 6$. *Therefore, Gene and Roberta traveled* 11 *hours on day one and* 6 *hours on day two.*

Practice 4. *The Miller family traveled a total of 1075 miles to see the Grand Canyon over a two day period. They averaged 55 miles per hour on day one and 60 miles per hour on day two. If the trip took* $18\frac{1}{2}$ *hours, determine their driving time on each day of the trip.* (Answer on page 93)

Example 5. *Roberta and Gene are 28 miles apart and begin to bicycle towards each other at the same moment. Gene's rate is 5 miles per hour greater than Roberta's rate, and after* 1.5 *hours they reach each other. Determine the distance each will travel before reaching the other.*

Solution 5. *The key is that Gene's and Roberta's unknown distances total 28 miles. Now even though we are looking for distances, the more crucial unknowns are the rates of Gene and Roberta. After all, distance comes from* $d = rt$. *So, if we let*

$$\begin{aligned} x &= \textit{Roberta's rate} \\ \textit{and} \quad y &= \textit{Gene's rate} \end{aligned}$$

then the first equation of the system is $y = x + 5$.

The second equation comes from the total distance being 28 miles. Since the time for both Roberta and Gene is 1.5 hours, the distance traveled by Roberta is $1.5x$ *and the distance traveled by Gene is* $1.5y$. *So, the second equation of the system is* $1.5x + 1.5y = 28$. *Therefore, the system that needs solved is*

$$\begin{cases} y = x + 5 \\ 1.5x + 1.5y = 28 \end{cases}$$

Since the first equation is already solved for y, *we use the substitution method. Thus,*

$$\begin{aligned} 1.5x + 1.5(x + 5) &= 28 \\ 1.5x + 1.5x + 7.5 &= 28 \\ 3x + 7.5 &= 28 \\ 3x &= 20.5 \\ x &= \frac{20.5}{3} \\ x &= \frac{41}{6} \end{aligned}$$

In the last step, the decimal was converted into a fraction to avoid a repeating decimal answer. Now that we know that Roberta's rate is $\frac{41}{6}$ *miles per hour, we find the distance she traveled to be* $\frac{41}{6} \times \frac{3}{2} = \frac{41}{4} = 10\frac{1}{4}$ *miles. Therefore, Genes's distance traveled is* $28 - 10\frac{1}{4} = 17\frac{3}{4}$.

Practice 5. *Roberta and Gene are in cities that are 600 miles apart and decide to leave at the same moment, traveling towards each other. Gene's rate is 30 mph slower than Roberta's rate. If they reach each other after* 6 *hours, determine the rate of each car.* (Answers on page 93)

Finally, we consider a mixture problem.

Example 6. *A* 40% *dye solution is to mixed with a* 70% *dye solution to get 120 liters of a* 50% *dye solution. How many liters of the* 40% *and* 70% *solutions will be needed to produce the desired result?*

Solution 6. *Let*

$$\begin{aligned} x &= \text{ the number of liters of 40\% dye solution} \\ \text{and } \ y &= \text{ the number of liters of 70\% dye solution} \end{aligned}$$

We will summarize the given information in the following table where the percents are written as a decimal.

% dye solution	*Liters*	*Liters of dye solution*
0.40	x	$0.40x$
0.70	y	$0.70y$
0.50	*120*	$0.50(120)$

Since $0.50(120) = 60$*, this yields the following system to solve*

$$\begin{cases} x + y = 120 \\ 0.40x + 0.70y = 60 \end{cases}$$

Solving the first equation for x *we obtain* $x = 120 - y$*. If we substitute this into the second equation, we find*

$$\begin{aligned} 0.40(120 - y) + 0.70y &= 60 \\ 48 - 0.40y + 0.70y &= 60 \\ 48 + 0.30y &= 60 \\ 0.30y &= 12 \\ y &= 40 \end{aligned}$$

Using $y = 40$*, we find* $x = 120 - y = 120 - 40 = 80$*. Therefore,* 40 *liters of the* 70% *dye solution must be mixed with* 80 *liters of the* 40% *dye solution to obtain the desired result.*

Practice 6. *A* 30% *alcohol solution is to be mixed with a* 80% *alcohol solution to produce 100 gallons of a* 50% *alcohol solution. How many gallons of the* 30% *and* 80% *solutions must be mixed to produce the desired result?* *(Answer on page 93)*

ANSWERS TO PRACTICE PROBLEMS

1. 23 and 56
2. 5 Star Wars figures; 12 Matchbox cars
3. $\$166,666.67$ at 9%; $\$333,333.33$ at 6%
4. 7 hrs on day one; 11.5 hrs on day two
5. Gene's rate is 35 mph; Roberta's rate is 65 mph
6. 60 gallons of 30%; 40 gallons of 80%

SECTION 3.4 EXERCISES

(Answers are found on page 235.)

1. A shopper pays \$20.75 for four pounds of a nut mixture consisting of almonds and pecans. If almonds cost \$3.50 per pound and pecans cost \$6.00 per pound. How much of each were purchased?

2. Kent Cinema took in $\$1,270$ in ticket sales for a movie that 220 people attended. If a childs ticket sold for \$4.50 and an adult ticket sold for \$6.50. How many children attended the movie?

3. A recent mailing of 56 letters and postcards cost \$21.72. If the letters cost 42 cents each to mail and postcards cost 27 cents each, how many of each were mailed?

4. There is a total of 22 coins in kitchen drawer consisting of only quarters and nickels. The total value of the coins is \$3.90. How many nickels and how many quarters are there?

5. Three times the larger of the two numbers is 10 more than twice the smaller. Five times the smaller is 11 less than four times the larger. What are the numbers?

6. The tuition plus room and board comes to $\$20,800$ per year. The room and board is \$550 more than half the tuition. How much is the tuition, and what does room and board cost?

7. Suppose $\$6,000$ is invested in two accounts. One account yields 8% annual interest while the other account yields $7\frac{1}{2}\%$ annual interest. If the annual return on the two investments is \$464, how much was invested at each rate?

8. Suppose $\$9,000$ was invested one year ago. Part of the money was invested at 6% annual interest and the rest at 10%. If the total interest for the year was \$652.80, how much was invested at each rate?

9. Walter invested some money at 5% annual interest and twice that amount at 6%. If his total annual interest was \$127.50, how much was invested at each rate?

10. Janice invested some of \$1,500 bonus into an account that paid $4\frac{1}{2}\%$ interest after one year. The rest of her bonus was invested in a mutual fund that suffered a 3% loss for the same year. If the net profit from both accounts was only \$15, how much was invested in each account?

11. Suppose \$10,000 is invested in two accounts paying 5% and 4% annual interest. The amount of interest at the end of one year is the same for both accounts. How much was invested in each account?

12. Suppose \$2,000 is invested in two accounts paying 3% and 5% annual interest. After 1 year the amount of interest earned in the 5% account is double the amount earned in the 3% account. How much was invested in each account?

13. A jet takes 2 hours and 40 minutes to travel 1,120 miles with a tail wind. The return trip against the wind takes 2 hours and 48 minutes. What is the speed of the jet in still air?

14. A salmon travels downstream in 1 hour and 20 minutes. The return trip takes him 4 hours against the current. If the river flows a 1.5 mph, find the distance the salmon travels in one direction.

15. Meg traveled 7 miles yesterday walking for one half-hour and biking for one half-hour. Today traveling at the same rates she traveled a total of 6 miles by walking for 40 minutes and biking for 20 minutes. What was her walking and biking rates?

16. A plane flying cross-country 2,400 miles with the wind takes 3.75 hours. Against the wind it takes 4 hours. Determine the wind speed.

17. If Will can row a boat 1 mile upstream in 24 minutes and can row the same distance downstream in 12 minutes find Wills rowing speed and the speed of the current.

18. Steve invests a \$20,000 bonus, part in a stock account and the rest in a bond fund. If the stock account grew at a yearly simple interest rate of 12% and the bond fund grew at a yearly simple interest rate of 7% and the total interest earned in one year is \$2,000, determine the amount invested in the stock fund.

19. Johnny has a piggy bank with just nickels and dimes. If he dumps his coins into a change counter and the counter says he has a total of 300 coins with a total value of \$27.30, how many nickels were in his piggy bank?

20. The Andersons travel from their home to Disney, a total distance of 850 miles, over two days. If on the first day they averaged 55 miles per hour and on the second day 65 miles per hour and the total driving time was 15 hours, determine the time traveled on day one.

21. Sue began her trip to the beach in a big hurry averaging 70 miles per hour. After a ticket for speeding, she dropped her speed to 65 miles per hour on the average. If the total distance to the beach was 265 miles and the total driving time was 4 hours, how long did she drive at the faster rate?

22. Roberta and Gene have the same starting point for a bike trip, but Roberta leaves 30 minutes before Gene. If Gene's rate is 12 miles per hour and Roberta's rate is 10 miles per hour, how long will it take Gene to reach Roberta?

23. Jen and Sarah live 400 miles apart and decide to meet at a predetermined point. If they meet after 4 hours, find the speed of each car if Jen's rate is 20 miles per hour faster than Sarah's rate.

24. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to make a 78% antifreeze solution. How many gallons of each solution must be mixed to finish with 120 gallons of the 78% solution?

25. A 40% saline solution is to be mixed with a 70% saline solution to yield 90 liters of a 50% saline solution. How many liters of the 40% and 70% solutions will be needed?

26. A lab technician needs 8 liters of a 55% alcohol solution. How many liters of a 40% alcohol solution and how many liters of a 60% alcohol solution does he need to mix?

27. Commemorative coins are to be forged from an alloy that is 40% silver. The plant has one alloy with 50% silver content and another one with 25% silver content. How many grams of each alloy should be used to obtain 20 grams of the alloy with 40% silver content?

28. How much of a drink with 40% grape juice must be mixed with a 70% grape juice drink in order to obtain 20 gallons of a 50% grape juice drink?

29. How much of a 4% hydrochloric acid solution must be mixed with a 12% hydrochloric acid solution to obtain 12 ounces of a 9% hydrochloric acid solution?

Chapter 4

Exponents and Polynomials

4.1 Positive Integer Exponents

In Core Math I, it was discussed that exponents can be used for repeated multiplication. (Click here to review exponents from Core Math I) For example,

$$3^4 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3}_{\text{4 factors}}$$

where 3 is the **base**, 4 is the **exponent**, and 3^4 is a **power**. The term **power** is also used to refer to just the exponent itself. An algebraic expression containing exponents is an **exponential expression**. When working with exponents it is very important to determine the base.

Example 1. *Determine the base of each exponential expression and evaluate.*

(a) 2^4 *(b)* $(-2)^4$ *(c)* -2^4

Solution 1.

(a) Here 2 *is the base and* 4 *is the exponent. Hence,*

$$2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16.$$

(b) Now, -2 *is the base while* 4 *is still the exponent. Thus,*

$$(-2)^4 = -2 \cdot -2 \cdot -2 \cdot -2 = 16.$$

(c) *It is important to note that* 2 *is the base. The lack of parentheses shows that the exponent of 4 applies only to that base of* 2 *and not* -2. *Therefore,*

$$-2^4 = -1 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = -16.$$

CAUTION: $-a^n$ and $(-a)^n$ do not always yield the same answers. While both have n as the exponent, $-a^n$ has base a; whereas, $(-a)^n$ has base $-a$.

Practice 1. *Evaluate each expression.* *(Answers on page 104)*

(a) -4^2 (b) $(-4)^2$ (c) -3^3 (d) $(-3)^3$

We next would like to develop the exponent rules for the integer exponents. To do this, we will begin with positive integer exponents and recall our definition of exponents as repeated multiplication. Consider,

$$\begin{aligned} 3^2 \cdot 3^5 &= \underbrace{3 \cdot 3}_{\text{2 factors}} \cdot \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{\text{5 factors}} \\ &= \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3}_{\text{7 factors}} \\ &= 3^7 \end{aligned}$$

Also,

$$\begin{aligned} 7^4 \cdot 7^5 &= \underbrace{7 \cdot 7 \cdot 7 \cdot 7}_{\text{4 factors}} \cdot \underbrace{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}_{\text{5 factors}} \\ &= \underbrace{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}_{\text{9 factors}} \\ &= 7^9 \end{aligned}$$

We see that $3^2 \cdot 3^5 = 3^{2+5} = 3^7$ and $7^4 \cdot 7^5 = 7^{4+5} = 7^9$. This suggests that when we multiply like bases we add the exponents. This is the product rule for exponents.

Product Rule for Exponents: Let m and n be integers. Then for any real number a,

$$a^m \cdot a^n = a^{m+n}.$$

(When multiplying powers with like bases, we keep the base and add the exponents.)

CAUTION: The bases must be the same before applying the product rule for exponents. For example, $x^2 \cdot y^4 \neq (xy)^6$. Remember, keep the base the same and only add the exponents. DO NOT multiply the bases together. For example, $3^2 \cdot 3^4 \neq 9^6$; instead $3^2 \cdot 3^4 = 3^{2+4} = 3^6$.

Example 2. *Use the rules of exponents to simplify each product.*

(a) $5^4 \cdot 5^9$ (b) $(-2)^2 \cdot (-2)^3$

(c) $3^2 \cdot 3^6 \cdot 3^7$ (d) $x^3 \cdot x^2 \cdot x^4$

Solution 2.

(a) $5^4 \cdot 5^9 = 5^{4+9} = 5^{13}$

(b) $(-2)^2 \cdot (-2)^3 = (-2)^{2+3} = (-2)^5 = -32$

(c) $3^2 \cdot 3^6 \cdot 3^7 = 3^{2+6+7} = 3^{15}$

(d) $x^3 \cdot x^2 \cdot x^4 = x^{3+2+4} = x^9$

Practice 2. *Use the rules of exponents to simplify each product.* *(Answers on page 104)*

(a) $2^4 \cdot 2^5$ (b) $7^2 \cdot 7^3 \cdot 7^6$ (c) $y^5 \cdot y^9 \cdot y^2$

We use the commutative and associative properties to multiply more complicated expressions.

Example 3. *Use the rules of exponents to simplify each product.*

(a) $2x^4 \cdot 5x^3$

(b) $(x^3y^5)(x^5y^8)$

(c) $(-2x^4y^7)(3x^5y^2)(-4xy^5)$

Solution 3.

(a) $2x^4 \cdot 5x^3 = (2 \cdot 5)\left(x^4 \cdot x^3\right) = 10x^7$

(b) $(x^3y^5)(x^5y^8) = \left(x^3 \cdot x^5\right)\left(y^5 \cdot y^8\right) = x^8y^{13}$

(c) $(-2x^4y^7)(3x^5y^2)(-4xy^5) = (-2 \cdot 3 \cdot -4)\left(x^4x^5x\right)\left(y^7y^2y^5\right) = 24x^{10}y^{14}$

Practice 3. *Use the rules of exponents to simplify each expression. (Answers on page 104)*

(a) $(-2x^3y^2z^5)(4xy^9z^2)$

(b) $(4x^4y^5)(3x^2y)(5x^3y^4)$

Now that we know the product rule for exponents, let us examine what happens when we have a base to a power raised to another power. For example, consider $(3^4)^3$ and $(5^3)^4$. Using the definition of an exponent along with the product rule for exponents, we get

$$(3^4)^2 = \underbrace{3^4 \cdot 3^4}_{\text{2 factors}} = 3^{4+4} = 3^8$$

and

$$(5^3)^4 = \underbrace{5^3 \cdot 5^3 \cdot 5^3 \cdot 5^3}_{\text{4 factors}} = 5^{3+3+3+3} = 5^{12}.$$

Notice in the first example, the product of the exponents, namely $4 \cdot 2$, gives us the exponent in the answer. In the second example, $3 \cdot 4$ gives us the exponent of 12 in the answer. This leads us to our next property of exponents.

Power Rule for Exponents: Let m and n be integers. Then for any real number a,

$$(a^m)^n = a^{mn}.$$

(When raising a base to a power to another power, we multiply the exponents.)

CAUTION: The powers are to be multiplied together. Do not raise the power inside the parenthesis to the outside power. We will see in the exercises that these are not the same.

Example 4. *Use the rules of exponents to simplify each product.*

(a) $(4^3)^4$ *(b)* $(2^5)^4$
(c) $(x^5)^3 \cdot (x^4)^2$ *(d)* $(2^3)^2 \cdot (2^6)^3 \cdot (2^5)^4$

Solution 4.

(a) $(4^3)^4 = 4^{3 \cdot 4} = 4^{12}$

(b) $(2^5)^4 = 2^{5 \cdot 4} = 2^{20}$

(c) $(x^5)^3 \cdot (x^4)^2 = x^{5 \cdot 3} x^{4 \cdot 2} = x^{15} x^8 = x^{15+8} = x^{23}$

(d) $(2^3)^2 \cdot (2^6)^3 \cdot (2^5)^4 = 2^{3 \cdot 2} \cdot 2^{6 \cdot 3} \cdot 2^{5 \cdot 4} = 2^6 \cdot 2^{18} \cdot 2^{20}$
$= 2^{6+18+20} = 2^{44}$

Practice 4. *Use the rules of exponents to simplify each expression. (Answers* *on page 104)*

(a) $\left(y^2\right)^5 \cdot \left(y^6\right)^2 \cdot \left(y^3\right)^4$ *(b)* $\left(3^2\right)^3 \cdot \left(3^4\right)^2$

If we recall our commutative and associative rules for multiplication we can obtain two more power rules for exponents. Consider

$$\begin{aligned}(2x)^3 &= (2x)(2x)(2x)\\ &= 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x\\ &= 2^3 \cdot x^3\\ &= 8x^3\end{aligned}$$

and

$$\begin{aligned}(2x^4)^3 &= (2x^4)(2x^4)(2x^4)\\ &= 2\cdot 2\cdot 2\cdot x^4\cdot x^4\cdot x^4\\ &= 2^3\cdot x^{4+4+4}\\ &= 8x^{12}\end{aligned}$$

Notice in the second example that $2^3 = 8$ and $(x^4)^3 = x^{12}$. This leads to our second power rule for exponents.

Product to a Power Rule for Exponents: Let n be any integer. Then for any real numbers a and b,

$$(ab)^n = a^n b^n.$$

(When we raise a product to a power, we raise each factor to the given power.)

CAUTION: Raising a Product to a Power Rule for Exponents does NOT apply to a sum. For example, $(3+4)^2 = 7^2 = 49$ but $3^2+4^2 = 9+16 = 25$. Hence, $(3+4)^2 \neq 3^2+4^2$.

Furthermore, using our properties of fractions we can obtain the third power rule of exponents. Consider

$$\left(\frac{2}{3}\right)^4 = \frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\cdot\frac{2}{3} = \frac{2\cdot 2\cdot 2\cdot 2}{3\cdot 3\cdot 3\cdot 3} = \frac{2^4}{3^4} = \frac{16}{84}.$$

The resulting rule is stated below.

Quotient to a Power Rule for Exponents: Let n be any integer. Then for any real numbers a and b where $b \neq 0$,

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

(When we raise a quotient to a power, we raise both the numerator and denominator to the given power.)

Example 5. *Use the rules of exponents to simplify each expression. Assume all variables represent non-zero real numbers.*

(a) $\left(x^2y^3\right)^5$ *(b)* $\left(\dfrac{x^2}{y^3}\right)^4 x^5$

(c) $\left(\dfrac{x^7}{y^3}\right)^3\left(\dfrac{x^2}{y^5}\right)^2$ *(d)* $(2x^2)^3\left(\dfrac{1}{2}x^5\right)^2(4x^6)$

Solution 5.

(a) $\left(x^2y^3\right)^5 = (x^2)^5(y^3)^5 = x^{2\cdot 5}y^{3\cdot 5} = x^{10}y^{15}$

(b) $\left(\dfrac{x^2}{y^3}\right)^4 x^5 = \dfrac{(x^2)^4}{(y^3)^4}\ x^5 = \dfrac{x^8x^5}{y^{12}} = \dfrac{x^{8+5}}{y^{12}} = \dfrac{x^{13}}{y^{12}}$

(c) $\left(\dfrac{x^7}{y^3}\right)^3\left(\dfrac{x^2}{y^5}\right)^2 = \dfrac{(x^7)^3}{(y^3)^3}\cdot\dfrac{(x^2)^2}{(y^5)^2} = \dfrac{x^{21}}{y^9}\cdot\dfrac{x^4}{y^{10}} = \dfrac{x^{21+4}}{y^{9+10}} = \dfrac{x^{25}}{y^{19}}$

(d) $(2x^2)^3\left(\dfrac{1}{2}x^5\right)^2(4x^6) = 2^3(x^2)^3\left(\dfrac{1}{2}\right)^2(x^5)^2\cdot 4x^6$

$= 8\cdot\dfrac{1}{4}\cdot 4\cdot x^6x^{10}x^6 = 8x^{6+10+6} = 8x^{22}$

Practice 5. *Use the rules of exponents to simplify each expression. (Answers* *on page 104)*

(a) $\left(2x^3y^2z\right)^2$

(b) $\left(\frac{y^2z}{x^4}\right)^3\left(\frac{z^2y}{x^3}\right)^4$

(c) $\left(3x^3\right)^3\left(\frac{1}{3}x^4\right)^2\left(-3x^2\right)$

Now we turn our attention to the meaning of a^0 for $a \neq 0$. It is important that we define a^0 to be sure that it still satisfies all of the integer properties we have discussed so far. For example, we still want $2^0 \cdot 2^4 = 2^{0+4} = 2^4$. Since $2^4 \neq 0$, 2^0 must be 1. Note that it is important here that $2^4 \neq 0$ because otherwise 2^0 could have been any number. The same reasoning would apply to any nonzero base so we are led to make the following definition.

Zero Exponent: For any nonzero real number a,

$$a^0 = 1.$$

Therefore, $8^0 = 1$, $(-3)^0 = 1$, and $(2x^2y^3)^0 = 1$ for nonzero variables x and y.

We will discuss the negative integer exponents in Section 4.3.

ANSWERS TO PRACTICE PROBLEMS

1. (a) -16
   (b) 16
   (c) -27
   (d) -27
2. (a) 2^9
   (b) 7^{11}
   (c) y^{16}
3. (a) $-8x^4y^{11}z^7$
   (b) $60x^9y^{10}$
4. (a) y^{34}
   (b) 3^{14}
5. (a) $4x^6y^4z^2$
   (b) $\frac{y^{10}z^{11}}{x^{24}}$
   (c) $-9x^{19}$

SECTION 4.1 EXERCISES

(Answers are found on page 236.)

Using exponents, rewrite the following expressions in a simpler form.

1. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
2. $2 \cdot 5 \cdot 5 \cdot 2 \cdot 2 \cdot 5$
3. $3 \cdot 2 \cdot 3 \cdot 2 \cdot 6$
4. $\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$

Evaluate each expression.

5. -24^0
6. $(-24)^0$
7. $(-2)^5$
8. 6^3
9. $(-3)^4$
10. -2^6
11. $\left(\frac{1}{2}\right)^3$
12. $(-1)^{34}$
13. $(1.5)^2$
14. -7^2
15. 0^{13}
16. $(-2)^4$
17. -4^4
18. $(-138)^0$
19. $-(-1)^{60}$
20. -8^0
21. $-(-5)^2$
22. $(0.001)^2$

Use the laws of exponents to calculate the following expressions mentally.

23. $24^4 \cdot \left(\frac{1}{8}\right)^4$

24. $20^7\,(0.5)^7$

25. $\left(\frac{1}{3}\right)^8 \cdot 9^{11} \cdot \left(\frac{2}{6}\right)^3$

26. $20^3 \cdot \left(\frac{1}{5}\right)^3 \cdot 2^3 \cdot \left(\frac{1}{4}\right)^3$

Simplify the following, leaving your answer as a single base raised to a single exponent. Assume all variables represent non-zero real numbers.

27. $x^7 \cdot x^3 \cdot x^4$

28. $128 \cdot (2^4)^3$

29. $2^3 \cdot 4^5 \cdot 8^3$

30. $(5^2)^4 \cdot 5^3 \cdot 5^2$

31. $9^2 \cdot 12^3 \cdot 2$

32. $16^7 \cdot 4^8 \cdot 8^3$

33. $125^2 \cdot 9^4 \cdot 27^2 \cdot 25^4$

34. $27^6 \cdot 3^8 \cdot 9^4 \cdot 81$

Use a combination of the rules of exponents to simplify each expression. Assume all variables represent nonzero real numbers.

35. $2x^2(2x)^2$

36. $(7x^5)^2$

37. $(4x^2)^3$

38. $(3x^4y^5)^3$

39. $(-6x^3y^9)^2$

40. $(2x^2y^5z^9)^4$

41. $\left(-3xy^3z^2\right)^3$

42. $(\frac{1}{2}x^4)(16x^5)$

43. $(2x^3)(-3x^5)(4x^7)$

44. $(-4b^3)(\frac{1}{6}b^2)(-9b^4)$

45. $(x^2y)(x^5y^3)$

46. $(2x^2y)\left(\frac{1}{4}x^3y^5\right)\left(\frac{1}{3}x^4y^2\right)$

47. $\left(xy^2\right)^2\left(x^4y^5\right)$

48. $\left(x^3y^4\right)^2\left(2x^2y^5\right)^3$

49. $(3x^5y^2)(-2x^3y^4)^2$

50. $\left(\dfrac{-3x^4}{2y^5}\right)^2$

51. $\left(\dfrac{5y^4z}{2x^3}\right)^3$

52. $\left(\dfrac{x^4}{9y^3}\right)\left(\dfrac{3x^6}{y^4}\right)$

53. $\left(\dfrac{x^3}{2y}\right)^3\left(\dfrac{4x}{y^2}\right)$

54. $\left(2x^3\right)^3\left(\dfrac{1}{16}x^5y\right)\left(8x^3y^2\right)^2$

55. $\left(3x^4y^3\right)^2\left(7xy^5\right)$

56. $\left(\dfrac{4x^2}{y}\right)\left(\dfrac{3x}{y^2}\right)^2$

57. $\left(-3x^4y\right)^2\left(-2x^2y^5\right)^3$

58. $\left(\dfrac{1}{4}xy^3\right)^2\left(2x^4y^5\right)^3$

59. $\left(3x^2y^3\right)^2\left(\dfrac{1}{9}x^4y^5\right)\left(\dfrac{1}{4}xy^3\right)$

60. $\left(49x^6y^9z\right)\left(\dfrac{-xy^2z}{7}\right)^2$

Using the properties of exponents, determine which is larger. State why.

61. 4^{28} or 8^{18}

62. $3^9+3^9+3^9$ or 9^6

63. 27^9 or 9^{14}

64. 6^{18} or 3^{36}

65. Is it true that $(3^4)^2 = 3^{(4^2)}$? Explain why or why not.

66. Is $(3+4)^3 = 3^3+4^3$? Are there any combinations of values of a and b for which $(a+b)^3 = a^3+b^3$? Are there any positive integer values of n for which $(a+b)^n = a^n+b^n$?

4.2 Polynomials

A polynomial in x is any expression which can be written as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ are any numbers and each exponent is a whole number. In other words, a polynomial in x is a sum of a fixed number of terms of the form ax^n where a can be any number and n is a whole number. The **degree** of a polynomial is the highest exponent of the polynomial. Examples of polynomials in one variable are:

$$3x^2 - 5x + 1 \qquad 7x^5 - 8x^3 + 2x^2 - 9 \qquad -9y^5 - 3$$

Furthermore,

$$\begin{aligned} 3x^2 - 5x + 1 &\text{ has degree } 2 \\ 7x^9 - 8x^3 + 2x^2 - 4 &\text{ has degree } 9 \\ \text{and} \quad -9y^5 - 3 &\text{ has degree } 5 \end{aligned}$$

Polynomials can also have more than one variable. A polynomial with more than one variable is the sum of terms in which all variables have whole number exponents. The **degree of a term** of a polynomial, is the sum of the exponents on the variables. Therefore, the degree of a polynomial with more than one variable is the highest degree of any term of the polynomial. Examples of polynomials with more than one variable are:

$$2x^2y - 3xy + 5 \qquad 4x^3y^2 - 3x^2y^4 \qquad 5x - 7xy + 2y.$$

Also, we see that

$$\begin{aligned} 2x^2y - 3xy + 5 &\text{ has degree } 3 \\ 4x^3y^2 - 3x^2y^4 &\text{ has degree } 6 \\ \text{and} \quad 5x - 7xy + 2y &\text{ has degree } 2 \end{aligned}$$

Polynomials can be classified according to the number of terms they have. For example, a polynomial with only one term is called a **monomial**. A polynomial with exactly two terms is called a **binomial**, and a polynomial with three terms is called a **trinomial**.

Example 1. *Determine the degree of each polynomial.*

(a) $8x^4 - 3x^3 + 5x^2 - 1$

(b) $7x^3y^4 + 3x^2y^3 - 6x^5y$

(c) $9x^4y^2 - 5x^9y$

Solution 1. (a) *The degree of* $8x^4 - 3x^3 + 5x^2 - 1$ *is* 4 *since this is the largest exponent.*

(b) *The degree of* $7x^3y^4 + 3x^2y^3 - 6x^5y$ *is the same as the highest degree of any term in the polynomial. The term* $7x^3y^4$ *has degree* 7*,* $3x^2y^3$ *has degree* 5 *and* $6x^5y$ *has degree* 6*. Therefore, the degree of the polynomial is* 7*.*

(c) *The degree of* $9x^4y^2 - 5x^9y$ *is also the same as the highest degree of any term in the polynomial. The term* $9x^4y^2$ *has degree* 6 *and* $5x^9y$ *has degree* 10*. Thus, the degree of the polynomial is* 10*.*

Practice 1. *Determine the degree of each polynomial.* (Answer on page 114.)

(a) $10x^6 + 3x^4 - 4x^3 + 7$ (b) $3a^4b^2 - 2a^5b^3 + 6ab^4$

A polynomial can assume different values depending on the value of the variable. We will use the function notation that was discussed in Section 1.1.

Example 2. *Let* $P(x) = 4x^2 - 2x + 1$*. Find*

(a) $P(3)$ (b) $P(-2)$

Solution 2. (a) *To evaluate this polynomial when* $x = 3$*, we replace every variable* x *with the value* 3*. Recalling the order of operations, we obtain*

$$P(3) = 4(3)^2 - 2(3) + 1 = 4(9) - 6 + 1 = 36 - 6 + 1 = 31$$

as the value of the polynomial when $x = 3$*.*

(b) To evaluate this polynomial when $x = -2$, we replace every variable x with the value -2. For this value, we obtain

$$P(-2) = 4(-2)^2 - 2(-2) + 1 = 4(4) + 4 + 1 = 16 + 4 + 1 = 21$$

Practice 2. *Let $P(x) = 3x^2 + 5x - 6$. Find (a) $P(2)$ (b) $P(-1)$* *(Answers on page 114.)*

In order to add or subtract polynomials, we combine like terms – terms with the same variable and same exponent. When we combine, we keep the variable the same and add (or subtract) the coefficients. For example, $3x^2$ and $8x^2$ are like terms (same variable and same exponent). As a result, $3x^2 + 8x^2 = 11x^2$ and $3x^2 - 8x^2 = -5x^2$. On the other hand, $3x^4$ and $8x^2$ are unlike terms (same variable but different exponents). Hence, these two terms cannot be combined. To add or subtract two or more polynomials with more than one term, we combine all like terms.

Example 3. *Simplify* $\left(4x^2 - 3x + 2\right) + \left(6x^2 + 7x - 5\right)$

Solution 3.

$$\begin{aligned}\left(4x^2 - 3x + 2\right) + \left(6x^2 + 7x - 5\right) &= (4x^2 + 6x^2) + (-3x + 7x) + (2 - 5)\\ &= 10x^2 + 4x - 3\end{aligned}$$

Practice 3. *Simplify* $\left(3x^2 - 5x - 7\right) + \left(8x^2 - 9x + 4\right)$ *(Answer on page 114.)*

To subtract polynomials, first distribute the negative and then combine like terms.

Example 4. *Simplify* $\left(5x^2 - 3x + 2\right) - \left(2x^2 - 7x - 3\right)$

Solution 4.

$$\begin{aligned}\left(5x^2 - 3x + 2\right) - \left(2x^2 - 7x - 3\right) &= 5x^2 - 3x + 2 - 2x^2 + 7x + 3\\ &= (5x^2 - 2x^2) + (-3x + 7x) + (2 + 3)\\ &= 3x^2 + 4x + 5\end{aligned}$$

Practice 4. *Simplify* $\left(3x^2+9x-6\right)-\left(7x^2+4x-3\right)$ *(Answer on page 114.)*

Example 5. *Simplify* $\left(3x^2-9x+7\right)-\left(4x^3+2x^2+4\right)+\left(9x^3+7x+1\right)$

Solution 5.

$$\begin{aligned}
&\left(3x^2-9x+7\right)-\left(4x^3+2x^2+4\right)+\left(9x^3+7x+1\right)\\
&=3x^2-9x+7-4x^3-2x^2-4+9x^3+7x+1\\
&=(-4x^3+9x^3)+(3x^2-2x^2)+(-9x+7x)+(7-4+1)\\
&=5x^3+x^2-2x+4
\end{aligned}$$

Practice 5. *Simplify* $\left(5x^3-7x^2+8x-3\right)+\left(4x^3-5x+7\right)-\left(9x^2+6x-11\right)$
(Answer on page 114.)

Example 6. *Simplify* $\left(7x^2y+3xy^2-6xy\right)+\left(4xy-3x^2y+9xy^2\right)-\left(2x^2y-5xy\right)$

Solution 6.

$$\begin{aligned}
&\left(7x^2y+3xy^2-6xy\right)+\left(4xy-3x^2y+9xy^2\right)-\left(2x^2y-5xy\right)\\
&=7x^2y+3xy^2-6xy+4xy-3x^2y+9xy^2-2x^2y+5xy\\
&=2x^2y+12xy^2+3xy
\end{aligned}$$

Practice 6. *Simplify*

$$\left(4x^2y^3+5x^3y^2-8xy^2\right)-\left(6xy^2-3x^2y^3-3x^3y^2\right)-\left(7x^2y^3+3x^3y^2\right)$$

(Answer on page 114.)

To multiply polynomials we multiply each term in the first polynomial by each term in the second polynomial. In order to do this, we need to recall the distributive property and the product rule for exponents: For any integers m and n, $a^m \cdot a^n = a^{m+n}$.

Example 7. *Simplify* $4x\left(3x^2+2\right)$

Solution 7. *We need to distribute* $4x$ *to every term in our parenthesis. Hence,*

$$\begin{aligned} 4x\left(3x^2+2\right) &= (4x)(3x^2)+(4x)(2) \\ &= 12x^3+8x \end{aligned}$$

Practice 7. *Simplify* $-2x^2\left(3x^2+4x-2\right)$ *(Answer on page 114.)*

Example 8. *Simplify* $(4x-3)(5x+2)$

Solution 8.

$$\begin{aligned} (4x-3)(5x+2) &= 4x(5x+2)-3(5x+2) \\ &= (4x)(5x)+(4x)(2)-(3)(5x)-3(2) \\ &= 20x^2+8x-15x-6 \\ &= 20x^2-7x-6 \end{aligned}$$

Be careful when combining like terms. Remember that the variable part remains unchanged.

Practice 8. *Simplify* $(7x-3)(5x+4)$ *(Answer on page 114.)*

There is also a rectangular array approach to multiplication of polynomials. To set up a rectangular array, consider $(3x+2)(4x-5)$. The terms of the first polynomial become the labels on each of the rows; whereas, the terms of the second polynomial become the labels on each of the columns, as shown in the next diagram.

	$4x$	-5
$3x$		
2		

To finish the problem and find the solution, we then multiply the term at the beginning of the row with the term at the top of the corresponding column. We then record the result in the appropriate cell of the table. This would look like

	$4x$	-5
$3x$	$12x^2$	$-15x$
2	$8x$	-10

Now to find the solution, we combine like terms of the cells **inside** the table to get $12x^2 - 7x - 10$. The benefits of the rectangular array is that it insures that you multiply every term in the first polynomial with every term in the second polynomial.

Example 9. *Use the rectangular array approach to multiply*

$$(3x^2 + 2x - 4)(2x^2 - 5x + 3).$$

Solution 9. *The set-up for the rectangular array approach is the following:*

	$2x^2$	$-5x$	3
$3x^2$			
$2x$			
-4			

Therefore, once the multiplication has been performed and recorded we have

	$2x^2$	$-5x$	3
$3x^2$	$6x^4$	$-15x^3$	$9x^2$
$2x$	$4x^3$	$-10x^2$	$6x$
-4	$-8x^2$	$20x$	-12

To finish, we combine like terms to get a solution of $6x^4 - 11x^3 - 9x^2 + 26x - 12$.

Practice 9. *Use the rectangular array approach to multiply*

$$\left(3x^3 - 4x + 2\right)\left(4x^2 - 5x - 3\right).$$

(Answer on page 114.)

The rectangular array approach is quite beneficial when the problem is fairly long and it is more likely that a mistake in the distributive property could arise. Once again, it guarantees that each term of the first polynomial will be multiplied by each term of the second polynomial.

Recall from Section 4.1 that it was determined that $(a+b)^n \neq a^n + b^n$ for all integers $n \geq 2$. So, how do we simplify $(3x-2)^2$? Recall, that the exponent tells us how many times the multiplication is to be repeated. Hence, using our definition of exponents we must rewrite the expression before multiplying it out.

Example 10. *Multiply* $(3x-2)^2$

Solution 10.

$$\begin{aligned}(3x-2)^2 &= (3x-2)(3x-2) \\ &= 9x^2 - 6x - 6x + 4 \\ &= 9x^2 - 12x + 4\end{aligned}$$

Practice 10. *Multiply* $(6x+5)^2$ *(Answer on page 114.)*

This same procedure can be used for any integer exponent value.

Example 11. *Multiply* $(3x-2)^3$

Solution 11. *From Example* 10*, we found that by multiplying the first two quantities together we have* $(3x-2)(3x-2) = 9x^2 - 12x + 4$*. Hence,*

$$\begin{aligned}(3x-2)^3 &= (3x-2)(3x-2)(3x-2) \\ &= (9x^2 - 12x + 4)(3x-2) \\ &= 27x^3 - 18x^2 - 36x^2 + 24x + 12x - 8 \\ &= 27x^3 - 54x^2 + 36x - 8\end{aligned}$$

Practice 11. *Multiply* $(6x+5)^3$ *(Answer on the next page.)*

ANSWERS TO PRACTICE PROBLEMS

1. (a) 6
   (b) 8
2. (a) 16
   (b) -8
3. $11x^2 - 14x - 3$
4. $-4x^2 + 5x - 3$
5. $9x^3 - 16x^2 - 3x + 15$
6. $2x^2y^3 + 5x^3y^2 - 14xy^2$
7. $-6x^4 - 8x^3 + 4x^2$
8. $35x^2 + 13x - 12$
9. $12x^5 - 15x^4 - 25x^3 + 28x^2 + 2x - 6$
10. $36x^2 + 60x + 25$
11. $216x^3 + 540x^2 + 450x + 125$

SECTION 4.2 EXERCISES

(Answers are found on page 237.)

1. Is it possible for the sum of two polynomials, both of degree three, to be a polynomial of degree two? If so, give an example. If not, explain why.
2. If we multiply a monomial (a polynomial with one term) by another monomial will we always get a monomial?
3. If we multiply a monomial by a binomial (a polynomial with two terms) will we ever get a trinomial (a polynomial with three terms)?
4. If we consider the product of a polynomial of degree two and a polynomial of degree three, what is the degree of the product? In general, if an n-th degree polynomial and an m-th degree polynomial are multiplied together, what is the degree of the product?

Determine the degree of each polynomial.

5. $P(x) = 6x^8 - 3x^5 + 2x + 9$
6. $P(x) = 4x^7 - 3x^6 + 2x^3 + 5x^2$
7. $P(x, y) = 2xy^3 - 3x^2y^4 + 5xy$
8. $P(a, b) = 3ab^2 - 2ab + 7a^2b$
9. $P(x, y) = 3x^3y^2 + 5xy^2 + 7x + 10$

10. $P(a,b) = 9a + 3a^2b - 6ab^2 + 4$

Evaluate each polynomial.

11. If $P(x) = 4x + 7$, find $P(-8)$.
12. If $P(x) = 3x - 4$, find $P\left(-\frac{5}{2}\right)$.
13. If $P(x) = 4x^2 - 6x + 8$, find $P\left(\frac{1}{2}\right)$.
14. If $P(x) = 5x^2 + 3x + 9$, find $P(2)$.
15. If $P(x) = x^3 - 4x^2 - 2x + 5$, find $P(-2)$.
16. If $P(x) = x^2 + x - 6$, find $P(-1)$.
17. If $P(x) = x^2 - x - 12$, find $P(-3)$.
18. If $P(x) = 11x^2 - 3x - 11$, find $P(2)$.
19. If $P(x) = -2x^2 - 8x + 6$, find $P(-3)$.
20. If $P(x) = x^3 - 10x^2 + 22x - 5$, find $P(-2)$.
21. If $P(x) = 7x^3 - 11x^2 + 6x - 9$, find $P(-1)$.
22. If $P(x) = -8x^3 + 6x^2 - 10x$, find $P(-5)$.
23. If $P(x) = 15x^4 - 6x^3 - 21x^2$, find $P(2)$.

Perform the indicated operations and simplify.

24. $(3x + 7) + (-4x + 9)$
25. $(2x + 7) - (-3x + 9)$
26. $(8x^3 + 12x^2) - (8x^3 + 8x^2)$
27. $(7a^2 + 8a - 9) + (-8a^2 - 9a + 8)$
28. $\big(5x^2 - 7x - 2\big) + \big(6x^2 + 4x - 9\big)$
29. $\big(6x^2 - 3x + 4\big) - \big(8x^2 + 5x - 2\big)$
30. $\big(3x^3 - 5x^2 + 6x - 7\big) - \big(4x^3 - 7x + 1\big) + \big(2x^3 - 5x^2 + 9x + 3\big)$

31. $\left(5x^3 - 9x^2 + 2x - 3\right) - \left(-6x^3 + 2x^2 - 8x - 1\right) - \left(4x^3 + 4x + 7\right)$

32. $(5a^3b^2 + 2a^2b^2 - 3ab^2) + (a^3b^2 - a^2b^2 + ab^2)$

33. $(17mn^2 - 6m^2n) + (2mn^2 + 10) - (10m^2n + 15)$

34. $4x\,(3x - 2)$

35. $5x^2\,(6x + 3)$

36. $-2x\left(4x^2 - 3x + 5\right)$

37. $3x^2\left(5x^2 - 2x - 7\right)$

38. $-3x^2y\left(5xy^2 - 7xy + 3x^2\right)$

39. $-2x^2y^4(5x^2y^2 - 3xy^3 + 2x^3y^4 - 7y)$

40. $(2x - 3)(2x + 5)$

41. $(3x - 2)(5x - 4)$

42. $(4x + 3)(7x - 2)$

43. $(5x - 3)\,(2x + 1)$

44. $(6x - 1)\,(4x + 2)$

45. $(2x - 3)\,(2x + 3)$

46. $(4x + 1)\,(4x - 1)$

47. $(5x + 2)\,(5x - 2)$

48. $(3x - 4)\,(3x + 4)$

49. $\left(\frac{3}{2}x - 5\right)\left(3x - \frac{2}{5}\right)$

50. $\left(3x - \frac{2}{3}\right)\left(2x + \frac{1}{4}\right)$

51. $(7x - 2)^2$

52. $(4x + 3)^2$

53. $(2x - 3)^2$

54. $(x-2)^3$

55. $(2x+1)^3$

56. $(4x+1)\left(3x^2-5x-2\right)$

57. $(x-4)\left(2x^2-3x-2\right)$

58. $x\,(3x-4)\,(7x+3)$

59. $2x^2\,(4x-3)\,(2x+5)$

60. $2x(3x-4)(2x+3)$

61. $(5x-2)\left(2x^2-3x+4\right)$

62. $(3x-5)\left(2x^3-4x^2+7x-2\right)$

Use a rectangular array to multiply.

63. $(3x-5)(2x^3-6x^2-3x+2)$

64. $\left(x^2-3x-1\right)\left(2x^2+5x-7\right)$

65. $\left(6x^2-x+3\right)\left(3x^2-2x-4\right)$

66. Give a counterexample to show that $x+2(x-3)\neq(x+2)(x-3)$

4.3 Negative Integer Exponents

In Section 4.1, we discussed how to work with zero and positive integer exponents. So, what about negative exponents? Our old definition of viewing exponents as repeated multiplication no longer holds for negative exponents. For example, we cannot evaluate 3^{-2} by taking the base 3 and multiplying it to itself -2 times. This makes no sense. However, we need to be sure that we define a^{-n} so that it still obeys all of the exponent rules we have discussed thus far. If $a \neq 0$,

$$a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1.$$

Thus, a^{-n} must be the reciprocal of a^n.

Negative Exponent Rule: For any nonzero real number a,

$$a^{-n} = \frac{1}{a^n}.$$

CAUTION: A positive constant raised to a negative power does NOT yield a negative number. For example, $3^{-2} \neq -6$, instead,

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$$

Furthermore, the above rule does not apply to negative constants. For example, $-4 \neq \frac{1}{4}$. The rule only applies to negative **exponents**.

We leave it as an exercise for the reader to verify that the product rule and power rules for exponents remain true with negative exponents. Thus, all the rules we discussed in Section 4.1 still hold for any integer exponent whether it is positive, negative, or zero.

Example 1. *Use the rules of exponents to simplify each expression. Write answers with positive exponents only.*

(a) $3^{-1} + 3^{-2}$

(b) $5^{-1} - 5^0$

(c) $(2xy)^2\left(3xy^7\right)^0$

(d) $\left(x^{-4}y^3\right)^{-2}$

(e) $\left(-2x^4y^3\right)^{-2}$

(f) $\left(8x^4y^{-3}\right)\left(\frac{1}{2}x^{-5}y^7\right)$

Solution 1.

(a) $3^{-1}+3^{-2}=\frac{1}{3}+\frac{1}{3^2}=\frac{1}{3}+\frac{1}{9}=\frac{3}{9}+\frac{1}{9}=\frac{4}{9}$

(b) $5^{-1}-5^0=\frac{1}{5}-1=\frac{1}{5}-\frac{5}{5}=-\frac{4}{5}$

(c) $(2xy)^2\left(3xy^7\right)^0=2^2x^2y^2\cdot 1=4x^2y^2$

(d) $\left(x^{-4}y^3\right)^{-2}=x^{-4\cdot -2}\cdot y^{3\cdot -2}=x^8y^{-6}=x^8\frac{1}{y^6}=\frac{x^8}{y^6}$

(e) $\left(-2x^4y^3\right)^{-2}=\frac{1}{(-2x^4y^3)^2}=\frac{1}{(-2)^2x^{4\cdot 2}y^{3\cdot 2}}=\frac{1}{4x^8y^6}$

(f) $\left(8x^4y^{-3}\right)\left(\frac{1}{2}x^{-5}y^7\right)=8\cdot\frac{1}{2}\cdot x^4x^{-5}y^{-3}y^7=4x^{4+(-5)}y^{(-3)+7}$

$=4x^{-1}y^4=4\frac{1}{x}y^4=\frac{4y^4}{x}$

Practice 1. *Use the rules of exponents to simplify each expression. Write answers with positive exponents only. (Answers on page 123)*

(a) $\left(r^{-2}s^3t^2\right)^{-2}$

(b) $\left(-2x^3y^{-2}z^{-1}\right)^2\left(-4x^{-2}yz^3\right)$

(c) $4^{-3}+4^{-2}$

Consider the example $\frac{2^{-4}}{3^{-5}}$. Using our negative exponent rule, we obtain

$$\frac{2^{-4}}{3^{-5}}=\frac{\frac{1}{2^4}}{\frac{1}{3^5}}=\frac{1}{2^4}\div\frac{1}{3^5}=\frac{1}{2^4}\cdot\frac{3^5}{1}=\frac{3^5}{2^4}.$$

Therefore, the negative exponent rule allows us to move factors in a fraction just by changing the sign of the exponent. Also, consider $\left(\frac{2}{5}\right)^{-3}$. Since $\frac{5}{2}$ is the reciprocal of $\frac{2}{5}$,

$$\left(\frac{2}{5}\right)^{-3} = \frac{1}{\left(\frac{2}{5}\right)^3} = \frac{1^3}{\left(\frac{2}{5}\right)^3} = \left(\frac{1}{\frac{2}{5}}\right)^3 = \left(\frac{5}{2}\right)^3.$$

Thus, if we have a fraction raised to a power, we can change the sign of the exponent by using the reciprocal of the fraction. We summarize these rules as follows:

> **Converting from Negative to Positive Exponents:** For any nonzero numbers a and b, and any integers m and n,
>
> $$\frac{a^{-m}}{b^{-n}} = \frac{b^n}{a^m} \qquad \text{and} \qquad \left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

Before examining some exercises we need to discuss our final exponent rule – the quotient rule. Consider, $\frac{2^6}{2^2}$ and $\frac{2^2}{2^6}$. Rewriting both expressions using the definition of an exponent, we obtain

$$\frac{2^6}{2^2} = \frac{2\cdot 2\cdot 2\cdot 2\cdot \cancel{2}\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}} = 2\cdot 2\cdot 2\cdot 2 = 2^4,$$

and

$$\frac{2^2}{2^6} = \frac{\cancel{2}\cdot \cancel{2}}{\cancel{2}\cdot \cancel{2}\cdot 2\cdot 2\cdot 2\cdot 2} = \frac{1}{2\cdot 2\cdot 2\cdot 2} = \frac{1}{2^4} = 2^{-4}.$$

For the first example, subtracting the exponents we have $6 - 2 = 4$ which is the resulting exponent. In the second example, subtracting the exponents, we have $2 - 6 = -4$ which again is the resulting exponent. These examples suggest the quotient rule for exponents.

> **Quotient Rule for Exponents:** For any nonzero real number a and any integers m and n,
>
> $$\frac{a^m}{a^n} = a^{m-n}.$$
>
> (When dividing powers with like bases, we keep the base and subtract the exponents.)

CAUTION: When using the quotient rule, the bases must be the same. Remember to keep the base the same and only subtract the exponents. The order of the subtraction is important since subtraction does not satisfy the commutative property. Therefore, when applying the quotient rule for exponents, remember that it is

(exponent in the numerator) – (exponent in the denominator).

When working with the rules of exponents, note that the steps can be performed in many different orders. As such, the examples below show one method of simplifying each expression. However, there are many equally correct ways to begin your solution. Regardless of what order you perform the steps, you will always get the same answer.

Example 2. *Use a combination of the exponent rules to simplify each exponential expression. Write all answers with only positive exponents. Assume all variables represent nonzero real numbers.*

(a) $\dfrac{(2x^2)^3}{4x^4}$

(b) $\dfrac{(6x^3)^2}{(2x^2)^3} \cdot (3x^2)^0$

(c) $\dfrac{\left(xy^2w^{-3}\right)^4}{(x^{-3}y^{-2}w)^3}$

(d) $\left(\dfrac{-3a^3b^{-5}}{6a^{-2}b^{-2}}\right)^3$

(e) $\dfrac{\left(2m^2n^{-3}\right)^{-2}}{(-3m^{-3}n^4)^{-1}}$

Solution 2.

(a) $\dfrac{(2x^2)^3}{4x^4} = \dfrac{2^3x^{2\cdot 3}}{4x^4} = \dfrac{8x^6}{4x^4} = 2x^{6-4} = 2x^2$

(b) $\dfrac{(6x^3)^2}{(2x^2)^3} \cdot (3x^2)^0 = \dfrac{6^2 x^{3\cdot 2}}{2^3 x^{2\cdot 3}} \cdot 1 = \dfrac{36x^6}{8x^6} = \dfrac{36}{8} = \dfrac{9}{2}$

(c) $\dfrac{\left(xy^2w^{-3}\right)^4}{\left(x^{-3}y^{-2}w\right)^3} = \dfrac{x^4y^{2\cdot 4}w^{-3\cdot 4}}{x^{-3\cdot 3}y^{-2\cdot 3}w^3} = \dfrac{x^4y^8w^{-12}}{x^{-9}y^{-6}w^3} = \dfrac{x^4x^9y^8y^6}{w^{12}w^3}$
$= \dfrac{x^{4+9}y^{6+8}}{w^{12+3}} = \dfrac{x^{13}y^{14}}{w^{15}}$

(d) $\left(\dfrac{-3a^3b^{-5}}{6a^{-2}b^{-2}}\right)^3 = \left(\dfrac{-3a^3a^2b^2}{6b^5}\right)^3 = \left(\dfrac{-3a^{3+2}b^{2-5}}{6}\right)^3 = \left(\dfrac{-a^5b^{-3}}{2}\right)^3$
$= \left(\dfrac{-a^5}{2b^3}\right)^3 = \dfrac{-a^{5\cdot 3}}{2^3b^{3\cdot 3}} = \dfrac{-a^{15}}{8b^9}$

(e) $\dfrac{\left(2m^2n^{-3}\right)^{-2}}{\left(-3m^{-3}n^4\right)^{-1}} = \dfrac{\left(-3m^{-3}n^4\right)}{\left(2m^2n^{-3}\right)^2} = \dfrac{-3m^{-3}n^4}{4m^4n^{-6}} = \dfrac{-3n^4n^6}{4m^4m^3}$
$= \dfrac{-3n^{10}}{4m^7}$

Practice 2. *Use a combination of the exponent rules to simplify each exponential expression. Write all answers with only positive exponents. Assume all variables represent nonzero real numbers.* (Answers on page 123)

(a) $\left(x^3y^{-1}z^2\right)\left(x^{-2}y^{-3}z^3\right)$

(b) $\dfrac{\left(3x^2y^3\right)^{-2}}{\left(2x^3y^4\right)^{-3}}$

(c) $\left(\dfrac{-2x^3y^{-2}z}{3x^{-5}y^{-1}z^3}\right)^{-3}$

In addition to simplifying expressions, the exponent rules can also be used to calculate expressions mentally by making the number smaller, and easier, to work with.

Example 3. *Use the laws of exponents to calculate the following expressions mentally.*

(a) $\dfrac{27^3}{9^3}$ *(b)* $16^4 \cdot (0.125)^4$ *(c)* $125^5 \cdot (25)^{-7}$

Solution 3.

(a) $\dfrac{27^3}{9^3} = \left(\dfrac{27}{9}\right)^3 = 3^3 = 27.$

(b) $16^4 \cdot (.125)^4 = 16^4 \cdot \left(\dfrac{1}{8}\right)^4 = \left(\dfrac{16}{8}\right)^4 = 2^4 = 16.$

(c) $125^5 \cdot (25)^{-7} = (5^3)^5 \left(5^2\right)^{-7} = 5^{15}\, 5^{-14} = 5^1 = 5.$

Practice 3. *Use the laws of exponents to calculate the following expressions mentally. (Answers below)*

(a) $\dfrac{4^9}{64^3}$

(b) $\dfrac{3^8 \cdot 9^3}{27^5}$

ANSWERS TO PRACTICE PROBLEMS

1. (a) $\frac{r^4}{s^6 t^4}$
   (b) $\frac{-16x^4 z}{y^3}$
   (c) $\frac{5}{64}$
2. (a) $\frac{xz^5}{y^4}$
   (b) $\frac{8x^5 y^6}{9}$
   (c) $\frac{-27y^3 z^6}{8x^{24}}$
3. (a) 1
   (b) $\frac{1}{3}$

SECTION 4.3 EXERCISES

(Answers are found on page 239.)

Evaluate each expression.

1. 3^{-4}
2. 8^{-2}
3. -5^{-4}
4. $\left(\dfrac{1}{2}\right)^{-3}$
5. $(-4)^{-2}$
6. -2^{-2}
7. $(-2)^{-2}$
8. $\dfrac{1}{5^{-2}}$
9. $\dfrac{1}{7^{-2}}$
10. $\left(\dfrac{4}{3}\right)^{-4}$
11. $\left(\dfrac{1}{6}\right)^{-3}$
12. $\left(\dfrac{2}{9}\right)^{-2}$

Evaluate each expression.

13. $2^{-2} + 4^{-2}$

14. $(-3)^{-2} + (-4)^{-1}$

15. $3^{-1} + 2^{-2}$

16. $(-4)^{-2} - 2^{-1}$

17. $2^2 - 2 + 2^{-1} - 2^{-2}$

18. $(-3)^2 + 3 - 3^{-1} + 3^{-2}$

19. $2^{-1} + 3^{-1} - 4^{-1}$

20. $(-2)^{-1} - (-2)^{-2} + (-2)^{-3}$

21. $\left(\frac{2}{3}\right)^{-2} + \left(\frac{4}{5}\right)^{-1}$

22. $\left(\frac{1}{3}\right)^{-1} + 3^{-1} + \left(\frac{1}{3}\right)^{-2} + 3^{-2}$

23. $\left(\frac{2}{3}\right)^{-2} + \left(\frac{3}{2}\right)^{-2} - \left(\frac{1}{3}\right)^{2} - \left(\frac{1}{2}\right)^{2}$

24. $\left(\frac{1}{2}\right)^{2} - \frac{1}{2} + \left(\frac{1}{2}\right)^{-1} - \left(\frac{1}{2}\right)^{-2}$

Simplify the following, leaving your answer as a single base raised to a single exponent. Assume all variables represent non-zero real numbers.

25. $x^7 \cdot x^3 \div x^4$

26. $128 \div 2^4$

27. $2^3 \cdot 4^5 \div 8^3$

28. $(5^2)^4 \div 5^3 \cdot 5^2$

29. $16^7 \cdot 4^8 \cdot 8^3 \div 2^{12}$

30. $27^6 \div 3^8 \cdot 9^4 \cdot 81$

Use a combination of the rules of exponents to simplify each expression. Write all answers using positive exponents only. Assume all variables represent nonzero real numbers.

31. $\dfrac{x^4y^5}{x^3y^2}$

32. $\dfrac{x^4y^3}{x^9y^2}$

33. $\dfrac{x^5y^{12}x^9}{xy^4}$

34. $\dfrac{x^2y^4x^5}{x^9y^8}$

35. $\dfrac{(x^2y^3)^2}{xy^5}$

36. $\dfrac{x^5y^{12}x^9}{(xy)^4}$

37. $\dfrac{(2x^3)(3x^2)}{(x^2)^3}$

38. $\dfrac{4x^3}{(2x^4)(3x)}$

39. $\dfrac{(4x^2y)(3x^3y^2)}{x^7y^4}$

40. $\left(\dfrac{4a^2b}{a^3b^2}\right)\left(\dfrac{5a^2b}{2b^4}\right)$

41. $\left(2x^{-2}y^3z^5\right)^3$

42. $\left(-4x^3y^{-5}z^{-1}\right)^2$

43. $\left(3x^4y^2z^{-3}\right)^{-2}$

44. $\left(5x^{-4}y^3z^{-6}\right)^{-3}$

45. $\dfrac{18x^{-5}y^{-2}}{10x^{-2}y^{-8}}$

46. $\dfrac{14x^{-3}y^{-2}}{24x^{-8}y^3}$

47. $\dfrac{\left(2x^{-3}y^4\right)^{-1}}{6x^{-5}y^{-2}}$

48. $\dfrac{3^{-1}x^4y^{-2}}{3^{-2}x^{-2}y}$

49. $\dfrac{6^{-2}x^7y^{-3}z^2}{6^{-3}x^{-4}y^5z^9}$

50. $\left(\dfrac{x^2}{x^4y^3}\right)^2$

51. $\left(\dfrac{xy^6}{x^4y^2}\right)^3$

52. $\left(\dfrac{4x^5y^2}{8x^4y^8}\right)^2$

53. $\left(\dfrac{a^2b^3c}{a^4b^{-1}c^2}\right)^2$

54. $\left(\dfrac{3x^2y^8}{9x^4y^5}\right)^{-3}$

55. $\left(\dfrac{x^3y^{-2}z^5}{x^{-2}y^5z}\right)^4$

56. $\left(2x^{-3}y^2\right)^3$

57. $\left(4x^{-1}y^{-3}\right)^2\left(5x^{-1}y^3\right)^{-1}$

58. $\left(4x^3y^2\right)^2\left(2x^2y^3\right)^{-3}$

59. $\left(3x^{-2}y^4\right)^{-2}$

60. $\left(6x^4y^{-3}\right)^{-1}\left(8x^6y^{-2}\right)$

61. $\dfrac{6x^5y^{-2}}{15x^{-3}y^8}$

62. $\dfrac{\left(4x^{-3}y^5\right)^{-1}}{8x^5y^{-3}}$

63. $\dfrac{9x^5y^{-3}}{24x^7y^8}$

64. $\left(2x^{-3}y^4\right)^3\left(3x^2y^{-1}\right)^{-2}$

Use a combination of the rules of exponents to simplify each expression. Write all answers using positive exponents only. Assume all variables represent nonzero real numbers.

65. $(2x^2y^{-5})(6x^{-3}y)(\frac{1}{3}x^{-1}y^3)$

66. $\left(\dfrac{xy^{-2}z^{-3}}{2x^{-2}yz^2}\right)^{-2}$

67. $\left(\dfrac{-2x^4y^{-4}}{3x^{-1}y^{-2}}\right)^4$

68. $\dfrac{(2x^3y^{-2})^3(-3xy^5)^{-2}}{4x^{-5}y^7}$

69. $\left(\dfrac{-4x^5y^{-2}z^{-3}}{8x^{-4}y^6z^{-4}}\right)^{-3}$

70. $\left(4x^3y^{-2}\right)^{-2}\left(2x^2y^3\right)^2\left(\dfrac{1}{3}x^4y^{-2}\right)^{-1}$

71. $\dfrac{(2x^3y^6z^{-1})^{-2}}{(3x^{-3}y^6z^2)^{-3}}$

72. A fellow student said that $8^5 \div 2^2 = 4^3$. How would you convince this student that this is incorrect? What is the correct answer?

4.4 Scientific Notation

One application of exponents is in the use of scientific notation. Scientific notation is convenient for writing very large numbers. When negative exponents are used with scientific notation, it can be used to represent very small numbers as well.

A number in **scientific notation** is written

$$a \times 10^n$$

where $1 \leq |a| < 10$ and n is an integer.

Two examples of scientific notation for small numbers are:

- The width of a strand of DNA is 2×10^{-9}
- The mass of a proton is 1.6724×10^{-27} kg

In scientific notation there is always exactly one digit before the decimal point. To rewrite a number using scientific notation the decimal point is moved so that only one nonzero digit is to the left of the decimal point. The number of places the decimal point was moved becomes the exponent on 10. This power is negative if the decimal point is moved to the right and positive if it is moved to the left.

Example 1. *Write each number in scientific notation.*

(a) $56,000,000$

(b) 0.0000000345

(c) *The speed of sound is* 4790 *feet per second*

Solution 1.

(a) We begin by moving the decimal point so that only one nonzero digit is in front of it, in other words, seven places to the left. Seven becomes the power on 10 and it is positive since it was moved to the left. Thus,

$$56,000,000 = 5.6 \times 10^7.$$

(b) *Since we move the decimal point eight places to the right in order to get only one nonzero digit is front of the decimal point, we get that*

$$0.0000000345 = 3.45 \times 10^{-8}.$$

(c) *The decimal place is moved three places to the left. Thus,* $4790 = 4.790 \times 10^3$.

Practice 1. *Write each number in scientific notation.* *(Answers on page 130)*

(a) 0.00000405

(b) *The population of Cuyahoga County is approximately* $1,360,000$ *people.*

Remember that the exponent on 10 tells you the direction and the number of decimal places the decimal point is moved. To change a number from scientific notation to standard notation, we reverse the process discussed before Example 1. In other words, if the exponent n is negative we move the decimal point $|n|$ places to the left. If the exponent n is positive, we move the decimal point n places to the right.

Example 2. *Convert the following numbers to standard notation (without exponents).*

(a) -9.75×10^8

(b) 1.4×10^{-5}

(c) *A gram is* 3.527×10^{-2} *ounces.*

(d) *The radius of the earth is* 6.4×10^6 *meters.*

Solution 2.

(a) $-9.75 \times 10^7 = -975,000,000$

(b) $1.4 \times 10^{-5} = 0.000014$

(c) $3.527 \times 10^{-2} = 0.03527$

(d) $6.4 \times 10^6 = 6,400,000$

Practice 2. *Convert the following numbers to standard notation (without exponents).* *(Answers on page 130)*

(a) *The number one all-time worldwide box office movie is Titanic (1997) which earned approximately* 1.84×10^9 *dollars.*

(b) 4.8×10^{-12}

Scientific notation also provides an efficient way to multiply and divide very small and very large numbers. Using the commutative and associative properties of real numbers, we multiply the numbers between 1 and 10 and use the rules of exponents to find the power of 10 in the final answer.

Example 3. *Perform the indicated operations and write all answers in scientific notation.*

(a) $\left(4 \times 10^5\right)\left(2 \times 10^9\right)$

(b) $\dfrac{9 \times 10^{12}}{3 \times 10^7}$

(c) $\left(5 \times 10^9\right)\left(3 \times 10^{-4}\right)$

(d) $\dfrac{\left(4 \times 10^7\right)\left(6 \times 10^3\right)}{5 \times 10^4}$

Solution 3.

(a) $\left(4 \times 10^5\right)\left(2 \times 10^9\right) = (4 \times 2)(10^5 \times 10^9) = 8 \times 10^{14}$

(b) $\dfrac{9 \times 10^{12}}{3 \times 10^7} = \dfrac{9}{3} \times \dfrac{10^{12}}{10^7} = 3 \times 10^5$

(c) $\left(5 \times 10^9\right)\left(3 \times 10^{-4}\right) = (5 \times 3)(10^9 \times 10^{-4}) = 15 \times 10^5 = (1.5 \times 10) \times 10^5 = 1.5 \times 10^6$

(d) $\dfrac{\left(4 \times 10^7\right)\left(6 \times 10^3\right)}{5 \times 10^4} = \dfrac{4 \times 6}{5} \times \dfrac{10^7 \times 10^3}{10^4} = \dfrac{24}{5} \times \dfrac{10^{10}}{10^4} = 4.8 \times 10^6$

Practice 3. *Perform the indicated operations and write all answers in scientific notation.* *(Answers on page 130)*

(a) $$\frac{\left(2.7 \times 10^{4}\right)\left(4 \times 10^{9}\right)}{\left(2.25 \times 10^{8}\right)}$$

(b) $$\frac{\left(7 \times 10^{12}\right)\left(8.1 \times 10^{-16}\right)}{\left(2 \times 10^{5}\right)\left(2.1 \times 10^{8}\right)}$$

ANSWERS TO PRACTICE PROBLEMS

1. (a) 4.05×10^{-6}
   (b) 1.36×10^{6}
2. (a) $1,840,000,000$
   (b) 0.0000000000048
3. (a) 4.8×10^{5}
   (b) 1.35×10^{-16}

SECTION 4.4 EXERCISES

(Answers are found on page 241.)

Convert the following numbers to standard notation (without exponents).

1. 9.46×10^{17}
2. 1.81×10^{6}
3. 10^{-10}
4. 4.3×10^{-3}
5. 3.2×10^{-5}
6. 5.789×10^{8}
7. 6.93×10^{11}
8. 3.4×10^{-8}
9. 2.75×10^{7}
10. 8.003×10^{5}
11. 9.02×10^{-6}
12. 5.486×10^{-4}

Convert the following numbers to scientific notation.

13. $8,340$
14. 0.00079
15. 0.000042
16. $4,360,000$
17. 0.003005
18. 0.000007201
19. $294,000,000.$
20. 0.000000345
21. 0.0000000703
22. $6,600,000,000$
23. $2,350,000,000,000$
24. 0.000000000532

Perform the indicated operations and write all answers in scientific notation.

25. $\left(2.4\times10^{-5}\right)\left(3.0\times10^{-7}\right)$

26. $\left(5.2\times10^{7}\right)\left(4.3\times10^{-3}\right)$

27. $\left(2.1\times10^{-4}\right)\left(3.5\times10^{8}\right)\left(1.2\times10^{6}\right)$

28. $\left(3.5\times10^{4}\right)\left(6.12\times10^{-11}\right)\left(4.1\times10^{8}\right)$

29. $\left(5.1\times10^{-6}\right)\left(3.02\times10^{11}\right)\left(2\times10^{-8}\right)$

30. $\dfrac{9.6\times10^{9}}{1.5\times10^{3}}$

31. $\dfrac{6.88\times10^{4}}{4.3\times10^{-8}}$

32. $\dfrac{4.8\times10^{13}}{(5\times10^{9})(2.4\times10^{15})}$

33. $\dfrac{\left(3.5\times10^{-3}\right)\left(5.2\times10^{8}\right)}{9.1\times10^{-7}}$

34. $\dfrac{6.7\times10^{4}}{5\times10^{-6}}\times\dfrac{7.1\times10^{3}}{4\times10^{10}}$

35. $\dfrac{3}{4\times10^{5}}\times\dfrac{8\times10^{6}}{2\times10^{8}}$

Chapter 5

Factoring

5.1 GCF and Factor by Grouping

Now we will develop some techniques to factor polynomials with integer coefficients. The first and most basic technique is to factor out the greatest common factor. Since factoring is the reverse process of multiplication, we will start this section with a multiplication problem.

Example 1. *Find the product:* $2x^2y\left(3x^3y^2 + 5xy^2 + 7x + 10\right)$

Solution 1. *We use the Distributive Law as follows:*

$$\begin{aligned} 2x^2y\left(3x^3y^2 + 5xy^2 - 10\right) &= 2x^2y \cdot 3x^3y^2 + 2x^2y \cdot 5xy^2 - 2x^2y \cdot 10 \\ &= 6x^5y^3 + 10x^3y^3 - 20x^2y. \end{aligned}$$

Note that the result is a polynomial, each of whose terms has the monomial $2x^2y$ as a factor. That is, $2x^2y$ is a ***common factor*** *of all of the terms of the polynomial.*

Recall that the ***greatest common factor*** of two integers is the largest integer that is a factor (or divisor) of both. We used this idea when writing a fraction in lowest terms. Similarly, we define the ***greatest common factor (GCF)*** of two or more monomials to be the monomial of highest degree and largest coefficient that is a factor of all. We have already used this idea when writing fractions containing variables in lowest terms. (Click here to review GCF from from Core Math I.)

Example 2. *Find the GCF of the list of integers:* 18, 63, *and* 36.

Solution 2. *You may recognize right away that* 9 *is the greatest common factor. If this isn't immediate, simply factor each into a product of prime factors.*

$$18 = 2 \cdot 3^2$$
$$63 = 3^2 \cdot 7$$
$$36 = 2^2 \cdot 3^2$$

So the GCF of the integers 18, 63, *and* 36 *is* $3^2 = 9$.

Practice 1. *Find the GCF of each triple.*

(a) $126,\ 42,\ 105$ *(Answers on page 139.)*

(b) $625,\ 81,\ 15$

Example 3. *Find the GCF of the list of powers of* x*:* x^5, x^3, *and* x^8.

Solution 3. *The smallest exponent of* x *appearing in the list is* 3. *This means that* x^3 *is a factor of each expression in the list. If it's not clear why, consider the following factorizations using the laws of exponents.*

$$x^5 = x^3 \cdot x^2$$
$$x^3 = x^3 \cdot 1$$
$$x^8 = x^3 \cdot x^5$$

So the GCF of x^5, x^3, *and* x^8 *is* x^3.

Practice 2. *Find the GCF of each triple.*

(a) $y^{12},\ y^5,\ y^6$ *(Answers on page 139.)*

(b) $w^2,\ w,\ w^4$

Example 4. *Find the GCF of the monomials* $18x^5yz^2$, $63x^3y^3z^3$, *and* $36x^8y^2$.

Solution 4. *We can consider the coefficients and variables one at a time.*

- *The GCF of the coefficients 18, 63, and 36 is 9.*
- *The highest power of x that divides x^5, x^3, and x^8 is x^3.*
- *The highest power of y that divides y, y^3, and y^2 is y.*
- *Finally, z is a factor of the first two monomials, but not the third, so the GCF will not contain z.*

So the GCF of the monomials $18x^5yz^2$, $63x^3y^3z^3$, and $36x^8y^2$ is $9x^3y$.

Practice 3. *Find the GCF of each triple*

(a) $22xy^2z$, $121x^3yz^2$, $33y^2z^3$ *(Answers on page 139.)*

(b) $17a^2b$, $3ab^2$, $15b^3$

We now apply this to the factoring of polynomials with more than one term. Find the GCF of all the terms of the polynomial and then factor it out using the distributive law.

Example 5. *Factor out the GCF of each of the following polynomials.*

(a) $P(x) = 8x + 4$

(b) $P(x) = 9x^3 - 12x^2 + 15x$

(c) $P(x, y, z) = 18x^5yz^2 + 63x^3y^3z^3 + 36x^8y^2$

Solution 5. *(a) The GCF of the terms $8x$ and 4 is 4. Thus, we obtain*

$$\begin{aligned} P(x) &= 8x + 4 \\ &= 4 \cdot 2x + 4 \cdot 1 \\ &= 4\,(2x + 1)\,. \end{aligned}$$

(b) The GCF of the terms $9x^3$, $12x^2$, and $15x$ is $3x$. Thus, we obtain

$$\begin{aligned} P(x) &= 9x^3 - 12x^2 + 15x \\ &= 3x \cdot 3x^2 - 3x \cdot 4x + 3x \cdot 5 \\ &= 3x\left(3x^2 - 4x + 5\right). \end{aligned}$$

(c) We saw in Example 4 that the GCF of the terms of this polynomial is $9x^3y$. Thus, we obtain

$$\begin{aligned} P(x,y,z) &= 18x^5yz^2 + 63x^3y^3z^3 + 36x^8y^2 \\ &= 9x^3y \cdot 2x^2z^2 + 9x^3y \cdot 7y^2z^3 + 9x^3y \cdot 4x^5y \\ &= 9x^3y\left(2x^2z^2 + 7y^2z^3 + 4x^5y\right). \end{aligned}$$

Practice 4. *Factor the GCF out of each polynomial.*

(a) $P(x) = x^2 - x$

(b) $P(y) = 5y^2 + 10y - 15$

(c) $P(x,y) = 18x^2y^2 - 36xy^3 + 60xy^5$

(Answers on page 139.)

If the leading coefficient of the polynomial is negative, it is customary to factor out the negative. It is important to factor -1 out of *each* term.

Example 6. *Factor out the GCF of $P(t) = -100t^5 + 75t^3 - 125t^2 - 25t$.*

Solution 6. *The GCF of the terms of this polynomial is $25t$. Since the leading coefficient, -100, is negative, we will factor out $-25t$. Thus, we obtain*

$$\begin{aligned} P(t) &= -100t^5 + 75t^3 - 125t^2 - 25t \\ &= (-25t)\left(4t^4\right) + (-25t)\left(-3t^2\right) + (-25t)(5t) + (-25t)(1) \\ &= -25t\left(4t^4 - 3t^2 + 5t + 1\right). \end{aligned}$$

Be careful with the last term! Since the expression being factored out is one of the terms of the polynomial, we are left with a factor of 1 for that term when the GCF is factored out.

Practice 5. *Factor the GCF out of each polynomial.*

(a) $P(x) = -9x^5 + 21x^4 - 15x^3 + 3x^2$

(b) $P(s,t) = -s^4t^2 + 5s^3t^3 - 11s^2t^4$

(Answers on page 139.)

In the previous examples, the GCF was a monomial. However, it is also possible to have a binomial as a common factor.

Example 7. *Factor out the GCF of* $P(x) = 3x(x+2) - 5(x+2)$.

Solution 7. *The GCF of the terms of this polynomial is the binomial* $(x+2)$. *Thus, we obtain*

$$\begin{aligned} P(x) &= 3x(x+2) - 5(x+2) \\ &= (x+2)(3x-5). \end{aligned}$$

Practice 6. *Factor the GCF out of the polynomial:* $P(y) = 2y^2(3y-1) + 7(3y-1)$ *(Answer on page 139.)*

We have seen from the previous examples that the GCF can be a number, variable, or algebraic expression. In the previous example, the GCF was easy to see because it was inside of parentheses. Now suppose that our polynomial has four terms such as $x^3 + 2x^2 + 2x + 4$. Note that this polynomial does not have a GCF. However, if we group the first two terms together then the GCF of the first two terms is x^2, giving us $x^3 + 2x^2 = x^2(x+2)$. The GCF of the last two terms is 2 which yields $2x + 4 = 2(x+2)$. Putting this together we have

$$\underbrace{x^3 + 2x^2} + \underbrace{2x + 4} = x^2(x+2) + 2(x+2).$$

Now the $(x+2)$ becomes the GCF as in our last example. Therefore, we can factor the polynomial as

$$\begin{aligned} x^3 + 2x^2 + 2x + 4 &= x^2(x+2) + 2(x+2) \\ &= (x+2)(x^2+2) \end{aligned}$$

This method is called **factor by grouping**. Note that in order for factor by grouping to succeed, the algebraic expression in the parentheses must be identical.

Example 8. *Factor:* $2x^3 + x^2 + 50x + 25$

Solution 8. *We will group the first two terms together and group the last two terms together. For the first two terms the GCF is* x^2 *and for the last two terms the GCF is* 25. *Factor by grouping yields*

$$\begin{aligned} \underbrace{2x^3 + x^2} + \underbrace{50x + 25} &= x^2(2x+1) + 25(2x+1) \\ &= (2x+1)(x^2+25) \end{aligned}$$

Practice 7. *Factor:* $3x^3 + 12x^2 - 5x - 20$ *(Answer on the next page.)*

Example 9. *Factor:* $18x^3 - 27x^2 + 8x - 12$

> **Solution 9.** *Again, we group the first two terms and also the last two terms. The GCF of the first two terms is* $9x^2$ *and the GCF of the last two terms is* 4*. Therefore, factor by grouping yields*
>
> $$\begin{aligned}\underbrace{18x^3 - 27x^2} + \underbrace{8x - 12} &= 9x^2(2x-3) + 4(2x-3)\\ &= (2x-3)(9x^2+4)\end{aligned}$$

Practice 8. *Factor:* $6x^3 - 4x^2 + 15x - 10$ *(Answer on the facing page.)*

In the previous two examples, we did not need to rearrange the terms of our polynomial. However, sometimes it is necessary to reorder the terms for factor by grouping to work.

Example 10. *Factor:* $10x^2 - 12y + 15x - 8xy$

> **Solution 10.** *The current order does not lead to a common factor since*
>
> $$\underbrace{10x^2 - 12y} + \underbrace{15x - 8xy} = 2(5x^2 - 6y) + x(15 - 8y).$$
>
> *Factoring cannot proceed because the polynomial inside each of the parentheses is different. Therefore, we must first rearrange the terms.*
>
> $$\begin{aligned}10x^2 - 12y + 15x - 8xy &= \underbrace{10x^2 - 8xy} - \underbrace{12y + 15x}\\ &= 2x(5x-4y) + 3(-4y+5x)\\ &= 2x(5x-4y) + 3(5x-4y)\\ &= (5x-4y)(2x+3)\end{aligned}$$

Practice 9. *Factor* $10x^3 - 2x^2y^2 - 5xy + y^3$ *(Answers on the next page.)*

ANSWERS TO §5.1 PRACTICE PROBLEMS

1. (a) 21
   (b) 1
2. (a) y^5
   (b) w
3. (a) $11yz$
   (b) b
4. (a) $x(x-1)$
   (b) $5(y^2+2y-3)$
   (c) $6xy^2(3x-6y+10y^3)$
5. (a) $-3x^2(3x^3-7x^2+5x-1)$
   (b) $-s^2t^2(s^2-5st+11t^2)$
6. $(3y-1)(2y^2+7)$
7. $(x+4)(3x-5)$
8. $(2x^2+5)(3x-2)$
9. $(2x^2-y)(5x-y^2)$

Section 5.1 EXERCISES:

(Answers are found on page 242.)

Find GCF of the following pairs and triples.

1. 52 and 78
2. 36 and 30
3. 100 and 140
4. 63 and 45
5. 6, 9, and 27
6. 10, 40, and 50
7. 20, 18, and 45
8. 64, 36, and 100
9. 24, 40, and 60
10. 15, 60, and 81

Find GCF of the following pairs and triples.

11. x^4, x^8, and x^7
12. y^2, y^5, and 1
13. x^3y^2, xy^5, and x^6y^3
14. x^5y^3, x^3y^2, and x^5y^4
15. x^3y^2z, $x^5y^2z^3$, and $x^2y^4z^2$
16. xy^2, y^3z^2, and x^4z^4
17. $14x^2y$ and $36x^5y^3$
18. $18x^5yz^2$, $15x^2y^2z^3$, and $24x^3y^2z^4$
19. $6x^3y^3z^2$, $10x^4y^2z$, and $8x^3y^4z^2$
20. $15x^2y^2$, $25x^2z^3$, and $35x^3z^3$

Factor out the GCF of the following polynomials.

21. $15x^2-10$
22. $26x^3-39x$
23. $10x^2+15x-45$
24. x^3+6x^2-17x

25. $-18x^6 + 18x^4 + 18x^2 - 18$
26. $-x^3y + 2x^2y + 3xy - 5y$
27. $16a^8 - 18a^6 - 30a^5$
28. $40x^4y^2 + 24x^2y^8 - 32x^3y^5$
29. $16x^2y^2 + 4x^2y - 6xy^2$
30. $x^3y^2z^2 + 3x^2y^3z^2 - 2x^2y^2z^3$

Factor out the GCF of the following polynomials.

31. $3(x+2) - x(x+2)$
32. $3x^2(x^2 - x) + 2x(x^2 - x)$
33. $x(y-3) + 5(y-3)$
34. $5x(x+y) + 4y(x+y)$
35. $3x^2(x+6) - 5x(x+6) + 7(x+6)$
36. $7x(x^2 - 5x - 4) + 8(x^2 - 5x - 4)$
37. $2y(xy+1) + 3x(xy+1)$
38. $x^2y(x^2 - y^2) - xy^2(x^2 - y^2)$
39. $5x^2(5x^2 - 6) - 6(5x^2 - 6)$
40. $3x(x^2 + 9x - 1) - (x^2 + 9x - 1)$

Factor out the GCF of the following polynomials.

41. $x^2(x^2 + 2x + 3) - 10x(x^2 + 2x + 3) - 2(x^2 + 2x + 3)$
42. $xy(2x^2 + 3xy + y) - 4x(2x^2 + 3xy + y) + 5y(2x^2 + 3xy + y)$
43. $(x^2 - 3x)(x^2 + 4x + 5) + (6x + 2)(x^2 + 4x + 5)$
44. $(2x + 3y)(6x - 5y) + 6(6x - 5y)$
45. $13x^2(x^2 + xy + y) - 26(x^2 + xy + y)$
46. $15(x+3)(3x^2 - 5) + 25(3x^2 - 5)$

Factor the following polynomials by grouping.

47. $2x^3 + 3x^2 + 8x + 12$
48. $5x^3 - x^2 + 45x - 9$
49. $3x^3 - 2x^2 + 3x - 2$
50. $4x^3 + 12x^2 + 9x + 27$
51. $18x^3 - 9x^2 + 2x - 1$
52. $12x^3 - 16x^2 + 3x - 4$
53. $5x^3 - x^2 + 20x - 4$
54. $7x^3 + 3x^2 + 63x + 27$
55. $16x^3 - 12x^2 + 4x - 3$
56. $24x^3 - 4x^2 - 18x + 3$
57. $16x^3 - 4x^2 - 20x + 5$
58. $18x^3 - 27x^2 + 8x - 12$
59. $16x^3 - 32x^2 - 3x + 6$
60. $25x^3 - 25x^2 - 7x + 7$

61. $2x^3 + x^2 + 50x + 25$

62. $9x^3 - 4x^2 + 9x - 4$

63. $2a^3 - b^2 + a^2b - 2ab$

64. $2xy + 12 - 3y - 8x$

65. $4x^3 - 5 + x - 20x^2$

66. $7x^2 - xy + 14x - 2y$

67. $6ab + 24b + a + 4$

68. $10x^2 - 8xy - 12y + 15x$

69. $5x^2 + 15xy - 2xz - 6yz$

70. $6x + 4xy + 3 + 2y$

5.2 Factoring Special Products

In this section, we will learn to recognize and factor polynomials which follow certain patterns.

First, we will factor trinomials which are the square of a binomial. Remembering that factoring is the reverse process of multiplication, we will first see how these patterns arise by squaring binomials.

Example 1. *Find each square and simplify the result.*

(a) $(x+5)^2$ *(b)* $(2x-3)^2$

Solution 1. *(a) We may use the Distributive Law as follows.*

$$\begin{aligned}(x+5)^2 &= (x+5)(x+5)\\ &= x(x+5)+5(x+5)\\ &= x^2+5x+5x+5^2\\ &= x^2+2(5x)+5^2\\ &= x^2+10x+25.\end{aligned}$$

(b) This time we will use the rectangular array approach that was introduced in Section 4.2. The set-up is:

	$2x$	-3
$2x$		
-3		

Multiplying each row entry by each column entry, we obtain:

	$2x$	-3
$2x$	$(2x)^2$	$-6x$
-3	$-6x$	$(-3)^2$

Thus,we obtain:

$$\begin{aligned}(2x-3)^2 &= (2x)^2+2(-6x)+(-3)^2\\ &= 4x^2-12x+9\end{aligned}$$

In each of these examples, we observe that the product of the binomial has three terms. The first term of the product is the square of the first term of the binomial; the third term of the product is the square of the last term of the binomial; and the middle term of the product is twice the product of the two terms of the binomial. This leads us to the following special product forms.

Special products:

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$a^2 - 2ab + b^2 = (a-b)^2$$

Example 2. *Factor:*

(a) $P(x) = x^2 + 14x + 49$

(b) $P(y) = 9y^2 - 12y + 4$

(c) $P(x, y) = 121x^2 + 110xy + 25y^2$

Solution 2. *(a) The first term of* $P(x)$ *is* x^2 *and the last term is* 7^2*. Now we must check if the middle term follows either of our patterns above. Twice the product of* x *and* 7 *is* $2 \cdot x \cdot 7 = 14x$*, which is what we have. Thus,*

$$P(x) = x^2 + 14x + 49 = (x+7)^2.$$

(b) The first term of $P(y)$ *is* $(3y)^2$ *and the last term is* 2^2*. Now we must check if the middle term follows either of our patterns above. Twice the product of* $3y$ *and* 2 *is* $2 \cdot 3y \cdot 2 = 12y$*. Since the middle term of* $P(y)$ *is* $-12y$*, we have the square of a* difference*. Thus,*

$$P(y) = 9y^2 - 12y + 4 = (3y-2)^2.$$

(c) The first term of $P(x, y)$ *is* $(11x)^2$ *and the last term is* $(5y)^2$*. Now we must check if the middle term follows either of*

our patterns above. Twice the product of $11x$ *and* $5y$ *is* $2 \cdot 11x \cdot 5y = 110xy$*, which is the middle term of* $P(x, y)$*. Thus,*

$$P(x, y) = 121x^2 + 110xy + 25y^2 = (11x + 5y)^2.$$

Practice 1. *Factor.* *(Answers on page* *148**.)*

(a) $P(x) = x^2 - 4x + 4$

(b) $P(t) = 49t^2 + 70t + 25$

(c) $P(x, y) = x^2 - 6xy + 9y^2$

Sometimes we will have to use more than one factoring technique for the same polynomial. Always check first if the terms have a GCF which can be factored out.

Example 3. *Factor:* $P(t) = t^3 + 12t^2 + 36t$.

Solution 3. *We see that* t *is the GCF of the three terms of* $P(t)$ *and so we first factor out* t.

$$P(t) = t^3 + 12t^2 + 36t = t\left(t^2 + 12t + 36\right)$$

Now we try to factor the polynomial that remains. The first term of $\left(t^2 + 12t + 36\right)$ *is* t^2 *and the last term is* 6^2*. Twice the product of* t *and* 6 *is* $2 \cdot t \cdot 6 = 12t$*, which is what we have. Thus,*

$$P(t) = t(t + 6)^2.$$

Practice 2. *Factor.* *(Answers on page* *148**.)*

(a) $P(x) = x^4 + 20x^3 + 100x^2$

(b) $P(u, v) = 2u^2v - 4uv + 2v$

In the previous section we studied perfect squares of binomials, that is, polynomials of the form $(a + b)^2$ and $(a - b)^2$. But what happens if we multiply $(a + b)$ by $(a - b)$?

Example 4. *Multiply and simplify the result.*

(a) $(x-2)(x+2)$ *(b)* $(3x+1)(3x-1)$

Solution 4. *(a) We may use the Distributive Law as follows.*

$$\begin{aligned}(x-2)(x+2) &= x(x+2) - 2(x+2)\\ &= x^2 + 2x - 2x - 2\cdot 2\\ &= x^2 + 0x - 2^2\\ &= x^2 - 4.\end{aligned}$$

There are two important observations to make. First, the two x terms in the product are additive inverses of one another, so the result when simplified has just two terms. Second, the resulting binomial is the difference of two perfect squares.

(b) This time we will use the rectangular array approach. The set-up is:

	$3x$	-1
$3x$		
1		

Performing the multiplication, we have:

	$3x$	-1
$3x$	$(3x)^2$	$-3x$
1	$3x$	$(1)(-1)$

Thus,we obtain,

$$\begin{aligned}(3x+1)(3x-1) &= (3x)^2 - 3x + 3x - 1\cdot 1\\ &= (3x)^2 + 0x - 1^2\\ &= 9x^2 - 1\end{aligned}$$

Once again, two of the four terms of the product are additive inverses of one another. Thus, the simplified form is a binomial. Also, the result is the difference of two squares.

In fact, if we multiply any factors of the form $(a-b)(a+b)$, we obtain

$$\begin{aligned}(a-b)(a+b) &= a(a-b) - b(a+b)\\ &= a\cdot a + ab - ba - b\cdot b\\ &= a^2 + 0 - b^2\\ &= a^2 - b^2\end{aligned}$$

We can interpret this as a factoring formula for the ***difference of two squares***.

> **Difference of Two Squares:**
>
> $$a^2 - b^2 = (a-b)(a+b)$$

Example 5. *Factor.*

(a) $P(t) = 4t^2 - 121$ *(b)* $P(x) = x^3 - x$ *(c)* $P(y) = y^4 - 81$

Solution 5. *(a) We observe that* $4t^2 = (2t)^2$ *and* $121 = 11^2$. *Thus,* $P(t)$ *is the difference of two squares.*

$$\begin{aligned}P(t) &= 4t^2 - 121\\ &= (2t)^2 - 11^2\\ &= (2t-11)(2t+11).\end{aligned}$$

(b) First we factor out the greatest common factor.

$$\begin{aligned}P(x) &= x^3 - x\\ &= x(x^2-1)\\ &= x(x^2-1^2)\\ &= x(x-1)(x+1).\end{aligned}$$

(c) Here we observe that $y^4 = (y^2)^2$.

$$\begin{aligned}P(y) &= y^4 - 81\\ &= (y^2)^2 - 9^2\\ &= (y^2-9)(y^2+9)\\ &= (y-3)(y+3)(y^2+9).\end{aligned}$$

> *Note that $y^2 + 9$ is the sum of two squares and does not factor further under the real numbers.*

We say that a polynomial such as $y^2 + 9$ in Example 55 is ***prime*** *(or **irreducible**)* ***over the real numbers*** since it cannot be factored as a product of nonconstant factors with real coefficients. In fact, the sum of two squares is always prime over the real numbers.

Practice 3. *Factor.* *(Answers on the next page.)*

(a) $P(x) = 25 - 49x^2$

(b) $P(y) = y^5 - 36y^3$

(c) $P(t) = 625t^4 - 1$

Finally, let us consider factor by grouping again. Remember to look for the difference of squares.

Example 6. *Factor:* $8x^3 + 4x^2 - 18x - 9$

> **Solution 6.** *Since this polynomial has four terms, we will use factoring by grouping.*
>
> $$\begin{aligned} \underbrace{8x^3 + 4x^2} - \underbrace{18x - 9} &= 4x^2(2x+1) - 9(2x+1) \\ &= (2x+1)(4x^2-9) \\ &= \boxed{(2x+1)(2x-3)(2x+3)} \end{aligned}$$

Practice 4. *Factor:* $2x^3 + 3x^2 - 8x - 12$ *(Answers on the following page.)*

ANSWERS TO PRACTICE PROBLEMS

1. (a) $(x-2)^2$
   (b) $(7t+5)^2$
   (c) $(x-3y)^2$

2. (a) $x^2(x+10)^2$
   (b) $2v(u-1)^2$

3. (a) $(5-7x)(5+7x)$
   (b) $y^3(y-6)(y+6)$
   (c) $(5t-1)(5t+1)(25t^2+1)$

4. $(2x+3)(x-2)(x+2)$

Section 5.2 EXERCISES:

(Answers are found on page 243.)

Multiply and simplify your answers.

1. $(x+4)^2$
2. $(x-4)^2$
3. $(x+4)(x-4)$
4. $(x-10)^2$
5. $(x+10)(x+10)$
6. $(x+10)(x-10)$
7. $(2x-1)(2x-1)$
8. $(2x+1)(2x-1)$
9. $(2x+1)(2x+1)$
10. $(4x-3)(4x+3)$
11. $(3x+7)^2$
12. $(4x-5)(4x-5)$
13. $(2x+3y)^2$
14. $(3x+5y)(3x-5y)$
15. $(4x-2z)^2$

Factor each of the following polynomials.

16. x^2-16
17. x^2-81
18. $9x^2-4$
19. $5x^2-20$
20. $36x^2-9$
21. $28x^2-63$
22. x^2+6x+9
23. $x^2-8x+16$
24. $4x^2+4x+1$
25. $2x^2-4x+2$
26. $12x^2+36x+27$
27. $5x^2+20x+20$

Find the value(s) of b which make each of the following polynomials a square.

28. $x^2 + 12x + b$
29. $x^2 - 6x + b$
30. $x^2 - 14x + b$
31. $x^2 + 8x + b$
32. $4x^2 - 4x + b$
33. $9x^2 + 6x + b$
34. $9x^2 + 12x + b$
35. $25x^2 - 30x + b$
36. $bx^2 + 2x + 1$
37. $bx^2 - 6x + 1$
38. $bx^2 + 10x + 25$
39. $bx^2 + 12x + 4$
40. $x^2 + bx + 1$
41. $x^2 + bx + 16$
42. $9x^2 + bx + 1$
43. $25x^2 + bx + 1$
44. $9x^2 + bx + 4$
45. $4x^2 + bx + 36$
46. $x^2 + bxy + y^2$
47. $x^2 + bxy + 9y^2$
48. $25x^2 + bxy + y^2$

Determine which of the following polynomials is a difference of squares. Factor those polynomials that are differences of squares.

49. $x^2 - 49$
50. $x^2 - 121$
51. $x^2 + 9$
52. $2x^2 - 8$
53. $16x^2 - 25$
54. $9x^2 + 4$

Determine which of the following polynomials is a square. Factor those polynomials that are squares.

55. $x^2 - 49$
56. $x^2 - 14x + 49$
57. $x^2 + 7x + 49$
58. $x^2 + 16x + 64$
59. $4x^2 - 2x + 1$
60. $9x^2 + 6x + 1$
61. $25x^2 + 20x + 4$
62. $4x^2 - 4x - 1$
63. $4x^2 - 12x + 9$

Factor the following differences of squares.

64. $x^2 - \frac{1}{4}$
65. $y^2 - \frac{4}{9}$
66. $\frac{9}{4}x^2 - 1$
67. $\frac{25}{16}y^2 - 1$
68. $\frac{1}{64}x^2 - 25$
69. $\frac{9}{49}x^2 - 16$
70. $\frac{16}{25}x^2 - \frac{4}{49}$
71. $\frac{1}{121}x^2 - \frac{49}{100}$
72. $x^2 - 169$
73. $25x^2 - 4$

Factor each of the following squares.

74. $x^2 + 20x + 100$
75. $y^2 + 14y + 49$
76. $x^2 - 22x + 121$
77. $4w^2 - 36w + 81$
78. $x^2y^2 + 4xy + 4$
79. $(xy)^2 + 10xy + 25$
80. $28x^2 + 84x + 63$
81. $x^2 - x + \frac{1}{4}$
82. $x^2 + \frac{2}{3}x + \frac{1}{9}$
83. $y^2 + \frac{6}{5}y + \frac{9}{25}$
84. $z^2 - \frac{12}{7}z + \frac{36}{49}$
85. $\frac{1}{81}x^2 - \frac{2}{9}x + 1$
86. $\frac{1}{16}r^2 - \frac{3}{2}r + 9$
87. $\frac{4}{9}x^2 + \frac{1}{3}x + \frac{1}{16}$
88. $\frac{25}{16}x^2 + \frac{5}{3}x + \frac{4}{9}$

Factor by grouping.

89. $5x^3 - x^2 - 45x + 9$
90. $3x^3 - 2x^2 - 3x + 2$
91. $4x^3 + 12x^2 - 9x - 27$
92. $18x^3 + 9x^2 - 2x - 1$
93. $12x^3 - 16x^2 - 3x + 4$
94. $5x^3 - x^2 - 20x + 4$

95. $7x^3 + 3x^2 - 63x - 27$

96. $16x^3 - 12x^2 - 4x + 3$

97. $24x^3 - 4x^2 - 6x + 1$

98. $16x^3 - 4x^2 - 4x + 1$

99. $18x^3 - 27x^2 - 8x + 12$

100. $16x^3 + 32x^2 - x - 2$

101. $25x^3 - 25x^2 - 4x + 4$

102. $2x^3 + x^2 - 50x - 25$

5.3 Factoring Trinomials

In this section, we will factor trinomials which are not perfect squares. As usual, we will begin with some multiplication problems.

Example 1. *Find each product and simplify the result.*

(a) $(x+3)(x-5)$ *(b)* $(2x+1)(x+8)$

Solution 1. *(a) We may use the Distributive Law as follows.*

$$\begin{aligned} (x+3)(x-5) &= x(x-5)+3(x-5) \\ &= x^2-5x+3x+3(-5) \\ &= x^2+(-5+3)x-15 \\ &= x^2-2x-15 \end{aligned}$$

(b) This time we will use the rectangular array approach. The set-up is:

	x	8
$2x$		
1		

Performing the multiplication, we have:

	x	8
$2x$	$2x^2$	$16x$
1	x	8

Thus,we obtain,

$$\begin{aligned} (2x+1)(x+8) &= 2x^2+16x+x+8 \\ &= 2x^2+(16+1)x+8 \\ &= 2x^2+17x+8 \end{aligned}$$

Practice 1. *Find each product and simplify the result.* *(Answers on page 159.)*

(a) $(x-7)(x-12)$ *(b)* $(3x-2)(2x+6)$

In each of these examples, the product of the binomials has three terms after like terms have been combined. A polynomial with three terms is called a trinomial. Now we wish to factor trinomials into products of two binomials. We will restrict our attention to factorizations **over the integers**; that is, factorizations where the coefficients of the factors are all integers.

Example 2. *Factor* $P(x) = x^2 + 11x + 30$.

Solution 2. *Since the leading coefficient is* 1*, we are looking for a factorization of the form*

$$P(x) = (x + a)(x + b)$$

where a *and* b *are integers and the product* $ab = 30$ *and the sum* $a + b = 11$*. If such a factorization does not come to mind immediately, we can systematically consider all factorizations of* 30 *into a product of two positive integers.*

ab	$a+b$
$1 \cdot 30$	$1 + 30 = 31$
$2 \cdot 15$	$2 + 15 = 17$
$3 \cdot 10$	$3 + 10 = 13$
$5 \cdot 6$	$5 + 6 = 11 \star$

Thus, we claim that

$$P(x) = x^2 + 11x + 30 = (x + 5)(x + 6).$$

We will check our factorization by multiplying.

	x	6
x	x^2	$6x$
5	$5x$	30

Thus,

$$\begin{aligned}(x + 5)(x + 6) &= x^2 + 6x + 5x + 30 \\ &= x^2 + 11x + 30,\end{aligned}$$

which is what we claimed.

Practice 2. *Factor* $P(x) = x^2 + 5x + 6$. *(Answer on page 159.)*

Example 3. *Factor* $P(x) = x^2 - x - 12$.

Solution 3. *We will first list all pairs of integer factors of* -12 *and their sums.*

ab	$a+b$
$-1 \cdot 12$	$-1 + 12 = 11$
$1 \cdot -12$	$1 - 12 = -11$
$-2 \cdot 6$	$-2 + 6 = 4$
$2 \cdot -6$	$2 - 6 = -4$
$-3 \cdot 4$	$-3 + 4 = 1$
$3 \cdot -4$	$3 + 4 = -1$ ⋆

We are hoping to find a pair of factors of -12 *whose sum is* -1, *since this is the coefficient of* x. *We see that the last pair in the table satisfies this condition. Therefore,*

$$\begin{aligned} P(x) &= x^2 - x - 12 \\ &= (x+3)(x-4). \end{aligned}$$

It's a good idea to check this by multiplying.

	x	-4
x	x^2	$-4x$
3	$3x$	-12

Thus,

$$\begin{aligned} (x+3)(x-4) &= x^2 - 4x + 3x - 12 \\ &= x^2 - x - 12, \end{aligned}$$

which is what we claimed.

We could have reduced our work by half in the previous example by observing that, since the middle coefficient was negative, it must be the negative factor of -12 which is largest in absolute value.

Practice 3. *Factor* $P(x) = x^2 - 2x - 35$. *(Answer on page 159.)*

Example 4. *Factor* $P(y) = y^2 - 16y + 28$.

Solution 4. *This time the constant term is positive, but the coefficient of* y *is negative. This means we should consider pairs of negative factors of* 28.

ab	$a+b$
$-1 \cdot -28$	$-1 - 28 = -29$
$-2 \cdot -14$	$-2 - 14 = -16$ ⋆
$-4 \cdot -7$	$-4 - 7 = -11$

Thus,

$$\begin{aligned} P(y) &= y^2 - 16y + 28 \\ &= (y-2)(y-14). \end{aligned}$$

We leave it to the reader to check this by multiplying.

Practice 4. *Factor* $P(w) = w^2 - 14w + 48$. *(Answer on page 159.)*

Example 5. *Factor* $P(t) = 2t^5 + 12t^4 - 14t^3$.

Solution 5. *We begin by factoring out the GCF which is* $2t^3$.

$$\begin{aligned} P(t) &= 2t^5 + 12t^4 - 14t^3 \\ &= 2t^3(t^2 + 6t - 7) \end{aligned}$$

Next, we consider pairs of integer factors of -7 *for which the largest factor in absolute value is positive. No need to make a chart; since* 7 *is prime, the only possibility is* $-1 \cdot 7$. *Thus,*

$$\begin{aligned} P(t) &= 2t^3(t^2 + 6t - 7) \\ &= 2t^3(t-1)(t+7). \end{aligned}$$

The reader should check this by multiplying.

Practice 5. *Factor* $P(x) = 3x^4 + 3x^3 - 330x^2$. *(Answer on page 159.)*

Example 6. *Factor* $P(x) = x^2 + 18x + 24$, *if possible.*

Solution 6. *Let us consider pairs of integer factors of* 24. *Since both the constant term* (24) *and the coefficient of* x (18) *are positive, it suffices to consider pairs of positive integers,*

ab	$a+b$
$1 \cdot 24$	$1 + 24 = 25$
$2 \cdot 12$	$2 + 12 = 14$
$3 \cdot 8$	$3 + 8 = 11$
$4 \cdot 6$	$4 + 6 = 10$

This is a complete list of all pairs of positive integers whose product is 24. *However, no pair in the list has a sum of* 18. *We conclude that* $P(x)$ *is prime over the integers.*

Practice 6. *Factor* $P(x) = x^2 - 10x + 30$, *or else show that it is prime over the integers.* *(Answer on page 159.)*

Example 7. *Find all values of* k *so that the polynomial* $P(x) = x^2 + kx - 10$ *will factor into a product of linear factors.*

Solution 7. *We wish to write* $P(x) = x^2 + kx - 10$ *as a product of the form* $(x + a)(x + b)$ *where* a *and* b *are integers. This means that* $ab = -10$ *and* $a + b = k$. *We will make a complete list of all pairs of integer factors of* -10.

ab	$a+b=k$
$1 \cdot -10$	$1 - 10 = -9$
$-1 \cdot 10$	$-1 + 10 = 9$
$2 \cdot -5$	$2 - 5 = -3$
$-2 \cdot 5$	$-2 + 5 = 3$

Therefore, if k *is* -9, 9, -3, *or* 3, *then* $P(x) = x^2 + kx - 10$ *will factor.*

Practice 7. *a. Write down the four polynomials that arise in Example 7 and factor each.*

b. Find all values of k so that the polynomial $P(x) = x^2 + kx - 55$ will factor into a product of linear factors. Then factor each of the resulting polynomials.

(Answers on page 159.)

Example 8. *Factor $P(x) = 14x^2 + 9x + 1$.*

Solution 8. *In this example, the leading coefficient is not* 1*, but the constant term is. Also, the coefficient of x is positive. Therefore our factorization will be of the form $(ax + 1)(bx + 1)$ for some integers a and b for which $ab = 14$ and $a + b = 9$. Let us consider pairs of (positive) integer factors of* 14.

ab	$a + b$
$1 \cdot 14$	$1 + 14 = 15$
$2 \cdot 7$	$2 + 7 = 9$ $\star$

Hence,

$$P(x) = 14x^2 + 9x + 1$$
$$P(x) = (2x + 1)(7x + 1).$$

We will check our factorization by multiplying.

	$7x$	1
$2x$	$14x^2$	$2x$
1	$7x$	1

Thus,

$$\begin{aligned}(2x + 1)(7x + 1) &= 14x^2 + 2x + 7x + 1 \\ &= 14x^2 + 9x + 1,\end{aligned}$$

which is what we claimed.

Practice 8. *Factor* $P(x) = 15x^2 - 8x + 1$. *(Answer on the next page.)*

Example 9. *Factor* $P(t) = 5t^2 + 14t - 3$.

Solution 9. *In this example, neither the leading coefficient nor the constant term is* 1. *Our factorization will be of the form* $(ax + c)(bx + d)$ *for some integers* a, b, c *and* d *for which* $ab = 5$, $cd = -3$, *and* $ad + bc = 14$. *That's an awful lot to keep track of! Fortunately,* 5 *and* 3 *are both prime, so there are not that many possibilities. We will use the factorization* $5 = 5 \cdot 1$ *for the coefficients of* x *in the factors. Then we need to consider four factorizations (counting order) of* -3*:*

$$\begin{aligned}
-3 &= 3 \cdot (-1), \\
-3 &= -3 \cdot 1, \\
-3 &= 1 \cdot (-3), \text{ and} \\
-3 &= -1 \cdot 3.
\end{aligned}$$

We will try all of the combinations to see which one (if any) works.

$$\begin{aligned}
(5x + 3)(1x - 1) &= 5x^2 - 2x - 3 \\
(5x - 3)(1x + 1) &= 5x^2 + 2x - 3 \\
(5x + 1)(1x - 3) &= 5x^2 - 14x - 3 \\
(5x - 1)(1x + 3) &= 5x^2 + 14x - 3\star
\end{aligned}$$

Hence,

$$\begin{aligned}
P(x) &= 5x^2 + 14x - 3 \\
P(x) &= (5x - 1)(x + 3).
\end{aligned}$$

Practice 9. *Factor* $P(x) = 7x^2 + 5x - 2$. *(Answer on the facing page.)*

Example 10. *Factor* $P(x, y) = 4x^2 + 24xy + 11y^2$.

Solution 10. *In this example, neither the leading coefficient nor the constant term is* 1*, and we have two variables. This time our factorization will be of the form* $(ax + cy)(bx + dy)$ *for some integers* a*,* b*,* c *and* d *for which* $ab = 4$*,* $cd = 11$*, and* $ad + bc = 24$*. Note that* a*,* b*,* c *and* d *will all be positive. We have two factorizations of* 4 *and just one of* 11*. However, order matters when the lead coefficients of the binomials are different, so we actually have 3 possibilities to try.*

$$(4x + 11y)(1x + 1y) = 4x^2 + 15xy + 11y^2$$
$$(4x + 1y)(1x + 11y) = 4x^2 + 45xy + 11y^2$$
$$(2x + 11y)(2x + 1y) = 4x^2 + 24xy + 11y^2\star$$

Hence,

$$P(x, y) = 4x^2 + 24xy + 11y^2$$
$$P(x) = (2x + 11y)(2x + y).$$

Practice 10. *Factor* $P(x) = 6x^2 + 13xy + 5y^2$*.* *(Answer below.)*

ANSWERS TO PRACTICE PROBLEMS

1. (a) $x^2 - 19x + 84$
   (b) $6x^2 + 14x - 12$
2. $(x + 2)(x + 3)$
3. $(x + 5)(x - 7)$
4. $(w - 6)(w - 8)$
5. $3x^2(x - 10)(x + 11)$
6. prime over the integers
7. (a) $P(x) = x^2 - 9x - 10 = (x + 1)(x - 10)$
   $P(x) = x^2 + 9x - 10 = (x - 1)(x + 10)$
   $P(x) = x^2 - 3x - 10 = (x + 2)(x - 5)$
   $P(x) = x^2 + 3x - 10 = (x - 2)(x + 5)$
   (b) $k = -54, 54, -6, 6$
   $P(x) = x^2 - 54x - 55 = (x + 1)(x - 55)$
   $P(x) = x^2 + 54x - 55 = (x - 1)(x + 55)$
   $P(x) = x^2 - 6x - 55 = (x + 5)(x - 11)$
   $P(x) = x^2 + 6x - 55 = (x - 5)(x + 11)$
8. $(3x - 1)(5x - 1)$
9. $(7x - 2)(x + 1)$
10. $(2x + y)(3x + 5y)$

Section 5.3 EXERCISES:

(Answers are found on page 245.)

Factor the following trinomials.

1. $x^2 + 3x + 2$
2. $x^2 - x - 2$
3. $x^2 + 2x - 3$
4. $x^2 + 6x - 7$
5. $x^2 - 12x + 11$
6. $x^2 - 10x - 11$
7. $x^2 + 5x + 6$
8. $x^2 - 2x - 8$
9. $x^2 - 7x + 12$
10. $-x^2 + 9x - 14$
11. $2x^2 - 3x - 2$
12. $5x^2 + 6x + 1$
13. $2x^2 - 4x + 2$
14. $5x^2 + 35x + 30$
15. $6x^2 + 8x + 2$
16. $x^2 + 4xy + 4y^2$
17. $x^2 - 7xy + 12y^2$
18. $5x^2 + 11xy + 2y^2$

Determine which of the following trinomials can be factored into a product of linear factors and which are prime.

19. $x^2 + 4x + 4$
20. $x^2 + 9x + 14$
21. $x^2 + x - 2$
22. $x^2 + x - 3$
23. $x^2 + x - 4$
24. $x^2 + 3x + 3$
25. $3x^2 + 4x + 1$
26. $3x^2 + 9x - 12$
27. $3x^2 + 5x + 2$
28. $3x^2 + 7x + 3$
29. $3x^2 + 9x + 3$
30. $5x^2 + 20x + 10$
31. $x^2 + 5xy + 7y^2$
32. $x^2 - 9xy + 18y^2$
33. $7x^2 + 2xy - 5y^2$

For each of the following polynomials, find all the values for b so that the polynomial will factor into a product of linear factors.

34. $x^2 + bx + 3$
35. $x^2 + bx - 3$
36. $x^2 + bx - 7$
37. $x^2 + bx + 7$
38. $x^2 + bx + 6$
39. $x^2 + bx - 6$
40. $2x^2 + bx + 1$
41. $2x^2 + bx - 1$
42. $x^2 + bx + 18$
43. $3x^2 + bx + 3$
44. $x^2 + bx + 8$
45. $6x^2 + bx + 5$
46. $x^2 + bxy + 9y^2$
47. $x^2 + bxy - 8y^2$
48. $7x^2 + bxy - 2y^2$

Factor the following polynomials.

49. $12x^2 + 20x - 8$
50. $5x^3 - 45x$
51. $16x^2 - 40xy + 16y^2$
52. $32x^4 - 200$
53. $-28x^4 + 84x^3 - 63x^2$
54. $\dfrac{9}{4}y^4 - 25$
55. $81t^4 - 256$
56. $2y^4 - 26y^2 + 72$
57. $8x^{10} - 98x^4$
58. $450 - 5a - 5a^2$
59. $-60n^3 - 300n^2 - 375n$
60. $6x^3 + 21x^2 - 12x$
61. $21x^2 - 48x - 45$
62. $12x^2 - 14x - 10$
63. $12x^2 - 14x - 6$
64. $12x^3 - 27x^2 - 27x$
65. $12x^3 - 62x^2 + 10x$
66. $4x^2 - 2x - 20$
67. $8x^2 + 46x - 12$
68. $8x^2 + 34x - 84$
69. $6x^2 - 2x - 60$
70. $10x^2 - 15x - 10$
71. $6x^3 - 16x^2 - 6x$
72. $28x^2 + 10x - 2$
73. $8x^2 - 20x + 12$
74. $30x^4 + 2x^3 - 4x^2$
75. $12x^2 + 15x - 18$
76. $18x^2 + 12x - 48$

5.4 Factoring Review

Now it is time to review all the factoring techniques we have just learned. The following is a general factoring strategy that can be followed.

1. Order the terms in either descending or ascending order according to the powers of the variable. If there is more than one variable, order according to the variable of your choice.

2. Look for a Greatest Common Factor (GCF). If the problem has one, then factor it out. Remember to include the GCF in your final answer.

3. Determine the number of terms in the expression.

    * If there are only two terms, determine if it is a difference of two squares. If so, then factor according to

    $$a^2 - b^2 = (a-b)(a+b)$$

    Remember that the sum of two squares cannot be factored further over the real numbers.

    * If there are three terms, determine if it is one of the special forms shown below:

    $$a^2 + 2ab + b^2 = (a+b)^2$$
    $$a^2 - 2ab + b^2 = (a-b)^2$$

    If it is not one of the special forms, factoring using any of the techniques discussed.

    * If there are four or more terms, try to factor by grouping.

4. Be sure to look to see if any factors can be factored further.

The best way to improve your factoring skills is to practice. The more factoring problems you do, the easier they will become and the quicker you will be able to see how an individual problem factors.

Example 1. *Factor Completely.*

(a) $4x^2y + 8xy - 12y$

(b) $12x^2 + 15x - 18$

(c) $12 + 5x - 2x^2$

(d) $64x^2 - 25y^2$

(e) $7x^3 - 3x^2 - 63x + 27$

(f) $30x^4 + 2x^3 - 4x^2$

(g) $2x^2 + 7x - 72$

(h) $48x^3 - 75x$

Solution 1.

(a) *Since the GCF is $4y$, we factor this out first. Continuing, we get*

$$4x^2y + 8xy - 12y = 4y(x^2 + 2x - 3) = 4y(x + 3)(x - 1).$$

(b) *Here the GCF is 3, so we begin by factoring this out.*

$$12x^2 + 15x - 18 = 3(4x^2 + 5x - 6) = 3(4x - 3)(x + 2).$$

(c) *There is no GCF and the terms are arranged in ascending order. Therefore, we use the techniques discussed to factor as*

$$12 + 5x - 2x^2 = (4 - x)(3 + 2x).$$

(d) *Since this problem only has two terms, we look to see if it is the difference of two squares. Note that $64x^2 = (8x)^2$ and $25y^2 = (5y)^2$. Therefore, using the difference of squares formula, we obtain*

$$64x^2 - 25y^2 = (8x - 5y)(8x + 5y).$$

(e) *Since there are four terms, we will attempt to factor this by grouping. We group the first two terms together and the last two terms together. Thus,*

$$\begin{aligned}\underbrace{7x^3 - 3x^2} - \underbrace{63x + 27} &= x^2(7x - 3) - 9(7x - 3)\\ &= (7x - 3)(x^2 - 9)\\ &= (7x - 3)(x - 3)(x + 3)\end{aligned}$$

Note that the difference of squares formula was also used in this problem.

(f) *The GCF of $2x^2$ is factored out first. Thus,*

$$\begin{aligned} 30x^4 + 2x^3 - 4x^2 &= 2x^2(15x^2 + x - 2) \\ &= 2x^2(5x+2)(3x-1) \end{aligned}$$

(g) *Since there is no GCF, we use the techniques discussed to obtain*

$$2x^2 + 7x - 72 = (2x-9)(x+8).$$

(h) *The GCF of $3x$ is factored out first. Then, the difference of squares formula is used to obtain*

$$\begin{aligned} 48x^3 - 75x &= 3x(16x^2 - 25) \\ &= 3x(4x-5)(4x+5) \end{aligned}$$

Practice 1. *Factor Completely . (Answers below)*

(a) $4x^3 + 8x^2 + 4x$

(b) $16x^3 - 4x^2 - 4x + 1$

(c) $40x^8y^7 - 16x^9y^5$

(d) $12x^3 - 62x^2 + 10x$

(e) $10x^2 + 17x + 3$

(f) $3x^2y - 6xy - 45y$

ANSWERS TO PRACTICE PROBLEMS

1. (a) $4x(x+1)^2$
   (b) $(4x-1)(2x-1)(2x+1)$
   (c) $8x^8y^5(5y^2 - 2x)$
   (d) $2x(6x-1)(x-5)$
   (e) $(5x+1)(2x+3)$
   (f) $3y(x-5)(x+3)$

SECTION 5.4 EXERCISES

(Answers are found on page 247.)

Factor Completely.

1. $4x^3y - xy^3$
2. $5x^2 + 10xy - 40y^2$
3. $x^4 - 2x^3 + 3x - 6$
4. $1 - (a+b)^2$
5. $6u^2 + 7uv - 3v^2$
6. $9x^2 - x^6y^2$

7. $2y^4 - 5y^2 - 12$
8. $3xy + 6xz - wy - 2wz$
9. $(x^2 - 5)^2 - 6(x^2 - 5) + 8$
10. $7a^2 + 17a - 12$
11. $8t^3 + 80t^2 + 150t$
12. $5x^2 - 45$
13. $a^3 - 13a^2 - 90a$
14. $27t^3 + 108t^2 + 36t + 144$
15. $(x - 3)^{10} - 1$
16. $3x^3y - 6x^2y^2 + 3xy^3$
17. $2z(x - 2y) - w(x - 2y)$
18. $b^2 + 20b - 800$
19. $12 - 8x - 7x^2$
20. $6ax - 2by + 4bx - 3ay$
21. $6x^2 + 48x + 72$
22. $81y^4 - 16x^4$
23. $a^2b^2 + b^2c^2 - c^2d^2 - a^2d^2$
24. $(3a + 5)a^2 + (3a + 5)a$
25. $4y^2 - 29yz - 24z^2$
26. $18x^3 - 8x$
27. $2x^3y - 6x^2y^2 - 8xy^3$
28. $15ac - 20ad + 3bc - 4bd$
29. $(x + 2)^2 - 9y^2$
30. $12y^3 + 44y^2 - 16y$
31. $18x^3 - 3x^2 - 6x + 1$
32. $3a^4 - 48a^2$
33. $40x^2 + 14x - 45$
34. $(3x + 2y)^2 + (3x + 2y) - 12$
35. $x^2y^2 - 225$
36. $81 - (a + 1)^2$
37. $2xy^2 - 6x^2 - 3y^3 + 9xy$
38. $4(x - 7)^3 - (x - 7)$
39. $4x^3 - 20x^2 + 25x$
40. $(a - 3)^2 - (a^2 + 3)^2$
41. $36x^2 - 49y^2$
42. $5x^3 - x^2 + 20x - 4$
43. $12x^2 - 14x - 10$
44. $2x^2 + 13x + 15$
45. $x^2 - 7xy + 12y^2$
46. $6x^2 - 11x - 10$
47. $2x^4y - 2x^4y^2 + 2x^3y^2$
48. $3x^3 + 18x^2 + 27x$
49. $6x^3y^2 - 2x^2y - 60x$
50. $9 + 13x - 10x^2$
51. $25x^3 - 25x^2 - 4xy + 4y$
52. $36x^3y - 16xy^3$
53. $30x^4 + 2x^3 - 4x^2$
54. $2x^2y^2 - 12xy + 18$
55. $2x^5 - 14x^4 + 24x^3$
56. $8x^2 - 17x + 9$

57. $8x^2 + 34x - 84$
58. $81x^2 - 49$
59. $14 + 3x - 2x^2$
60. $2x^3 + 3x^2 - 8x - 12$
61. $16x^4 - 81$
62. $12x^2 + 15x - 18$
63. $4x^3y - 12x^2y^2 + 8x^2y$
64. $4x^2 - 4x - 48$
65. $8x^2 + 46x - 12$
66. $x^3 - 13x^2 - 90x$
67. $8x^3 + 80x^2 + 150x$
68. $3x^4y + 6x^3y - 72x^2y$
69. $8x^2 + 6xy - 5y^2$
70. $1 - 8x - 20x^2$

5.5 Solving Equations by Factoring

The key idea in finding the solution to an equation by factoring is the ***Zero Product Property***. This is because zero has a special property. It is impossible to multiply two nonzero numbers to obtain a product of zero. In other words, if a product is zero, then at least one of the factors must be zero.

> **Zero Product Property:** Let a and b be real numbers. If $ab = 0$, then
> $$a = 0 \text{ or } b = 0 \text{ (or both).}$$

Example 1. *Solve:* $(x-3)(x+2) = 0$

Solution 1. *We have a product equal to zero. By the Zero Product Property, at least one of the factors must be zero. That is,*

$$x - 3 = 0 \qquad \textit{or} \qquad x + 2 = 0.$$

Solving each of the resulting linear equations for x, we obtain

$$x = 3 \qquad \textit{or} \qquad x = -2.$$

We leave it to the reader to check each solution in the original equation.

Practice 1. *Solve for the variable:* $(2x-1)(x+4) = 0$ *(Answers on page 174.)*

Example 2. *Solve* $-16t^2 - 16t + 96 = 0$ *by factoring.*

Solution 2. *Factoring the left-hand side, we obtain:*

$$-16\left(t^2 + t - 6\right) = 0$$
$$-16(t+3)(t-2) = 0$$

So the solutions to this equation are precisely the values of t for which the product of -16, $t+3$, and $t-2$ is equal to zero.

Thus, we conclude that

$$-16 = 0 \qquad \textit{or} \qquad t+3=0 \qquad \textit{or} \qquad t-2=0.$$

Since $-16 \neq 0$, we must have

$$t+3=0 \qquad \textit{or} \qquad t-2=0.$$

That is,

$$t=-3 \qquad \textit{or} \qquad t=2.$$

Practice 2. *Solve by factoring:* $6x + 28x - 10 = 0$ *(Answers on page 174.)*

Example 3. *Solve each of the following equations for the variable.*

(a) $x^2 - 13x = 0$

(b) $x(x-13) = -30$

(c) $3t(t+2) + 4 = t + 6$

Solution 3. *(a) Since the right-hand side is* 0*, we start by factoring the left-hand side.*

$$\begin{aligned} x^2 - 13x &= 0 \\ x(x-13) &= 0 \end{aligned}$$

Using the Zero Product Property, we obtain:

$$\begin{aligned} &x=0 \qquad \textit{or} \qquad x-13=0, \qquad \textit{and so} \\ &x=0 \qquad \textit{or} \qquad x=13. \end{aligned}$$

(b) Although the left-hand side is factored, we cannot *conclude that*

$$x=-30 \qquad \textit{or} \qquad x-13=-30.$$

This is because there is no "–30 Product Property" analogous to the Zero Product Property! It is quite possible to *have a product of two numbers equal to* -30 *even though neither number is itself* -30*. (Consider* -5×6*, for example.) Thus, we must first do a little algebra to get* 0 *on one side of the equation.*

$$\begin{aligned} x(x-13) &= -30 \\ x^2 - 13x &= -30 \\ x^2 - 13x + 30 &= -30 + 30 \\ x^2 - 13x + 30 &= 0 \\ (x-3)(x-10) &= 0 \end{aligned}$$

Now using the Zero Product Property, we obtain:

$$\begin{array}{rccl} x - 3 = 0 & \textit{or} & x - 10 = 0, & \textit{and so} \\ x = 3 & \textit{or} & x = 10. & \end{array}$$

(c) Again, we have a bit of algebra to do before we can factor.

$$\begin{aligned} 3t(t+2) + 4 &= t + 6 \\ 3t^2 + 6t + 4 &= t + 6 \\ 3t^2 + 6t + 4 - t - 6 &= t + 6 - t - 6 \\ 3t^2 + 5t - 2 &= 0 \\ (3t-1)(t+2) &= 0 \end{aligned}$$

Using the Zero Product Property, we obtain:

$$\begin{array}{rccl} 3t - 1 = 0 & \textit{or} & t + 2 = 0, & \textit{and so} \\ 3t = 1 & \textit{or} & t = -2, & \textit{and so} \\ t = \dfrac{1}{3} & \textit{or} & t = -2. & \end{array}$$

Practice 3. *Solve by factoring:* $2x(x+3) = -4$ *(Answers on page 174.)*

Example 4. *Solve:* $25y^2 + 20y + 4 = 0$

Solution 4. *We note that the right-hand side is* 0 *and the left-hand side is a perfect square.*

$$\begin{aligned} 25y^2 + 20y + 4 &= 0 \\ (5y)^2 + 20y + 2^2 &= 0 \\ (5y+2)^2 &= 0 \end{aligned}$$

Using the Zero Product Property, we obtain:

$$\begin{aligned} 5y + 2 &= 0 \\ 5y &= -2 \\ y &= -\frac{2}{5} \end{aligned}$$

In this example, we had just one solution since the quadratic polynomial was the square of a binomial.

Practice 4. *Solve by factoring:* $9t^2 - 42t + 49 = 0$ *(Answers on page 174.)*

Example 5. *Solve for the variable by factoring:* $4x^2 = 9$

Solution 5. *We first need one side to be zero, and then factor using the difference of squares formula.*

$$\begin{aligned} 4x^2 &= 9 \\ 4x^2 - 9 &= 0 \\ (2x)^2 - 3^2 &= 0 \\ (2x-3)(2x+3) &= 0 \end{aligned}$$

Using the Zero Product Property, we obtain:

$$\begin{aligned} 2x - 3 &= 0 \qquad \textit{or} \qquad 2x + 3 = 0, \qquad \textit{and so} \\ 2x &= 3 \qquad \textit{or} \qquad 2x = -3, \qquad \textit{and so} \\ x &= \frac{3}{2} \qquad \textit{or} \qquad x = -\frac{3}{2}. \end{aligned}$$

Practice 5. *Solve for the variable by factoring:* $16w^2 = 1$ *(Answers on page 174.)*

Example 6. *Solve for the variable.*

a. $(x-7)\left(2x^2+5x+3\right)=0$

b. $u^3 - 7u^2 = -12u$

Solution 6. *We will solve each of these by factoring.*

a. This polynomial is already partially factored.

$$\begin{aligned}(x-7)(2x^2+5x+3)&=0\\(x-7)(2x+3)(x+1)&=0\end{aligned}$$

And so by Zero Product Property,

$$\begin{array}{lllllll} x-7=0 & \textit{or} & 2x+3=0 & \textit{or} & x+1=0 & \textit{and so} \\ x=7 & \textit{or} & 2x=-3 & \textit{or} & x=-1 & \textit{and so} \\ x=7 & \textit{or} & x=-\dfrac{3}{2} & \textit{or} & x=-1. & \end{array}$$

b. Here we will first factor out the GCF.

$$\begin{aligned}u^3-7u^2&=-12u\\u^3-7u^2+12u&=0\\u(u^2-7u+12)&=0\\u(u-3)(u-4)&=0\end{aligned}$$

And so by Zero Product Property,

$$\begin{array}{llllll} u=0 & \textit{or} & u-3=0 & \textit{or} & u-4=0 & \textit{and so} \\ u=0 & \textit{or} & u=3 & \textit{or} & u=4. & \end{array}$$

Practice 6. *Solve for the variable by factoring.* *(Answers on page 174.)*

a. $(4t-1)(t^2+6t-16)=0$

b. $5y^4 = 7y^3 - 2y^2$

Example 7. *Solve for x:* $4x^3+6x^2-40x=0$

Solution 7. *Since one side is already zero, we factor to obtain*

$$4x^3+6x^2-40x=0$$

$$2x(2x^2+3x-20)=0$$

$$2x(2x-5)(x+4)=0$$

So, the Zero Product Property gives us

$$\begin{array}{rcrcr} 2x=0 & or & 2x-5=0 & or & x+4=0 \\ x=0 & or & 2x=5 & or & x=-4 \\ x=0 & or & x=\dfrac{5}{2} & or & x=-4 \end{array}$$

Practice 7. *Solve by factoring:* $12x^3+12x^2-9x=0$ *(Answers on page 174.)*

Example 8. *Solve for x:* $2x(x-3)+3(x-5)=x^2+3$

Solution 8. *We must first remove all parentheses and then set the equation equal to zero.*

$$2x(x-3)+3(x-5)=x^2+3$$

$$2x^2-6x+3x-15=x^2+3$$

$$2x^2-3x-15=x^2+3$$

$$x^2-3x-18=0$$

$$(x-6)(x+3)=0$$

Using the Zero Product Property, we obtain

$$x-6=0 \qquad or \qquad x+3=0$$

$$x=6 \qquad or \qquad x=-3$$

Practice 8. *Solve by factoring:* $3x(x+1)+2(x+2)=2x(x+5)$ *(Answers on the next page.)*

Example 9. *Solve for x:* $5x^3-2x^2-20x+8=0$

Solution 9. *Since the left hand side of the equation has four terms, we will factor by grouping.*

$$\underbrace{5x^3-2x^2}-\underbrace{20x+8}=0$$

$$x^2(5x-2)-4(5x-2)=0$$

$$(5x-2)(x^2-4)=0$$

$$(5x-2)(x-2)(x+2)=0$$

Using the Zero Product Property, we get

$$\begin{aligned} 5x-2=0 &\quad or \quad & x-2=0 &\quad or \quad & x+2=0 \\ 5x=2 &\quad or \quad & x=2 &\quad or \quad & x=-2 \\ x=\frac{2}{5} &\quad or \quad & x=2 &\quad or \quad & x=-2 \end{aligned}$$

Practice 9. *Solve by factoring:* $36x^3-9x^2-4x+1=0$ *(Answers below.)*

ANSWERS TO PRACTICE PROBLEMS

1. $x=-4, \frac{1}{2}$
2. $x=-5, \frac{1}{3}$
3. $x=-2,-1$
4. $t=\frac{7}{3}$
5. $w=\pm\frac{1}{4}$
6. (a) $t=-8, \frac{1}{4}, 2$
   (b) $y=0, \frac{2}{5}, 1$
7. $x=-\frac{3}{2}, 0, \frac{1}{2}$
8. $x=1,4$
9. $x=\pm\frac{1}{3}, \frac{1}{4}$

Section 5.5 EXERCISES:

(Answers are found on page 249.)

Solve the following equations.

1. $(x-3)(x+2)=0$
2. $(x-7)(x-2)=0$
3. $(x+3)(x+4)=0$
4. $x(x-6)=0$
5. $(x+4)(x+4)=0$
6. $5(x+1)(x-4)=0$
7. $(2x+1)(x+6)=0$
8. $(3x-2)(4x-1)=0$
9. $3(10x+1)(7x-2)=0$
10. $x^2+3x-4=0$
11. $x^2-7x+12=0$
12. $x^2+10x+25=0$
13. $3x^2+18x+27=0$
14. $5x^2+6x+1=0$
15. $4x^2-8x+3=0$
16. $x(x-3)=4$

17. $(x+1)(x+2)=12$
18. $x^2+10x=-25$
19. $x^2+5x=x+8$
20. $3x^2+5x=2x^2-3x$
21. $5x^2+3x+9=4x^2+3x-7$
22. $2x^3-2x^2-12x=0$
23. $2x^3-18x^2+36x=0$
24. $x^4-81=0$
25. $x^4-4x^2+3=0$
26. $x^2-12x+11=0$
27. $10x^2-15x-10=0$
28. $4x(x-2)=3(x-2)$
29. $2x^2-5x-3=0$
30. $x^2+6x+9=0$
31. $3x^2=2x$
32. $4x^2-2x-12=0$
33. $x(x+7)=5(x+3)$
34. $4x^2-2x-20=0$
35. $(2x+1)(x-3)=-5$
36. $3x(3x+1)-2=x(3x+4)$
37. $12x^2-14x-10=0$
38. $5x^2-7x=0$
39. $x^2-8x-9=0$
40. $(x-4)(x+2)=4x(x-3)$
41. $16x^2-25=0$
42. $2x(3x-1)+2=2x(x+1)+5$
43. $2x^3-9x^2-5x=0$
44. $18x^3+9x^2-2x-1=0$
45. $9x^4-13x^2+4=0$
46. $9x^3-4x^2-9x+4=0$
47. $10x^3+17x^2+3x=0$
48. $4x^4-13x^2+9=0$
49. $24x^3-4x^2-6x+1=0$
50. $16x^4-17x^2+1=0$
51. $2x^3-14x^2+12x=0$
52. $3x^3+9x^2-30x=0$
53. $4x^3+12x^2-9x-27=0$
54. $4x^4-17x^2+4=0$
55. $8x^3-20x^2+12x=0$

Find the value(s) for b so that the following equations have integer solutions.

56. $x^2+bx+5=0$
57. $x^2+bx-5=0$
58. $x^2+bx+9=0$
59. $x^2+bx-9=0$
60. $x^2+bx+12=0$
61. $x^2+bx-8=0$

62. $x(x+b) = -10$
63. $y^2 + by = 17$
64. $y^2 + by = 18$

Find the value(s) for b so that the given value a is a solution of the the equation.

65. $x^2 + bx + 16 = 0; \quad a = 2$
66. $x^2 + bx + 16 = 0; \quad a = 4$
67. $x^2 + bx - 16 = 0; \quad a = -1$
68. $x^2 + bx - 16 = 0; \quad a = 4$

Exploration: Factors and x-Intercepts

Note to the Instructor: *Students will need a simple graphing device with zoom and trace functionality. This could be a graphing calculator or a graphing application on the Web such as gcalc.net. You will need to spend a few minutes demonstrating how to graph functions and how to zoom and trace on the graph. Ideally the material in this exploration will be covered in two lab days: one day for linear functions and one day for quadratic functions. You might wish to have students work together in pairs or small groups. Because this is a discovery activity, no answers are provided in the text. Therefore, it is crucial to provide ample assistance during the activity and feedback afterward.*

In this exploration you will be lead to discover one of the fundamental ideas of algebra: the connection between linear factors, x-intercepts, and solutions of polynomial equations. This is a central idea to which you will return many times throughout this course and as you continue your algebra studies. Therefore, it is crucial for you to actually *do* all of the activities in this section, participate in class discussions, and get feedback from your instructor.

x-intercepts of linear polynomials

Recall from Core Math I that the equation $y = mx + b$ represents a line with slope m and y-intercept $(0, b)$. (Click here to review this material from Core Math I) Here, we want to switch our focus to x-intercepts, that is, the points where a graph crosses the x-axis. Remember that the y-coordinate of an x-intercept is 0.

Each row in the following table represents a linear function. Use your graphing device and algebra skills from Core Math I to fill in the blanks in each row. As you do so, observe the patterns that emerge. These patterns will help you to fill in the remaining rows. Your instructor may require you to turn in your work, written neatly on a separate piece of paper.

	$y = mx + b$ form	**$y = m(x - c)$ form**	**Slope**	**x-intercept**	**Solution of equation $y = 0$**
1	$y = x - 2$	$y = 1(x - 2)$			
2	$y = x - 13$	$y = 1(x - 13)$			
3			1	$(35, 0)$	
4			1		$x = 7$
5	$y = x + 8$	$y = 1(x - (-8))$			
6	$y = x + 4$				
7			1	$(-12, 0)$	
8			1		$x = -10$
9	$y = 3x - 18$	$y = 3(x - 6)$			
10	$y = 2x - 1$				
11			5	$(2, 0)$	
12			-1		$x = 1$
13	$y = 4x + 12$	$y = 4(x - (-3))$			
14	$y = -7x - 14$				
15			-3	$(-10, 0)$	
16			9		$x = -3$
17			m		$x = c$

x-intercepts of quadratic polynomials

From the above exploration, we know that the graph of $y = x - 2$ is a line with slope 1 and x-intercept 2 and the graph of $y = x - 5$ is a line with slope 1 and x-intercept 5. (Graph these on your graphing device to confirm.) What happens if we create a new polynomial function by taking the product of $x - 2$ and $x - 5$, that is $y = (x - 2)(x - 5)$? Do you have a guess as to what the shape of this graph is? What do you think the x-intercepts are? Graph $y = (x - 2)(x - 5)$ to check your guesses. When you have found the x-intercepts correctly, write them in the first row of the table on the next page.

Next, we wish to write $y = (x - 2)(x - 5)$ in expanded form. To do this, perform the multiplication and combine like terms. Write the result in the table. Also write the leading coefficient of the polynomial in the table. Is it obvious from the expanded form of the equation what the x-intercepts are? If you wish to determine the x-intercepts, would you rather have the factored form $y = (x - 2)(x - 5)$ or the expanded form $y = x^2 - 7x + 10$?

To finish this example, we wish to solve the equation $(x - 2)(x - 5) = 0$ for x. If you have not yet learned an algebraic method to do this, you can use your grapher instead as follows. Remember that the expression $(x-2)(x-5)$ is equal to y. So we want to know what x is when $y = 0$. Trace along your graph until the y-coordinate is 0. What is the x-coordinate? (Be careful: there might be more than one answer!) Write this in the table and you have filled in the first row.

Each row in the table represents a quadratic function. As you fill in the blanks in each row, observe the patterns that emerge. These patterns will help you to fill in the remaining rows. Your instructor may require you to turn in your work, written neatly on a separate piece of paper.

	Expanded Form	**Factored Form**	**Lead Coeff**	**x-int(s)**	**Solutions of equation $y = 0$**
1		$y = (x-2)(x-5)$			
2	$y = x^2 - 3x + 2$	$y = 1(x-2)(x-1)$			
3			1	$(5, 0)$, $(3, 0)$	
4			1		$x = 7, 2$
5	$y = x^2 + 5x - 24$	$y = 1(x+8)(x-3)$			
6	$y = x^2 + 12x + 35$				
7			1	$(-20, 0)$, $(4, 0)$	
8			1		$x = -5, -4$
9	$y = 2x^2 - 12x - 14$	$y = 2(x+1)(x-7)$			
10	$y = 5x^2 + 25x + 30$				
11			-3	$(-1, 0)$, $(1, 0)$	
12			10		$x = 2, 12$
13	$y = 2x^2 + 5x - 3$	$y = 2(x - \frac{1}{2})(x+3)$			
14	$y = 5x^2 + 12x + 4$				
15			3	$(-2, 0)$, $(\frac{2}{3}, 0)$	
16			2		$x = -\frac{7}{2}, 1$
17			a		$x = d_1, d_2$

Chapter 6

Radicals

6.1 Introduction to Radicals

Recall the following definition for square roots which was discussed in Core Math I.

> **Square roots of a**: The square root of a, denoted $\sqrt{a}$, is the number whose square is a. In other words,
>
> $$\sqrt{a} = b \quad \text{means} \quad b^2 = a.$$

Remember that the radical sign implies the **principal**, or nonnegative, square root.

Example 1. *Find each square root.*

(a) $\sqrt{121}$	*(c)* $\sqrt{-16}$	*(e)* $\sqrt{x^4}$
(b) $\sqrt{\frac{49}{64}}$	*(d)* $-\sqrt{16}$	*(f)* $\sqrt{4x^8}$

Solution 1. *(a)* $\sqrt{121} = 11$ *since* $(11)^2 = 121$.

(b) $\sqrt{\frac{49}{64}} = \frac{7}{8}$ *since* $\left(\frac{7}{8}\right)^2 = \frac{49}{64}$

(c) $\sqrt{-16}$ *is not a real number. This is because there is no real number a such that $a^2 = -4$.*

(d) $-\sqrt{16} = -4$ *since* $-\sqrt{16} = -1 \cdot \sqrt{16}$ *and* $\sqrt{16} = 4$ *since* $4^2 = 16$.

(e) $\sqrt{x^4} = x^2$ *since* $(x^2)^2 = x^4$.

(f) $\sqrt{4x^8} = 2x^4$ *since* $(2x^4)^2 = 4x^8$.

Practice 1. *Find each square root.* *(Answers on page 184)*

(a) $\sqrt{\dfrac{25}{81}}$ *(b)* $\sqrt{9x^{12}}$ *(c)* $\sqrt{-36}$

We now generalize our square root definition to general n-th roots.

n-th roots of a: The n-th root of a, denoted $\sqrt[n]{a}$, is a number whose n-th power equals a. In other words,

$$\sqrt[n]{a} = b \quad \text{means} \quad b^n = a.$$

The number n is called the **index** and a is called the **radicand**.

Remember that the index two is generally omitted for square roots. As with square roots, if n is even, the radical $\sqrt[n]{\ }$ represents the nonnegative n-th root.

Example 2. *Find each root.*

(a) $\sqrt[3]{27}$ *(c)* $\sqrt[4]{-81}$ *(e)* $\sqrt[5]{-32}$

(b) $\sqrt[3]{\dfrac{8}{125}}$ *(d)* $\sqrt[4]{\dfrac{16}{81}}$ *(f)* $\sqrt[6]{x^{12}}$

Solution 2. *(a)* $\sqrt[3]{27} = 3$ *since* $3^3 = 27$.

(b) $\sqrt[3]{\dfrac{8}{125}} = \dfrac{2}{5}$ *since* $\left(\dfrac{2}{5}\right)^3 = \dfrac{8}{125}$.

(c) $\sqrt[4]{-81}$ *is not a real number since there does not exist a real number a such that $a^4 = -81$.*

(d) $\sqrt[4]{\dfrac{16}{81}} = \dfrac{2}{3}$ *since* $\left(\dfrac{2}{3}\right)^4 = \dfrac{16}{81}$.

(e) $\sqrt[5]{-32} = -2$ *since* $(-2)^5 = -32$.

(f) $\sqrt[6]{x^{12}} = x^2$ *since* $(x^2)^6 = x^{12}$.

Practice 2. *Find each root.* *(Answers on page 184)*

(a) $\sqrt[3]{-\dfrac{64}{27}}$ (b) $\sqrt[4]{16x^8}$ (c) $\sqrt[6]{64}$

When working with variables, it is not clear whether the variable represents a positive or negative real number. Because when n is even the radical $\sqrt[n]{}$ represents the nonnegative n-th root, when variables are present it may be necessary to include absolute values in the answer. We state this rule next.

Evaluating $\sqrt[n]{a^n}$:

- If n is even, then $\sqrt[n]{a^n} = |a|$.
- If n is odd, then $\sqrt[n]{a^n} = a$.

Note that the absolute value sign is not required unless the index is even.

Example 3. *Find each root.*

(a) $\sqrt[4]{(-2)^4}$ (c) $\sqrt{x^{10}}$

(b) $\sqrt[3]{(-6)^3}$ (d) $\sqrt[5]{x^{15}}$

Solution 3. (a) $\sqrt[4]{(-2)^4} = |-2| = 2$.

(b) $\sqrt[3]{(-6)^3} = -6$. *No absolute value sign is necessary since the index is odd.*

(c) $\sqrt{x^{10}} = |x^5|$. *The absolute value sign is needed because it is not clear if x^5 is positive.*

(d) $\sqrt[5]{x^{15}} = x^3$. *No absolute value sign is necessary since the index is odd.*

Practice 3. *Find each root.* *(Answer below.)*

(a) $\sqrt[3]{x^9}$ (b) $\sqrt[4]{(-8)^4}$ (c) $\sqrt[6]{x^{18}}$

ANSWERS TO PRACTICE PROBLEMS

1. (a) $\frac{5}{9}$
   (b) $3x^6$
   (c) not a real number
2. (a) $-\frac{4}{3}$
   (b) $2x^2$
   (c) 2
3. (a) x^3
   (b) 8
   (c) $|x^3|$

SECTION 6.1 EXERCISES

(Answers are found on page 250.)

Evaluate.

1. $\sqrt{81}$
2. $-\sqrt{121}$
3. $\sqrt{\frac{9}{16}}$
4. $-\sqrt{400}$
5. $-\sqrt{\frac{100}{49}}$
6. $\sqrt{-4}$
7. $\sqrt[3]{-8}$
8. $-\sqrt[4]{16}$
9. $\sqrt[3]{1000}$
10. $\sqrt[3]{-\frac{125}{27}}$
11. $-\sqrt[3]{64}$
12. $\sqrt[4]{-256}$

13. $\sqrt[5]{-32}$

14. $\sqrt[3]{-\dfrac{1}{8}}$

15. $-\sqrt{\dfrac{1}{4}}$

16. $\sqrt[3]{125}$

17. $-\sqrt[3]{-64}$

18. $\sqrt[3]{\dfrac{27}{8}}$

19. $\sqrt{0.64}$

20. $\sqrt{0.36}$

21. $\sqrt{0.25}$

22. $\sqrt[3]{0.008}$

23. $\sqrt[3]{-0.027}$

24. $\sqrt{(-5)^2}$

25. $\sqrt{7^2}$

26. $\sqrt[3]{(-8)^3}$

27. $\sqrt[3]{9^3}$

28. $\sqrt[4]{(-10)^4}$

29. $\sqrt[4]{7^4}$

30. $\sqrt{\left(-\dfrac{3}{5}\right)^2}$

31. $\sqrt[7]{(-3)^7}$

32. $\sqrt{x^{14}}$

33. $\sqrt{x^{20}}$

34. $\sqrt[4]{x^4}$

35. $\sqrt[4]{x^{16}}$

36. $\sqrt{100x^4}$

37. $\sqrt{49y^6}$

38. $\sqrt[3]{216x^9}$

39. $\sqrt{64x^{12}}$

40. $\sqrt[3]{64x^{12}}$

41. $\sqrt[4]{81x^{12}}$

42. $\sqrt{81x^{12}}$

43. $\sqrt{16x^{16}}$

44. $\sqrt[4]{16x^{16}}$

45. $\sqrt[3]{64x^9}$

46. $\sqrt[4]{16x^8y^{12}}$

47. $\sqrt{121x^6y^2}$

48. $\sqrt[3]{125x^{15}y^{21}}$

49. $\sqrt[5]{-243x^{10}y^{25}}$

50. $\sqrt{36x^4y^{10}}$

6.2 Simplifying Radical Expressions

Before we can perform any operations with radical expressions, we need to be able to simplify radicals. We know that

$$\sqrt{4}\sqrt{9} = 2 \cdot 3 = 6 \quad \text{and} \quad \sqrt{4 \cdot 9} = \sqrt{36} = 6.$$

Since both cases result in 6, we can conclude that $\sqrt{4}\sqrt{9} = \sqrt{4 \cdot 9}$. Furthermore, we know that

$$\sqrt[3]{\frac{27}{8}} = \frac{3}{2} \quad \text{and} \quad \frac{\sqrt[3]{27}}{\sqrt[3]{8}} = \frac{3}{2}.$$

Since both cases result if $\frac{3}{2}$, we can again conclude that $\sqrt[3]{\frac{27}{8}} = \frac{\sqrt[3]{27}}{\sqrt[3]{8}}$. It is left to the reader to examine examples with other roots. The two results are stated below in general.

Product rule for radicals: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$

In other words, the radical of a product is the product of the radicals.

Quotient rule for radicals: If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers and n is a positive integer,

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

In other words, the radical of a quotient is the quotient of the radicals.

We will use the above rules to simplify a radical. Unless otherwise stated, assume that all variables represent positive real numbers. This means that there is no need to include the absolute value sign in the answer.

Example 1. *Simplify. Assume each variable represents a positive real number.*

(a) $\sqrt{4x^6y^8}$

(b) $\sqrt{\frac{x^4}{16}}$

(c) $\sqrt[3]{-27x^3y^9z^6}$

(d) $\sqrt[3]{\dfrac{64x^3}{y^6}}$

Solution 1. *(a) Using the product rule for radicals, we have*

$$\begin{aligned}\sqrt{4x^6y^8} &= \sqrt{4}\sqrt{x^6}\sqrt{y^8}\\ &= 2x^3y^4\end{aligned}$$

(b) Using the quotient rule for radicals, we obtain

$$\sqrt{\frac{x^4}{16}} = \frac{\sqrt{x^4}}{\sqrt{16}} = \frac{x^2}{4}$$

(c) Using the product rule for radicals, we have

$$\begin{aligned}\sqrt[3]{-27x^3y^9z^6} &= \sqrt[3]{-27}\,\sqrt[3]{x^3}\,\sqrt[3]{y^9}\,\sqrt[3]{z^6}\\ &= -3xy^3z^2\end{aligned}$$

(d) Using the quotient rule and then the product rule for radicals, we get

$$\begin{aligned}\sqrt[3]{\frac{64x^3}{y^6}} &= \frac{\sqrt[3]{64x^3}}{\sqrt[3]{y^6}}\\ &= \frac{\sqrt[3]{64}\,\sqrt[3]{x^3}}{\sqrt[3]{y^6}}\\ &= \frac{4x}{y^2}\end{aligned}$$

Practice 1. *Simplify. Assume all variables represent positive real numbers.*

(a) $\sqrt{49x^4x^{10}}$ *(b)* $\sqrt[3]{\dfrac{8x^{12}}{y^{15}}}$ *(c)* $\sqrt[4]{16x^4y^{12}z^{16}}$

(Answers on page 191)

Not all the radical problems we encounter will work out to be perfect n-th roots. As a result, we need to be able to simplify any radical. The definition of a radical in simplified form follows.

Simplifying radicals: A radical is in simplest form when the following conditions are satisfied:

- There is no fraction under the radical.
- There is no radical in the denominator.
- There is no common factor, other than 1, between the exponents on factors under the radical and the index.
- The quantity under the radical has no factor raised to a power greater than or equal to the index.

Example 2. *Simplify* $\sqrt{12x^9}$. *Assume the variable represents a positive real number*

Solution 2. *Since both* 12 *and* x^9 *are not perfect squares, we need to rewrite each as a product of two terms: one a perfect square and one that is not only not a perfect square, but contains no factors that are perfect squares. Hence,* $12 = 4 \cdot 3$ *and* $x^9 = x^8 \cdot x$. *Using the product rule for radicals, we get*

$$\sqrt{12x^9} = \sqrt{4 \cdot 3 \cdot x^8 \cdot x} = \sqrt{4x^8}\sqrt{3x} = 2x^4\sqrt{3x}.$$

Notice that we placed each term that is a perfect square in the first radical and everything that would be left over in the second radical.

Practice 2. *Simplify* $\sqrt{32x^7}$. *Assume the variable represents a positive real number.* *(Answers on page 191)*

Example 3. *Simplify. Assume all variables represent positive real numbers.*

(a) $\sqrt{32x^5y^{11}}$ (c) $\sqrt{147x^6y^3z^4}$ (e) $\sqrt[3]{54x^3y^5z^4}$

(b) $\sqrt{75x^{13}y^7z^4}$ (d) $\sqrt{128x^7y^2z^{19}}$ (f) $\sqrt[4]{32x^5y^7z^9}$

Solution 3. *(a)*

$$\begin{aligned}\sqrt{32x^5y^{11}} &= \sqrt{16\cdot 2\cdot x^4\cdot x\cdot y^{10}\cdot y}\\ &= \sqrt{16x^4y^{10}}\sqrt{2xy}\\ &= 4x^2y^5\sqrt{2xy}\end{aligned}$$

(b)

$$\begin{aligned}\sqrt{75x^{13}y^7z^4} &= \sqrt{25\cdot 3\cdot x^{12}\cdot x\cdot y^6\cdot y\cdot z^4}\\ &= \sqrt{25x^{12}y^6z^4}\sqrt{3xy}\\ &= 5x^6y^3z^2\sqrt{3xy}\end{aligned}$$

(c)

$$\begin{aligned}\sqrt{147x^6y^3z^4} &= \sqrt{49\cdot 3\cdot x^6\cdot y^2\cdot y\cdot z^4}\\ &= \sqrt{49x^6y^2z^4}\sqrt{3y}\\ &= 7x^3yz^2\sqrt{3y}\end{aligned}$$

(d)

$$\begin{aligned}\sqrt{128x^7y^2z^{19}} &= \sqrt{64\cdot 2\cdot x^6\cdot x\cdot y^2\cdot z^{18}\cdot z}\\ &= \sqrt{64x^6y^2z^{18}}\sqrt{2xz}\\ &= 8x^3yz^9\sqrt{2xz}\end{aligned}$$

(e)

$$\begin{aligned}\sqrt[3]{54x^3y^5z^4} &= \sqrt[3]{27\cdot 2\cdot x^3\cdot y^3\cdot y^2\cdot z^3\cdot z}\\ &= \sqrt[3]{27x^3y^3z^3}\,\sqrt[3]{2y^2z}\\ &= 3xyz\,\sqrt[3]{2y^2z}\end{aligned}$$

(f)

$$\begin{aligned}\sqrt[4]{32x^5y^7z^9} &= \sqrt[4]{16\cdot 2\cdot x^4\cdot x\cdot y^4\cdot y^3\cdot z^8\cdot z}\\ &= \sqrt[4]{16x^4y^4z^8}\,\sqrt[4]{2xy^3z}\\ &= 2xyz^2\,\sqrt[4]{2xy^3z}\end{aligned}$$

Practice 3. *Simplify. Assume the variable represents a positive real number. (Answers on page 191)*

(a) $\sqrt{72x^9y^3}$ (b) $\sqrt[3]{108x^5y^9z^4}$ (c) $\sqrt[4]{162x^4y^9z^5}$

For a radical to be in simplest form, there can be no common factors, other than one, between the exponents of factors under the radical and the index. So, $\sqrt[8]{x^6y^2}$ would not be in simplest form since the exponents on the factors of the radicand and the index have a common factor of 2. In order to simplify this, we need to use the Index Rule for radicals.

Index Rule for Radicals: If k, m and n are positive integers, then

$$\sqrt[kn]{a^{km}} = \sqrt[n]{a^m}.$$

Example 4. *Simplify* $\sqrt[8]{x^6y^2}$.

Solution 4. *Note that the exponents on the factors under the radical and the index have a common factor of 2. Therefore, using the Index Rule for radicals, we get*

$$\begin{aligned}\sqrt[8]{x^6y^2} &= \sqrt[4\cdot 2]{x^{3\cdot 2}y^{1\cdot 2}}\\ &= \sqrt[4]{x^3y}\end{aligned}$$

Practice 4. *Simplify* $\sqrt[12]{x^3y^6}$ *(Answer below)*

ANSWERS TO PRACTICE PROBLEMS

1. (a) $7x^2y^5$
   (b) $\frac{2x^4}{y^5}$
   (c) $2xy^3z^4$
2. $4x^3\sqrt{2x}$
3. (a) $6x^4y\sqrt{2xy}$
   (b) $3xy^3z\sqrt[3]{4x^2z}$
   (c) $3xy^2z\sqrt[4]{2yz}$
4. $\sqrt[4]{xy^2}$

SECTION 6.2 EXERCISES

(Answers are found on page 251.)

Simplify. Assume all variables represent positive real numbers.

1. $\sqrt{8x^3}$
2. $\sqrt{162x^2}$
3. $\sqrt[3]{-24x^6}$
4. $\sqrt{72x^3}$
5. $\sqrt[4]{x^9}$
6. $\sqrt[4]{x^2}$
7. $\sqrt[8]{x^4}$
8. $\sqrt[12]{x^4}$
9. $\sqrt[6]{x^4y^8}$
10. $\sqrt[4]{x^2y^8}$
11. $\sqrt[6]{x^{18}y^4}$
12. $\sqrt[4]{16x^2y^{10}}$
13. $\sqrt{24x^2}$
14. $\sqrt{18x^5y^7}$
15. $\sqrt{32a^5b^{15}}$
16. $\sqrt{40x^{11}y^7}$
17. $2\sqrt{16a^2b^3}$
18. $4\sqrt{20a^4b^7}$
19. $3x\sqrt{12x^2y^7}$
20. $4y\sqrt{18x^5y^4}$
21. $2x^2\sqrt{8x^2y^3}$
22. $3y\sqrt{28x^3y^4}$
23. $5\sqrt{18xy^3}$
24. $2x\sqrt{48x^5y^2}$
25. $\sqrt[3]{16x^4y^9}$
26. $\sqrt[3]{54x^5y^7}$
27. $\sqrt[3]{8x^6y^4}$
28. $\sqrt[3]{16x^5y^7}$
29. $\sqrt[3]{40x^4y^8}$
30. $\sqrt{243x^9y^{25}z^{18}}$
31. $\sqrt[3]{135x^9y^{14}z^{25}}$
32. $\sqrt[4]{324x^{13}y^{11}z^{22}}$
33. $\sqrt{252x^{10}y^{13}z^{21}}$
34. $\sqrt[3]{108x^7y^9z^{14}}$
35. $\sqrt{147x^{10}y^6z^3}$
36. $\sqrt{320x^5y^9z^{12}}$
37. $\sqrt[4]{32x^8y^{11}z^{12}}$
38. $\sqrt{192x^8y^{15}z^6}$
39. $\sqrt[3]{-108x^{15}y^7z^{11}}$
40. $\sqrt[4]{48x^8y^{14}z^{11}}$
41. $\sqrt[5]{128x^{19}y^{23}z^{10}}$
42. $\sqrt[3]{-81x^4y^{10}}$

6.3 Addition and Subtraction of Radicals

Now that we know how to simplify radicals we turn our attention to adding and subtracting radicals. Before we do this, recall that we performed addition and subtraction of algebraic expressions in Section 4.2 by combining like terms. For example, $3x + 5x = (3 + 5)x = 8x$. The same procedure can be used for the addition and subtraction of radicals. When adding and subtracting radicals, you may only combine radicals with the same index and exactly the same quantity under the radical (same radicand). For example, $3\sqrt{7} + 5\sqrt{7} = (3 + 5)\sqrt{7} = 8\sqrt{7}$. However, $3\sqrt{7}$ and $5\sqrt[3]{7}$ cannot be combined since their indices are different. Likewise, $3\sqrt{7}$ and $5\sqrt{3}$ cannot be combined because the quantity under the radical is different.

Example 1. *Simplify:* $7\sqrt{11} + 5\sqrt{11} - 4\sqrt{11}$

Solution 1. *Since all the radical parts are identical, we have*

$$\begin{aligned} 7\sqrt{11} + 5\sqrt{11} - 4\sqrt{11} &= (7 + 5 - 4)\sqrt{11} \\ &= 8\sqrt{11} \end{aligned}$$

Practice 1. *Simplify:* $8\sqrt{3} - 3\sqrt{3} + 2\sqrt{3}$ *(Answer on page 198)*

Remember you cannot add two radicals together if they have different radical parts. Although we saw that there is a product and quotient rule for radicals, as stated in Section 6.2, there is no addition property of radicals. In other words,

$$\sqrt[n]{a} + \sqrt[n]{b} \neq \sqrt[n]{a + b}.$$

Example 2. *Simplify:* $3\sqrt{5} - 7\sqrt{2} + 6\sqrt{2}$

Solution 2. *Since only the last two terms have identical radical parts, these are the only two terms that we can combine. Hence,*

$$\begin{aligned} 3\sqrt{5} - 7\sqrt{2} + 6\sqrt{2} &= 3\sqrt{5} + (-7 + 6)\sqrt{2} \\ &= 3\sqrt{5} - \sqrt{2} \end{aligned}$$

Practice 2. *Simplify:* $7\sqrt{6}+9\sqrt{2}-11\sqrt{6}$ *(Answer on page 198)*

Of course, not every problem will have the same radical parts. What happens if the radicands do not match? The next example illustrates that some simplification might be needed before we can combine the radicals.

Example 3. *Simplify:* $3\sqrt{48}+5\sqrt{27}$

Solution 3.

$$\begin{aligned} 3\sqrt{48}+5\sqrt{27} &= 3\sqrt{16\cdot 3}+5\sqrt{9\cdot 3} \\ &= 3\sqrt{16}\sqrt{3}+5\sqrt{9}\sqrt{3} \\ &= 3\cdot 4\sqrt{3}+5\cdot 3\sqrt{3} \\ &= 12\sqrt{3}+15\sqrt{3} \\ &= 27\sqrt{3} \end{aligned}$$

Practice 3. *Simplify:* $4\sqrt{32}-5\sqrt{50}$ *(Answer on page 198)*

Example 4. *Simplify:* $3\sqrt{12}-7\sqrt{72}+5\sqrt{50}$

Solution 4.

$$\begin{aligned} 3\sqrt{12}-7\sqrt{72}+5\sqrt{50} &= 3\sqrt{4\cdot 3}-7\sqrt{36\cdot 2}+5\sqrt{25\cdot 2} \\ &= 3\sqrt{4}\sqrt{3}-7\sqrt{36}\sqrt{2}+5\sqrt{25}\sqrt{2} \\ &= 3\cdot 2\sqrt{3}-7\cdot 6\sqrt{2}+5\cdot 5\sqrt{2} \\ &= 6\sqrt{3}-42\sqrt{2}+25\sqrt{2} \\ &= 6\sqrt{3}-17\sqrt{2} \end{aligned}$$

Practice 4. *Simplify:* $-2\sqrt{63}+2\sqrt{28}+3\sqrt{7}$ *(Answer on page* *198**)*

Example 5. *Simplify:* $\sqrt{\dfrac{3}{16}}+\sqrt{\dfrac{75}{4}}$

Solution 5. *Applying the Quotient rule for radicals, we have*

$$\begin{aligned}
\sqrt{\frac{3}{16}}+\sqrt{\frac{75}{4}} &= \frac{\sqrt{3}}{\sqrt{16}}+\frac{\sqrt{75}}{\sqrt{4}}\\
&= \frac{\sqrt{3}}{4}+\frac{\sqrt{25\cdot 3}}{2}\\
&= \frac{\sqrt{3}}{4}+\frac{\sqrt{25}\sqrt{3}}{2}\\
&= \frac{\sqrt{3}}{4}+\frac{5\sqrt{3}}{2}\\
&= \frac{\sqrt{3}}{4}+\frac{10\sqrt{3}}{4}\\
&= \frac{11\sqrt{3}}{4}
\end{aligned}$$

Practice 5. *Simplify:* $\sqrt{\dfrac{75}{64}}+\sqrt{\dfrac{3}{16}}$ *(Answer on page* *198**)*

When adding or subtracting radicals that contain variables, remember that in order to have like terms, the radical parts and the variable parts must be identical.

Example 6. *Simplify:* $2x\sqrt[3]{54x^4}-5\sqrt[3]{16x^7}$

Solution 6.

$$\begin{aligned}
2x\sqrt[3]{54x^4} - 5\sqrt[3]{16x^7} &= 2x\sqrt[3]{27 \cdot 2x^3x} - 5\sqrt[3]{8 \cdot 2x^6x} \\
&= 2x\sqrt[3]{27x^3}\sqrt[3]{2x} - 5\sqrt[3]{8x^6}\sqrt[3]{2x} \\
&= 2x \cdot 3x\sqrt[3]{2x} - 5 \cdot 2x^2\sqrt[3]{2x} \\
&= 6x^2\sqrt[3]{2x} - 10x^2\sqrt[3]{2x} \\
&= -4x^2\sqrt[3]{2x}
\end{aligned}$$

Practice 6. *Simplify:* $5x\sqrt[3]{54x^3y} - 2\sqrt[3]{250x^6y} + 9x^2\sqrt[3]{128y}$ *(Answer on page 198)*

Example 7. *Simplify:* $\dfrac{\sqrt{50}}{3} + \dfrac{3\sqrt{8}}{2} + \dfrac{\sqrt{3}}{\sqrt{4}}$

Solution 7.

$$\begin{aligned}
\frac{\sqrt{50}}{3} + \frac{3\sqrt{8}}{2} + \frac{\sqrt{3}}{\sqrt{4}} &= \frac{\sqrt{25 \cdot 2}}{3} + \frac{3\sqrt{4 \cdot 2}}{2} + \frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{25}\sqrt{2}}{3} + \frac{3\sqrt{4}\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \\
&= \frac{5\sqrt{2}}{3} + \frac{6\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \\
&= \frac{5\sqrt{2}}{3} + 3\sqrt{2} + \frac{\sqrt{3}}{2} \\
&= \frac{5\sqrt{2}}{3} + \frac{9\sqrt{2}}{3} + \frac{\sqrt{3}}{2} \\
&= \frac{14\sqrt{2}}{3} + \frac{\sqrt{3}}{2}
\end{aligned}$$

Practice 7. *Simplify:* $\dfrac{2\sqrt{3}}{4} - \dfrac{3\sqrt{27}}{2} + \dfrac{\sqrt{3}}{3}$ *(Answer on page 198)*

ANSWERS TO PRACTICE PROBLEMS

1. $7\sqrt{3}$
2. $-4\sqrt{6}+9\sqrt{2}$
3. $-9\sqrt{2}$
4. $\sqrt{7}$
5. $\frac{7\sqrt{3}}{8}$
6. $41x^2\sqrt[3]{2y}$
7. $-\frac{11\sqrt{3}}{3}$

SECTION 6.3 EXERCISES

(Answers are found on page 252.)

Perform the indicated operations and simplify. Assume all variables represent positive real numbers.

1. $\sqrt{12}+\sqrt{75}$
2. $\sqrt[3]{16}+\sqrt[3]{54}$
3. $\sqrt{27}-\sqrt{192}$
4. $3\sqrt{20}+5\sqrt{45}$
5. $4\sqrt[3]{16}-2\sqrt[3]{2}+2\sqrt{2}-2\sqrt{16}$
6. $\sqrt{\frac{8}{49}}-\sqrt{\frac{50}{9}}$
7. $5\sqrt{27x}-\sqrt{75x}$
8. $\sqrt{\frac{3}{25}}+4\sqrt{\frac{3}{100}}$
9. $3\sqrt{8x^2}-\sqrt{50x^2}$
10. $\sqrt{2x^2}+6\sqrt{32x^2}-2\sqrt{x^2}$
11. $\sqrt{12y^3}+\sqrt{27y^3}+3y\sqrt{3y}$
12. $\sqrt{28x^2}-\sqrt{63x^2}+\sqrt[3]{56x^3}$
13. $5\sqrt{4x}-3\sqrt{9x}$
14. $3\sqrt{3x^2}-5\sqrt{27x^2}$
15. $-2\sqrt{8x^2}+5\sqrt{32x^2}$
16. $5\sqrt{18}-2\sqrt{75}$
17. $3x\sqrt{12x}-5\sqrt{27x^3}$
18. $8\sqrt{8}-4\sqrt{32}-9\sqrt{50}$
19. $-2\sqrt{3}+5\sqrt{27}-4\sqrt{45}$

20. $\sqrt{25x} - \sqrt{9x} + \sqrt{16x}$

21. $3\sqrt{3x} + \sqrt{27x} - 8\sqrt{75x}$

22. $x\sqrt{3y^2} - 2y\sqrt{12x^2} + xy\sqrt{3}$

23. $\sqrt{18} + \sqrt{8} - \sqrt{32}$

24. $\sqrt{48} + \sqrt{20} - \sqrt{27} + 2\sqrt{20}$

25. $\sqrt{18x^3} + x\sqrt{32x} + \sqrt{50x^2}$

26. $\sqrt{25} + \sqrt{12}$

27. $\sqrt{49} - \sqrt{90}$

28. $8\sqrt{18} - 2\sqrt{50} - 3\sqrt{98}$

29. $\sqrt[3]{375} + 2\sqrt[3]{81} - 7\sqrt[3]{24}$

30. $\sqrt{\dfrac{5}{4}} - \sqrt{\dfrac{45}{16}}$

31. $5\sqrt[3]{16} + 9\sqrt[3]{128}$

32. $7\sqrt{8} + 9\sqrt{18} - 4\sqrt{162}$

33. $-5\sqrt{63x^3} + 9x\sqrt{28x} + 6\sqrt{7x^3}$

34. $3\sqrt{8} - 6\sqrt{50} + 2\sqrt{200}$

35. $5\sqrt[3]{16} + 9\sqrt[3]{54} - 7\sqrt[3]{2}$

36. $-2\sqrt[4]{48} + 3\sqrt[4]{243} + 5\sqrt[4]{3}$

37. $3x\sqrt{75x^3} + 7\sqrt{12x^5} - x\sqrt{27x}$

38. $\sqrt{18x} - 2\sqrt{12x^3} + \sqrt{32x} + 3x\sqrt{27x}$

39. $4\sqrt[3]{16} - 5\sqrt[3]{54} + 7\sqrt[3]{128}$

40. $3\sqrt{48} - 2\sqrt{300}$

41. $-2\sqrt{63} + 3\sqrt{7} + 4\sqrt{28}$

42. $\sqrt{\dfrac{3}{16}} + \sqrt{\dfrac{75}{4}}$

43. $3\sqrt{48} + 5\sqrt{27} - 6\sqrt{108}$

44. $5\sqrt[3]{16} - 7\sqrt[3]{54}$

45. $12\sqrt{8} - 5\sqrt{18} + 3\sqrt{162}$

46. $2\sqrt[3]{16} - 5\sqrt[3]{54} + 3\sqrt[3]{128}$

47. $4\sqrt{18x} - \sqrt{72x} + \sqrt{50x}$

48. $\sqrt{\dfrac{8}{81}} - \sqrt{\dfrac{32}{9}}$

49. $-8\sqrt{27} + 7\sqrt{48} - 3\sqrt{108}$

50. $2\sqrt{18} - 5\sqrt{50} + 3\sqrt{32}$

6.4 Multiplication of Radicals

We can multiply radicals using many of the same properties and techniques that we used in Section 4.2 to multiply polynomials.

Example 1. *Simplify:* $\left(3\sqrt{3}\right)\left(2\sqrt{6}\right)$

Solution 1.

$$\begin{aligned}\left(3\sqrt{3}\right)\left(2\sqrt{6}\right) &= (3\cdot 2)\left(\sqrt{3}\sqrt{6}\right)\\ &= 6\sqrt{18}\\ &= 6\sqrt{9}\sqrt{2}\\ &= 6\cdot 3\sqrt{2}\\ &= 18\sqrt{2}\end{aligned}$$

Practice 1. *Simplify:* $\left(5\sqrt{2}\right)\left(2\sqrt{6}\right)$ *(Answer on page 204)*

Be careful not to combine factors on the outside of the radical with the numbers that appear in the radicand. For example, $3x\sqrt{2x} \neq \sqrt{6x^2}$.

Example 2. *Simplify:* $3\sqrt{2}\left(5\sqrt{3}-4\right)$

Solution 2. *Using the distributive property, we obtain*

$$\begin{aligned}3\sqrt{2}\left(5\sqrt{3}-4\right) &= \left(3\sqrt{2}\right)\left(5\sqrt{3}\right)-\left(3\sqrt{2}\right)(4)\\ &= 15\sqrt{6}-12\sqrt{2}\end{aligned}$$

Since the radicands are different, they cannot be combined.

Practice 2. *Simplify:* $4\sqrt{3}\left(2+6\sqrt{5}\right)$ *(Answer on page 204)*

Example 3. *Simplify:* $5x\sqrt[3]{9}\left(2\sqrt[3]{3}-6\sqrt[3]{9}\right)$

Solution 3. *Once again, using the distributive property, we obtain*

$$\begin{aligned}5x\sqrt[3]{9}\left(2\sqrt[3]{3}-6\sqrt[3]{9}\right)&=10x\sqrt[3]{27}-30x\sqrt[3]{81}\\&=10x\cdot 3-30x\sqrt[3]{27\cdot 3}\\&=30x-30x\sqrt[3]{27}\sqrt[3]{3}\\&=30x-90x\sqrt[3]{3}\end{aligned}$$

Practice 3. *Simplify:* $3x\sqrt[3]{2}\left(5\sqrt[3]{4}+7\sqrt[3]{2}\right)$ *(Answer on page 204)*

Next, we illustrate multiplying a radical expression containing two terms with another radical expression containing two terms. As in Section 4.2, we use the distributive property twice.

Example 4. *Simplify:* $\left(3\sqrt{5}+2\right)\left(2\sqrt{5}-7\right)$

Solution 4.

$$\begin{aligned}\left(3\sqrt{5}+2\right)\left(2\sqrt{5}-7\right)&=6\sqrt{25}-21\sqrt{5}+4\sqrt{5}-14\\&=6\cdot 5-21\sqrt{5}+4\sqrt{5}-14\\&=30-21\sqrt{5}+4\sqrt{5}-14\\&=16-17\sqrt{5}\end{aligned}$$

Be careful in the preceding example. A common mistake is to combine the $16-17$. Remember that the $-17\sqrt{5}$ is considered one term. Therefore, since the 16 does not have a radical, these two terms cannot be combined.

Practice 4. *Simplify:* $\left(4\sqrt{2}-3\right)\left(2\sqrt{2}+5\right)$ *(Answer on page 204)*

Recall from Section 4.2 that $(a+b)^2=(a+b)(a+b)$.

Example 5. *Simplify:* $\left(2\sqrt{3}+\sqrt{6}\right)^2$

Solution 5.

$$\begin{aligned}\left(2\sqrt{3}+\sqrt{6}\right)^2 &= \left(2\sqrt{3}+\sqrt{6}\right)\left(2\sqrt{3}+\sqrt{6}\right)\\ &= 4\sqrt{9}+2\sqrt{18}+2\sqrt{18}+\sqrt{36}\\ &= 4\cdot 3+4\sqrt{18}+6\\ &= 12+4\sqrt{9}\sqrt{2}+6\\ &= 18+4\cdot 3\sqrt{2}\\ &= 18+12\sqrt{2}\end{aligned}$$

Practice 5. *Simplify:* $\left(\sqrt{8}-3\right)^2$ *(Answer on page 204)*

Example 6. *Simplify:* $\left(3\sqrt{5}-4\right)\left(3\sqrt{5}+4\right)$

Solution 6.

$$\begin{aligned}\left(3\sqrt{5}-4\right)\left(3\sqrt{5}+4\right) &= 9\sqrt{25}+12\sqrt{5}-12\sqrt{5}-16\\ &= 9\sqrt{25}-16\\ &= 9\cdot 5-16\\ &= 45-16\\ &= 29\end{aligned}$$

The preceding example could also be simplified using the difference of squares formula, $(a-b)(a+b)=a^2-b^2$, stated in Section 5.2. Using this formula, the previous example is simplified as

$$\begin{aligned}\left(3\sqrt{5}-4\right)\left(3\sqrt{5}+4\right) &= \left(3\sqrt{5}\right)^2-4^2\\ &= 9\sqrt{25}-16\\ &= 45-16\\ &= 29\end{aligned}$$

Practice 6. *Simplify:* $\left(2\sqrt{7}+3\right)\left(2\sqrt{7}-3\right)$ *(Answer on page 204)*

Example 7. *Simplify:* $\left(2\sqrt{8}-\sqrt{3}\right)\left(2\sqrt{48}-\sqrt{2}\right)$

Solution 7. *Since* $\sqrt{8}$ *and* $\sqrt{48}$ *can be simplified, we will do this first.*

$$\begin{aligned}\left(2\sqrt{8}-\sqrt{3}\right)\left(2\sqrt{48}-\sqrt{2}\right) &= \left(2\sqrt{4\cdot 2}-\sqrt{3}\right)\left(2\sqrt{16\cdot 3}-\sqrt{2}\right)\\ &= \left(2\cdot 2\sqrt{2}-\sqrt{3}\right)\left(2\cdot 4\sqrt{3}-\sqrt{2}\right)\\ &= \left(4\sqrt{2}-\sqrt{3}\right)\left(8\sqrt{3}-\sqrt{2}\right)\end{aligned}$$

Finally, using the distributive property, we obtain

$$\begin{aligned}\left(2\sqrt{8}-\sqrt{3}\right)\left(2\sqrt{48}-\sqrt{2}\right) &= \left(4\sqrt{2}-\sqrt{3}\right)\left(8\sqrt{3}-\sqrt{2}\right)\\ &= 32\sqrt{6}-4\sqrt{4}-8\sqrt{9}+\sqrt{6}\\ &= 32\sqrt{6}-4\cdot 2-8\cdot 3+\sqrt{6}\\ &= 32\sqrt{6}-8-24+\sqrt{6}\\ &= -32+33\sqrt{6}\end{aligned}$$

In the previous example, we could also distribute first and then simplify your answer.

Practice 7. *Simplify:* $\left(2\sqrt{27}-\sqrt{2}\right)\left(2\sqrt{3}+\sqrt{8}\right)$ *(Answer on page 204)*

ANSWERS TO PRACTICE PROBLEMS

1. $20\sqrt{3}$
2. $8\sqrt{3}+24\sqrt{5}$
3. $30x+21x\sqrt[3]{4}$
4. $1+14\sqrt{2}$
5. $17-12\sqrt{2}$
6. 19
7. $32+10\sqrt{6}$

SECTION 6.4 EXERCISES

(Answers are found on page 253.)

Perform the indicated operations and simplify. Assume all variables represent positive real numbers.

1. $\sqrt{6}\sqrt{15}$
2. $\left(3\sqrt{2}\right)\left(5\sqrt{32}\right)$
3. $\sqrt[3]{9}\sqrt[3]{-9}$
4. $\left(4\sqrt{5}\right)\left(-3\sqrt{15}\right)$
5. $\left(-3\sqrt{7}\right)\left(6\sqrt{21}\right)$
6. $\sqrt{3a^2b^5}\sqrt{6ab^7}$
7. $\sqrt{5x^3y}\sqrt{10x^2y}$
8. $\sqrt{2}(\sqrt{2}-\sqrt{3})$
9. $3(\sqrt{12}-\sqrt{3})$
10. $\sqrt{8}(\sqrt{2}-\sqrt{5})$
11. $\sqrt{3a}(\sqrt{3a}-\sqrt{3b})$
12. $\sqrt{5x}(\sqrt{10x}-\sqrt{x})$
13. $(3\sqrt{x}-2y)(5\sqrt{x}-4y)$
14. $(5\sqrt{x}+2\sqrt{y})(3\sqrt{x}-\sqrt{y})$
15. $(\sqrt{3x}+y)(\sqrt{3x}-y)$
16. $(\sqrt{x}+5)^2$
17. $(2\sqrt{x}-3)^2$
18. $(3+\sqrt{2})(2-\sqrt{3})$

19. $(\sqrt{x}-4)(\sqrt{x}+4)$

20. $(\sqrt{x}+3)^2$

21. $(\sqrt{x}+4)(\sqrt{x}-1)$

22. $(4\sqrt{x}+\sqrt{3})^2$

23. $\left(3\sqrt{2}-4\sqrt{3}\right)\left(5\sqrt{2}+\sqrt{3}\right)$

24. $4\sqrt{3}\left(2\sqrt{6}-3\right)$

25. $\left(3\sqrt{5}+4\right)^2$

26. $3\sqrt{7}\left(2\sqrt{7}-3\sqrt{2}\right)$

27. $\left(2\sqrt{3}-5\sqrt{2}\right)\left(2\sqrt{3}+5\sqrt{2}\right)$

28. $\left(4\sqrt{12}-2\right)\left(5\sqrt{3}-3\right)$

29. $\left(3\sqrt{5}+4\right)\left(2\sqrt{5}-9\right)$

30. $(\sqrt{8}-\sqrt{3})^2$

31. $4\sqrt{3}\left(2\sqrt{27}-5\sqrt{3}\right)$

32. $(2\sqrt{3}-5\sqrt{6})(3\sqrt{3}+4\sqrt{6})$

33. $3\sqrt{2x}(4\sqrt{6x}-\sqrt{8x})$

34. $(3\sqrt{2}-5)(4\sqrt{2}+7)$

35. $(2x\sqrt{3x^3})(5x\sqrt{12x})$

36. $(3\sqrt{2}-4)^2$

37. $\left(2\sqrt{8}-5\right)^2$

38. $\left(3\sqrt{5}-2\right)\left(2\sqrt{5}-7\right)$

39. $\left(3\sqrt{5}-4\right)\left(3\sqrt{5}+4\right)$

40. $4\sqrt{2}\left(5\sqrt{6}-3\sqrt{10}\right)$

41. $\left(-8\sqrt[4]{8}\right)\left(2\sqrt[4]{4}\right)$

42. $3\sqrt{7}\left(2\sqrt{7}+6\sqrt{2}\right)$

43. $\left(2\sqrt{3}+5\sqrt{6}\right)\left(3\sqrt{3}+4\sqrt{6}\right)$

44. $\left(2\sqrt{3}-5\sqrt{2}\right)^2$

45. $\left(4\sqrt{12}-2\right)\left(5\sqrt{3}-5\right)$

46. $4\sqrt{5}\left(2\sqrt{5}-\sqrt{10}\right)$

47. $\left(2\sqrt{5}+3\right)\left(3\sqrt{5}-6\right)$

48. $\left(4\sqrt{x}+x\sqrt{7}\right)^2$

49. $\left(-4\sqrt[3]{9}\right)\left(3\sqrt[3]{9}\right)$

50. $\left(2\sqrt{12}-4\right)\left(5\sqrt{3}+5\right)$

6.5 Rationalizing the Denominator

In Section 6.2, we noted that a simplified radical expression can have no fraction under the radical and no radical in the denominator. Therefore, if either of these occur we need to rewrite the expression so that there is only a rational number in the denominator. **Rationalizing the denominator** is the process of removing radicals from the denominator so that the expression will be in simplified form.

Example 1. *Simplify:* $\dfrac{2}{\sqrt{3}}$

> **Solution 1.** *Since we do not have an equation, the only number that we can multiply by is one. We need to multiply both the numerator and denominator each by* $\sqrt{3}$*. Hence, we are multiplying the expression by* $\dfrac{\sqrt{3}}{\sqrt{3}}$ *or one in disguise.*
>
> $$\begin{aligned}\frac{2}{\sqrt{3}} &= \frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}\\ &= \frac{2\sqrt{3}}{\sqrt{9}}\\ &= \frac{2\sqrt{3}}{3}\end{aligned}$$
>
> *Since the denominator is a rational number, we have successfully simplified the radical expression.*

Be careful not to divide (or cancel) terms that are inside a radical with those that are outside a radical. For example, $\dfrac{\sqrt{6}}{15} \neq \dfrac{\sqrt{2}}{5}$.

Practice 1. *Simplify:* $\dfrac{7}{\sqrt{5}}$ *(Answer on page 212)*

Example 2. *Simplify:* $\dfrac{-12}{\sqrt{18}}$

Solution 2. *First, note that* $\sqrt{18} = \sqrt{9}\sqrt{2} = 3\sqrt{2}$. *Therefore, we really only need to multiply the numerator and denominator by* $\sqrt{2}$ *in order to rationalize the denominator. Therefore, multiplying by* $\frac{\sqrt{2}}{\sqrt{2}}$, *we get*

$$\begin{aligned}
\frac{-12}{\sqrt{18}} &= \frac{-6}{3\sqrt{2}} \\
&= \frac{-12}{3\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\
&= \frac{-12\sqrt{2}}{3\sqrt{4}} \\
&= \frac{-12\sqrt{2}}{3 \cdot 2} \\
&= \frac{-12\sqrt{2}}{6} \\
&= -2\sqrt{2}
\end{aligned}$$

The previous example can also be solved by multiplying both the numerator and denominator by $\sqrt{18}$ first and then simplifying later.

Practice 2. *Simplify:* $\dfrac{9}{\sqrt{27}}$ *(Answer on page 212)*

Example 3. *Simplify:* $\sqrt{\dfrac{5}{6}}$

Solution 3. *First, we will use the quotient rule for radicals and then multiply both numerator and denominator by* $\sqrt{6}$.

$$\begin{aligned}\sqrt{\frac{5}{6}} &= \frac{\sqrt{5}}{\sqrt{6}} \\ &= \frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} \\ &= \frac{\sqrt{30}}{\sqrt{36}} \\ &= \frac{\sqrt{30}}{6}\end{aligned}$$

Practice 3. *Simplify:* $\sqrt{\frac{3}{5}}$ *(Answer on page 212)*

Now, suppose it is not a square root in the denominator, but $\sqrt[3]{2}$. Notice that multiplying by $\sqrt[3]{2}$ will NOT eliminate the radical from the denominator since $\sqrt[3]{2}\sqrt[3]{2} = \sqrt[3]{4}$. In Section 6.1 we noted that if the index is odd, then $\sqrt[n]{a^n} = a$ and if the index is even then $\sqrt[n]{a^n} = |a|$. We will use this property to eliminate higher order radicals from the denominator. Therefore, to eliminate $\sqrt[3]{2}$ we need to multiply by $\sqrt[3]{2^2}$, or $\sqrt[3]{4}$ in order for it to work, since

$$\sqrt[3]{2} \cdot \sqrt[3]{2^2} = \sqrt[3]{2^3} = 2.$$

Example 4. *Simplify:* $\frac{5}{\sqrt[3]{2}}$

Solution 4. *In order to eliminate the radical in the denominator, we need to multiply by* $\sqrt[3]{4}$ *on both top and bottom.*

$$\begin{aligned}\frac{5}{\sqrt[3]{2}} &= \frac{5}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}} \\ &= \frac{5\sqrt[3]{4}}{\sqrt[3]{8}} \\ &= \frac{5\sqrt[3]{4}}{2}\end{aligned}$$

Practice 4. *Simplify:* $\sqrt[3]{\frac{4}{9}}$ *(Answer on page 212)*

Whenever a radical expression contains a sum or difference involving radicals in the denominator, we rationalize the denominator by multiplying both numerator and denominator by the conjugate of the denominator. The **conjugate** contains exactly the same numbers in exactly the same order with the operation sign changed. For example, the conjugate of $2+\sqrt{3}$ is $2-\sqrt{3}$ and the conjugate of $\sqrt{3}+7$ is $\sqrt{3}-7$. This clears the radicals from the denominator as a result of the difference of square formula $(a+b)(a-b)=a^2-b^2$.

Example 5. *Simplify:* $\frac{2}{\sqrt{5}-3}$

Solution 5. *We need to multiply both numerator and denominator by* $\sqrt{5}+3$ *which is the conjugate of the denominator in order to simplify the radical expression.*

$$\begin{aligned}
\frac{2}{\sqrt{5}-3} &= \frac{2}{(\sqrt{5}-3)} \cdot \frac{(\sqrt{5}+3)}{(\sqrt{5}+3)} \\
&= \frac{2(\sqrt{5}+3)}{\sqrt{25}+3\sqrt{5}-3\sqrt{5}-9} \\
&= \frac{2(\sqrt{5}+3)}{5-9} \\
&= \frac{2(\sqrt{5}+3)}{-4} \\
&= \frac{-(\sqrt{5}+3)}{2}
\end{aligned}$$

Practice 5. *Simplify:* $\frac{12}{3-\sqrt{6}}$ *(Answer on page 212)*

Example 6. *Simplify:* $\dfrac{4}{5+2\sqrt{3}}$

Solution 6. *Here we need to multiply both numerator and denominator by* $5-2\sqrt{3}$ *which is the conjugate of the denominator.*

$$\begin{aligned}
\frac{4}{5+2\sqrt{3}} &= \frac{4}{5+2\sqrt{3}} \cdot \frac{5-2\sqrt{3}}{5-2\sqrt{3}} \\
&= \frac{4(5-2\sqrt{3})}{25-10\sqrt{3}+10\sqrt{3}-4\sqrt{9}} \\
&= \frac{4(5-2\sqrt{3})}{25-12} \\
&= \frac{4(5-2\sqrt{3})}{13}
\end{aligned}$$

Practice 6. *Simplify:* $\dfrac{3}{2\sqrt{5}+3}$ *(Answer on page 212)*

Multiplying by the conjugate works even when both of the terms in the denominator contain a radical.

Example 7. *Simplify:* $\dfrac{15}{\sqrt{7}+\sqrt{2}}$

Solution 7. *We will rationalize by multiplying the numerator and denominator by* $\sqrt{7}-\sqrt{2}$.

$$
\begin{aligned}
\frac{15}{\sqrt{7}+\sqrt{2}} &= \frac{15}{(\sqrt{7}+\sqrt{2})} \cdot \frac{(\sqrt{7}-\sqrt{2})}{(\sqrt{7}-\sqrt{2})} \\
&= \frac{15(\sqrt{7}-\sqrt{2})}{\sqrt{49}+\sqrt{14}-\sqrt{14}-\sqrt{4}} \\
&= \frac{15(\sqrt{7}-\sqrt{2})}{7-2} \\
&= \frac{15(\sqrt{7}-\sqrt{2})}{5} \\
&= 3(\sqrt{7}-\sqrt{2})
\end{aligned}
$$

Practice 7. *Simplify:* $\dfrac{-6}{\sqrt{3}+2\sqrt{5}}$ *(Answer on page 212)*

Example 8. *Simplify:* $\dfrac{1-\sqrt{2}}{\sqrt{8}+\sqrt{6}}$

Solution 8. *We need to multiply both numerator and denominator by* $\sqrt{8}-\sqrt{6}$ *which is the conjugate of the denominator.*

$$
\begin{aligned}
\frac{1-\sqrt{2}}{\sqrt{8}+\sqrt{6}} &= \frac{(1-\sqrt{2})}{(\sqrt{8}+\sqrt{6})} \cdot \frac{(\sqrt{8}-\sqrt{6})}{(\sqrt{8}-\sqrt{6})} \\
&= \frac{(1-\sqrt{2})(\sqrt{8}-\sqrt{6})}{\sqrt{64}+\sqrt{48}-\sqrt{48}-\sqrt{36}} \\
&= \frac{(1-\sqrt{2})(\sqrt{8}-\sqrt{6})}{8-6} \\
&= \frac{(1-\sqrt{2})(\sqrt{4}\sqrt{2}-\sqrt{6})}{2} \\
&= \frac{(1-\sqrt{2})(2\sqrt{2}-\sqrt{6})}{2}
\end{aligned}
$$

Practice 8. *Simplify:* $\dfrac{4+\sqrt{3}}{\sqrt{12}-\sqrt{2}}$ *(Answer below)*

ANSWERS TO PRACTICE PROBLEMS

1. $\dfrac{7\sqrt{5}}{5}$
2. $\sqrt{3}$
3. $\dfrac{\sqrt{15}}{5}$
4. $\dfrac{\sqrt[3]{12}}{3}$
5. $4(3+\sqrt{6})$
6. $\dfrac{3(2\sqrt{5}-3)}{11}$
7. $\dfrac{6(\sqrt{3}-2\sqrt{5})}{17}$
8. $\dfrac{(4+\sqrt{3})(2\sqrt{3}+\sqrt{2})}{10}$

SECTION 6.5 EXERCISES

(Answers are found on page 254.)

Rationalize the denominator in each expression. Assume that all variables represent positive real numbers.

1. $\dfrac{8}{\sqrt{6}}$
2. $\dfrac{3}{\sqrt{3}}$
3. $\dfrac{1}{\sqrt{8}}$
4. $\dfrac{6}{\sqrt{12x}}$
5. $\sqrt{\dfrac{11}{4}}$
6. $\sqrt{\dfrac{8}{9}}$
7. $\sqrt{\dfrac{2}{3}}$
8. $\sqrt{\dfrac{11}{5}}$
9. $\dfrac{6}{\sqrt{18}}$
10. $\dfrac{4}{\sqrt{6}}$
11. $\dfrac{18}{\sqrt{12}}$
12. $\dfrac{3}{\sqrt{12}}$
13. $\sqrt{\dfrac{5}{9}}$
14. $\sqrt{\dfrac{8}{16}}$
15. $\sqrt{\dfrac{3}{8}}$
16. $\sqrt{\dfrac{2}{7}}$
17. $\dfrac{16}{\sqrt{28}}$
18. $\dfrac{9}{\sqrt{108}}$
19. $\dfrac{4\sqrt{5}}{\sqrt{50}}$
20. $\dfrac{6}{\sqrt{45}}$
21. $\dfrac{4\sqrt{2}}{\sqrt{24}}$

22. $\dfrac{7}{\sqrt{72}}$

23. $\dfrac{21}{\sqrt{7}}$

24. $\dfrac{5}{\sqrt[4]{x}}$

25. $\sqrt{\dfrac{5x}{8}}$

26. $\dfrac{15}{\sqrt{48}}$

27. $\dfrac{6\sqrt{2}}{5\sqrt{3}}$

28. $\dfrac{4\sqrt{3}}{\sqrt{6}}$

29. $\dfrac{20\sqrt{15}}{\sqrt{5}}$

30. $\dfrac{8\sqrt{2}}{\sqrt{24}}$

31. $\dfrac{7}{\sqrt[3]{2}}$

32. $\sqrt[3]{\dfrac{2}{3}}$

33. $\dfrac{3}{\sqrt[5]{x^2}}$

34. $\dfrac{4}{\sqrt[4]{8}}$

35. $\dfrac{8}{\sqrt[7]{x^3}}$

36. $\dfrac{5}{\sqrt[3]{9}}$

Rationalize the denominator in each expression. Assume that all variables represent positive real numbers.

37. $\dfrac{6}{\sqrt{7}+3}$

38. $\dfrac{2}{3-\sqrt{2}}$

39. $\dfrac{4}{1+\sqrt{3}}$

40. $\dfrac{10}{\sqrt{7}+\sqrt{5}}$

41. $\dfrac{6}{\sqrt{5}+1}$

42. $\dfrac{15}{\sqrt{3}-6}$

43. $\dfrac{11}{5-\sqrt{3}}$

44. $\dfrac{12}{\sqrt{7}-4}$

45. $\dfrac{6}{\sqrt{5}+\sqrt{3}}$

46. $\dfrac{5}{\sqrt{7}-\sqrt{3}}$

47. $\dfrac{15}{\sqrt{6}+\sqrt{3}}$

48. $\dfrac{12}{\sqrt{7}+4}$

49. $\dfrac{12}{2-\sqrt{6}}$

50. $\dfrac{-14}{\sqrt{3}+\sqrt{7}}$

51. $\dfrac{8}{1-\sqrt{5}}$

52. $\dfrac{2}{\sqrt{7}-\sqrt{3}}$

53. $\dfrac{2}{\sqrt{5}-3}$

54. $\dfrac{8}{2+\sqrt{6}}$

55. $\dfrac{21}{2\sqrt{2}-\sqrt{5}}$

56. $\dfrac{12}{3-\sqrt{6}}$

57. $\dfrac{18}{\sqrt{10}+\sqrt{6}}$

58. $\dfrac{6}{\sqrt{7}-4}$

59. $\dfrac{15}{\sqrt{11}-4}$

60. $\dfrac{2}{3\sqrt{2}+4}$

61. $\dfrac{-4}{\sqrt{3}-2\sqrt{5}}$

62. $\dfrac{14}{\sqrt{11}-3\sqrt{2}}$

63. $\dfrac{24}{5+2\sqrt{3}}$

64. $\dfrac{2+\sqrt{2}}{5-\sqrt{3}}$

65. $\dfrac{1-\sqrt{7}}{4+3\sqrt{2}}$

6.6 Rational Exponents

We have already defined the exponent rules for integer exponents. However, what about exponential expressions that contain rational exponents such as $2^{1/3}$, $27^{2/3}$, and $32^{-1/5}$. We need a definition that will still obey all the rules discussed in sections 4.1 and 4.3.

Consider $x = 2^{1/3}$. Then using the rules of exponents we see that

$$x^3 = \left(2^{1/3}\right)^3 = 2^{1/3} \cdot 2^{1/3} \cdot 2^{1/3} = 2^1 = 2.$$

Since $x^3 = 2$ we know that $x = \sqrt[3]{2}$. However, we also know that $x = 2^{1/3}$. Hence,

$$2^{1/3} = \sqrt[3]{2}.$$

We leave it to the reader to examine other examples. This leads to the following definitions.

Rational exponents: If m and n are positive integers with m/n in lowest terms, then

- $x^{1/n} = \sqrt[n]{x}$
- $x^{m/n} = \sqrt[n]{x^m}$ OR $x^{m/n} = (\sqrt[n]{x})^m$

If n is even then we require $x \geq 0$.

In other words, in a rational exponent, the numerator indicates the power and the denominator indicates the root.

Example 1. *Write using radicals (do not rationalize):*

$$5x^{1/2} + 3x^{2/3} + 7x^{-2/5} + 8x^{-1/3}$$

Solution 1.

$$\begin{aligned}5x^{1/2}+3x^{2/3}+7x^{-2/5}+8x^{-1/3} &= 5\sqrt{x}+3\sqrt[3]{x^2}+7\left(\frac{1}{x^{2/5}}\right)+8\left(\frac{1}{x^{1/3}}\right)\\ &= 5\sqrt{x}+3\sqrt[3]{x^2}+\frac{7}{\sqrt[5]{x^2}}+\frac{8}{\sqrt[3]{x}}\end{aligned}$$

Practice 1. *Write using radicals (do not rationalize).* *(Answer on page 223)*

$$\frac{2}{5}x^{1/3}+\frac{3}{7}x^{-1/2}+\frac{5}{8}x^{-2/3}$$

Example 2. *Write using rational exponents:* $3\sqrt{x}-2\sqrt[3]{x}+\dfrac{8}{\sqrt[4]{x^3}}+7\sqrt[5]{x^2}$

Solution 2.

$$\begin{aligned}3\sqrt{x}-2\sqrt[3]{x}+\frac{8}{\sqrt[4]{x^3}}+7\sqrt[5]{x^2} &= 3x^{1/2}-2x^{1/3}+\frac{8}{x^{3/4}}+7x^{2/5}\\ &= 3x^{1/2}-2x^{1/3}+8x^{-3/4}+7x^{2/5}\end{aligned}$$

Practice 2. *Write using rational exponents.* *(Answer on page 223)*

$$\frac{4}{\sqrt[5]{x^2}}-3\sqrt[4]{x^5}+7\sqrt{x}-\frac{9}{\sqrt[3]{x^2}}$$

Example 3. *Simplify:* $16^{3/4}$

Solution 3. *Since $\frac{3}{4}$ is the exponent, we know that we need to use the fourth root and the third power. So,*

$$\begin{aligned} 16^{3/4} &= \left(\sqrt[4]{16}\right)^3 \\ &= (2)^3 \\ &= 8 \end{aligned}$$

Practice 3. *Simplify:* $32^{4/5}$ *(Answer on page 223)*

As in Section 4.3, we need to be very careful with negatives. The placement of the negative sign does matter.

Example 4. *Simplify: (a)* $-16^{3/4}$ *and (b)* $(-16)^{3/4}$.

Solution 4. *(a) Note that for $-16^{3/4}$ the negative is not raised to the three fourths power. So, we view $-16^{3/4}$ as $-1 \cdot 16^{3/4}$. From the previous example we see that $-16^{3/4} = -8$.*

(b) Here, the negative is raised to the three fourths power. So, switching to radical form, we have $(-16)^{3/4} = \left(\sqrt[4]{-16}\right)^3$. However, we know that we cannot take the even root of a negative number in the Real Number System. Hence, $(-16)^{3/4}$ is not a real number.

Practice 4. *Simplify: (a)* $-8^{2/3}$ *and (b)* $(-8)^{2/3}$ *(Answer on page 223)*

Example 5. *Simplify:* $\left(\dfrac{16}{81}\right)^{3/4}$

Solution 5.

$$\begin{aligned}\left(\frac{16}{81}\right)^{3/4} &= \left(\sqrt[4]{\frac{16}{81}}\right)^3\\ &= \left(\frac{2}{3}\right)^3\\ &= \frac{8}{27}\end{aligned}$$

Practice 5. *Simplify:* $\left(\frac{25}{36}\right)^{3/2}$ *(Answer on page 223)*

Recall from section 4.3 that $x^{-n} = \frac{1}{x^n}$. Since the exponent rules must be valid for rational exponents, we have

$$x^{-m/n} = \frac{1}{x^{m/n}}.$$

Example 6. *Simplify:* $32^{-2/5}$

Solution 6.

$$\begin{aligned}32^{-2/5} &= \frac{1}{32^{2/5}}\\ &= \frac{1}{\left(\sqrt[5]{32}\right)^2}\\ &= \frac{1}{(2)^2}\\ &= \frac{1}{4}\end{aligned}$$

Practice 6. *Simplify:* $64^{-2/3}$. *(Answer on page 223)*

We now turn our attention to operations involving rational exponential expressions. Remember that according to the order of operations, exponents are evaluated before multiplication.

Example 7. *Evaluate:* $3(27)^{1/3} - 3(8)^{1/3}$

Solution 7.

$$\begin{aligned} 3(27)^{1/3} - 3(8)^{1/3} &= 3\sqrt[3]{27} - 3\sqrt[3]{8} \\ &= 3(3) - 3(2) \\ &= 9 - 6 \\ &= 3 \end{aligned}$$

Practice 7. *Simplify:* $5(36)^{1/2} - 5(4)^{1/2}$ *(Answer on page 223)*

Example 8. *Evaluate:* $\frac{4}{3}(9)^{3/2} - \frac{4}{3}(1)^{3/2}$

Solution 8.

$$\begin{aligned} \frac{4}{3}(9)^{3/2} - \frac{4}{3}(1)^{3/2} &= \frac{4}{3}(\sqrt{9})^3 - \frac{4}{3}(\sqrt{1})^3 \\ &= \frac{4}{3}(3)^3 - \frac{4}{3}(1)^3 \\ &= \frac{4}{3}(27) - \frac{4}{3}(1) \\ &= \frac{108}{3} - \frac{4}{3} \\ &= \frac{104}{3} \end{aligned}$$

Practice 8. *Evaluate:* $\frac{3}{5}(8)^{2/3} - \frac{3}{5}(1)^{2/3}$ *(Answer on page 223)*

Example 9. *Evaluate:* $3(-1)^{-2/3} - 3(-8)^{-2/3}$

Solution 9.

$$\begin{aligned}
3(-1)^{-2/3} - 3(-8)^{-2/3} &= 3\left(\frac{1}{(\sqrt[3]{-1})^2}\right) - 3\left(\frac{1}{(\sqrt[3]{-8})^2}\right) \\
&= 3\left(\frac{1}{(-1)^2}\right) - 3\left(\frac{1}{(-2)^2}\right) \\
&= 3\left(\frac{1}{1}\right) - 3\left(\frac{1}{4}\right) \\
&= 3 - \frac{3}{4} \\
&= \frac{12}{4} - \frac{3}{4} \\
&= \frac{9}{4}
\end{aligned}$$

Practice 9. *Evaluate:* $5(-1)^{-2/5} - 5(-32)^{-2/5}$ *(Answer on page 223)*

In Sections 4.1 and 4.3 we discussed the exponent rules as they applied to integer exponent values. These same rules apply to rational exponents. The exponent rules are stated here, again, for convenience.

Exponent Rules: Let m and n be real numbers.

- $a^m \cdot a^n = a^{m+n}$
- $\dfrac{a^m}{a^n} = a^{m-n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $a^0 = 1$
- $a^{-n} = \dfrac{1}{a^n}$
- $\dfrac{a^{-m}}{b^{-n}} = \dfrac{b^n}{a^m}$
- $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

Example 10. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write answer using positive exponents only.*

$$\frac{(x^{2/3})^2}{(x^2)^{7/3}}$$

Solution 10.

$$\begin{aligned}\frac{(x^{2/3})^2}{(x^2)^{7/3}} &= \frac{x^{4/3}}{x^{14/3}}\\ &= x^{4/3-14/3}\\ &= x^{-10/3}\\ &= \frac{1}{x^{10/3}}\end{aligned}$$

Practice 10. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only.* *(Answer on page 223)*

$$\frac{\left(x^4\right)^{2/3}}{\left(x^{3/4}\right)^2}$$

Example 11. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only.*

$$\left(2x^4y^{-4/5}\right)^3\left(8y^2\right)^{2/3}$$

Solution 11.

$$\begin{aligned}\left(2x^4y^{-4/5}\right)^3\left(8y^2\right)^{2/3} &= \left(8x^{12}y^{-12/5}\right)\left(8^{2/3}y^{4/3}\right)\\ &= \left(8x^{12}y^{-12/5}\right)\left(4y^{4/3}\right)\\ &= 32x^{12}y^{-12/5+4/3}\\ &= 32x^{12}y^{-36/15+20/15}\\ &= 32x^{12}y^{-16/15}\\ &= \frac{32x^{12}}{y^{16/15}}\end{aligned}$$

Practice 11. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only.* *(Answer on page 223)*

$$\left(-3x^{-2/3}y^2\right)^3(4x)^{-1/2}$$

Example 12. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only.*

$$\frac{\left(x^{15}y^{-5}\right)^{1/5}}{\left(x^{-2}y^3\right)^{1/3}}$$

Solution 12.

$$\begin{aligned}
\frac{\left(x^{15}y^{-5}\right)^{1/5}}{\left(x^{-2}y^{3}\right)^{1/3}} &= \frac{x^3y^{-1}}{x^{-2/3}y} \\
&= \frac{x^3x^{2/3}}{yy} \\
&= \frac{x^{3+2/3}}{y^2} \\
&= \frac{x^{11/3}}{y^2}
\end{aligned}$$

Practice 12. *Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only. (Answer below)*

$$\frac{\left(x^{-3}y^{4}\right)^{1/2}}{\left(x^{8}y^{-12}\right)^{1/4}}$$

ANSWERS TO PRACTICE PROBLEMS

1. $\frac{2}{5}\sqrt[3]{x} + \frac{3}{7\sqrt{x}} + \frac{5}{8\sqrt[3]{x^2}}$
2. $4x^{-2/5} - 3x^{5/4} + 7x^{1/2} - 9x^{-2/3}$
3. 16
4. (a) -4
   (b) 4
5. $\frac{125}{216}$
6. $\frac{1}{16}$
7. 20
8. $\frac{9}{5}$
9. $\frac{15}{4}$
10. $x^{7/6}$
11. $-\frac{27y^6}{2x^{5/2}}$
12. $\frac{y^5}{x^{7/2}}$

SECTION 6.6 EXERCISES

(Answers are found on page 256.)

Write using radicals (do not rationalize).

1. $4x^{2/3}+3x^{1/3}-2x^{1/2}$
2. $5x^{1/2}+2x^{2/3}+3x^{3/4}$
3. $4x^{-2/3}+2x^{-1/3}+6x^{1/2}$
4. $\frac{3}{4}x^{-2/3}+\frac{3}{8}x^{-1/2}+\frac{7}{5}x^{1/3}$
5. $7x^{4/7}-3x^{-3/5}+\frac{11}{12}x^{1/4}$
6. $\frac{5}{6}x^{2/3}-3x^{1/5}+6x^{3/4}$
7. $2x^{3/2}-6x^{2/3}-4x^{-1/2}$
8. $9x^{-1/4}+6x^{3/5}-2x^{-1/3}$

Write using rational exponents.

9. $\sqrt[5]{x^2}+\sqrt[3]{x}-2\sqrt{x}$
10. $5\sqrt[7]{x^4}+8\sqrt{x}+2\sqrt[3]{x^2}-5\sqrt[4]{x}$
11. $7\sqrt[5]{x^2}+3\sqrt[3]{x^2}-4\sqrt[4]{x^3}$
12. $9\sqrt[3]{x}-\frac{5}{\sqrt[4]{x^3}}-2\sqrt{x^5}$
13. $\frac{6}{5\sqrt{x}}-\frac{2}{\sqrt[3]{x}}+8\sqrt[4]{x}$
14. $\frac{1}{2\sqrt[3]{x}}+3\sqrt{x^3}-\frac{4}{\sqrt[3]{x}}$
15. $4\sqrt[5]{x^4}-\frac{1}{3\sqrt{x}}+5\sqrt[4]{x^3}$
16. $\frac{\sqrt[3]{x}}{4}+3\sqrt[5]{x^2}-6\sqrt[4]{x^5}$

Evaluate.

17. $36^{1/2}$
18. $(-64)^{2/3}$
19. $9^{3/2}$
20. $25^{-1/2}$
21. $4^{-3/2}$
22. $8^{-1/3}$
23. $(-27)^{-2/3}$
24. $4^{5/2}$
25. $256^{3/4}$
26. $9^{-3/2}$
27. $(-1)^{2/5}$
28. $4(-8)^{-1/3}$
29. $5(4)^{3/2}$
30. $32^{-3/5}$
31. $-16^{1/2}$

32. $25^{-3/2}$

33. $81^{-3/4}$

34. $(-125)^{-2/3}$

35. $-49^{1/2}$

36. $243^{3/5}$

37. $81^{1/4}$

38. $-3\left(\dfrac{8}{27}\right)^{2/3}$

39. $\left(\dfrac{8}{27}\right)^{4/3}$

40. $\left(\dfrac{81}{16}\right)^{-3/4}$

41. $\left(\dfrac{27}{8}\right)^{2/3}$

42. $\left(\dfrac{169}{25}\right)^{-1/2}$

Evaluate.

43. $3(8)^{2/3} - 3(-1)^{2/3}$

44. $2(9)^{3/2} - 2(4)^{3/2}$

45. $2(16)^{1/2} - 3(9)^{1/2}$

46. $\dfrac{3}{2}(36)^{1/2} - \dfrac{3}{2}(16)^{1/2}$

47. $\dfrac{2}{5}(27)^{1/3} - \dfrac{2}{5}(8)^{1/3}$

48. $6(9)^{-1/2} - 2(4)^{-1/2}$

49. $(8)^{-1/3} - (1)^{-1/3}$

50. $\dfrac{1}{2}(16)^{-1/2} - \dfrac{1}{2}(4)^{-1/2}$

51. $\dfrac{2}{3}(4)^{-1/2} - \dfrac{2}{3}(1)^{-1/2}$

52. $5(-1)^{1/3} - 5(-8)^{1/3}$

53. $\dfrac{2}{5}(4)^{5/2} - \dfrac{2}{5}(1)^{5/2}$

54. $-\dfrac{3}{2}(8)^{-2/3} + \dfrac{3}{2}(1)^{-2/3}$

Use the rules of exponents to simplify the following. Assume that all variable represent positive real numbers. Write using positive exponents only.

55. $\left(2x^{1/3}y^{1/2}\right)^6$

56. $\dfrac{x^{3/4}}{x^{1/2}x}$

57. $(8x^2)^{1/3}(16x)^{-1/4}$

58. $(25x^3)^{-1/2}(4x^3)^{1/2}$

59. $\dfrac{x^{2/3}x^{-3/4}x^{7/6}}{x^{3/2}}$

60. $\left(x^{3/4}x^{-3/8}\right)^{-2}x^{5/2}$

61. $\dfrac{x^{3/5}x^{-2}}{x^{3/10}}$

62. $\left(-64a^{3/4}b^{6}\right)^{2/3}$

63. $\left(27x^{9}y^{12}\right)^{2/3}$

64. $\left(16x^{4}y^{6}\right)^{3/2}$

65. $\dfrac{x^{1/3}}{x^{2/3}\cdot x^{-1}}$

66. $\left(\dfrac{4x^{1/3}y}{9xy^{2}}\right)^{1/2}$

67. $\dfrac{x^{3/4}y^{-1/2}}{x^{-1/3}y^{1/6}}$

68. $\dfrac{\left(4x^{3}y\right)^{1/2}}{8x^{1/4}y^{-1/2}}$

Chapter 7

Answers to Exercises

Chapter 1

Section 1.1 (Exercises on page 10.)

1. function;
   Domain $= \{3, 4, 5, 7, 10\}$;
   Range $= \{-8, -2, 6\}$
2. not a function
3. function;
   Domain $= \{-2, -1, 1, 2\}$;
   Range $= \{7\}$
4. function;
   Domain $= \{-5, -3, -1, 1, 4\}$;
   Range $= \{-125, -27, 1, 64\}$
5. function;
   Domain $= \{-1, 1, 2, 3, 7\}$;
   Range $= \{-10, 2, 3, 4, 8\}$
6. function;
   Domain $= \{1, 2, 3, 4, 5\}$;
   Range $= \{3\}$
7. function;
   Domain $= \{-2, -1, 0, 1, 2\}$;
   Range $= \{0, 1, 4\}$
8. not a function
9. function;
   Domain $= \{-3, 5, 6, 8\}$;
   Range $= \{-5, 3, 12\}$
10. function;
    Domain $= \{-3, -2, 2, 3\}$;
    Range $= \{4, 9\}$
11. not a function
12. function;
    Domain $= \{2, 4, 6\}$;
    Range $= \{12\}$
13. function
14. not a function
15. function
16. not a function
17. function
18. not a function

19. not a function

20. function

21. not a function

22. function

23. function

24. not a function

25. (a) -7
    (b) -22
    (c) 2
    (d) -31

26. (a) -4
    (b) -6
    (c) -4
    (d) $-\dfrac{21}{4}$

27. (a) $-\dfrac{1}{8}$
    (b) $-\dfrac{3}{8}$
    (c) 0
    (d) -12

28. (a) -2
    (b) 6
    (c) $\dfrac{14}{5}$
    (d) $\dfrac{2}{3}$

29. (a) 22
    (b) 9
    (c) -5
    (d) -6

30. (a) -13
    (b) -13
    (c) -13
    (d) -13

31. (a) 22
    (b) 16
    (c) 10
    (d) 31

32. (a) 2
    (b) -2
    (c) $\dfrac{5}{2}$
    (d) $-\dfrac{17}{4}$

33. (a) -7
    (b) -3
    (c) -19
    (d) -39

34. (a) 5
    (b) 11
    (c) 39
    (d) 3

35. (a) 4
    (b) 10
    (c) 13
    (d) 16

Chapter 2

Section 2.1 (Exercises on page 22.)

1. $x = -\dfrac{1}{32}$
2. $x = \dfrac{7}{4}$
3. no solution
4. $x = -3$
5. $x = \dfrac{24}{17}$
6. $x = -\dfrac{1}{6}$
7. all real numbers
8. $x = \dfrac{3}{14}$
9. $x = \dfrac{11}{17}$
10. $x = \dfrac{33}{16}$
11. $x = -\dfrac{19}{11}$
12. $x = -\dfrac{34}{25}$
13. no solution
14. $x = 7$
15. no solution
16. $x = 5$
17. $x = 0$
18. all real numbers
19. $x = 0$
20. $x = \dfrac{3}{25}$
21. $x = -\dfrac{7}{5}$
22. $x = -6$
23. $x = 4$
24. all real numbers

Section 2.2 (Exercises on page 31.)

1. 6
2. 8
3. 5
4. 47
5. 10
6. 2
7. 5
8. 12
9. 9
10. all real numbers
11. 60
12. -16
13. 7 and 8
14. -2

15. $12, 24, 45$

16. 180

17. 89, 91

18. 12, 14

19. 5,015, 5,017

20. 3,342, 3,344, 3,346

21. 3,343, 3,344, 3,345

22. $22, 24$

23. $11, 12, 13$

24. 5 feet and 12 feet

25. 152 adults

26. Savanna pitched 3 games, Kayla pitched 15 games, Cheyenne pitched 10 games

27. Roger sold 15, Will sold 24, Jacob sold 27

28. cashier for 3.5 hours, stocked shelves for 2.75 hours, trained new employees for 1.75 hours

29. $w = 5$, $l = 13$

30. width is 14 ft, length is 49 ft

31. $x = 25$ ft, $y = 50$ ft

32. 25 ft by 25 ft

33. (a) 3 gallons for one coat.
    (b) 5 gallons for two coats.

34. first piece: 8.5 in, second piece: 11 in; third piece: 8.5 in; fourth piece: 11 in

35. 8 ft, 28 ft

36. 70 yd, 130 yd

37. 25 inches

38. 15 minutes

39. first song: 30 min; second song: 60 min; third song: 90 min

40. 700 adults

41. 150 children, 1858 adults

42. 18 yards

43. 20 feet

44. each angle is 139°

45. 48° and 132°

46. 54° and 126°

47. 48° and 42°

48. each angle is 73°

49. 35° and 55°

50. 85°

51. 65°

Section 2.3 (Exercises on page 43.)

1. They are the same.
2. They are the same.
3. \$5.00
4. \$45.00
5. \$8.50
6. \$140
7. \$800
8. 36 liters
9. 19.2 liters
10. 18 cds
11. 9 cups
12. 96%
13. 40%
14. 24 students
15. 180 women
16. 7 students
17. 20%
18. 25%
19. 120 oz
20. 20%
21. \$540
22. \$17
23. 25%
24. 62.5%
25. 75%
26. 87.5%
27. \$6000
28. \$12000
29. 2.5%
30. Job 2
31. Discount of $\frac{1}{4}$ is the same as paying $\frac{3}{4}$ of the original amount.
32. Candidate B won by one vote
33. \$1500
34. $100°,\ 50°,\ 30°$
35. \$6467.89
36. 125 square feet
37. \$230.50
38. \$21.18
39. \$23.08
40. \$233.16
41. \$180
42. 15%
43. $\$13,200$
44. $\$34,170.74$
45. $\$9,910$
46. 130 games

47. 15.56%

48. 32%

49. $149.45

50. (a) $637.50
    (b) $1, 912.50
    (c) $318.75

51. (a) interest is $480; account balance is $12, 480
    (b) $12, 979.20
    (c) $19.20

52. 6.5%

Section 2.4 (Exercises on page 52.)

1. (a) 16 ounces
   (b) 24 ounces
   (c) 11 ounces

2. 450 flashes

3. 50 feet

4. 3000 sheets

5. 4.5 miles

6. 10 times

7. 75 gallons

8. 3 miles

9. 40 pens

10. $4\frac{3}{8}$ cups

11. $62.25

12. $4\frac{1}{6}$ cups of flour

13. Missed 16 goals

14. (a) $2\frac{1}{2}$
    (b) $\frac{5}{8}$
    (c) $1\frac{9}{16}$

15. 7 inches

16. 2800 markers

17. $2\frac{1}{4}$ cups

18. 110 miles

19. 85.5 inches

20. 75 teachers

21. 960 female students

22. 500 fish

23. 7 lbs cement, 21 lbs gravel

24. 21 lbs

25. 20.32 cm

26. 35 gallons

27. 14 lbs

28. $1, 174.90

Chapter 3

Section 3.1 (Exercises on page 64.)

1. $(4,-2)$
2. $(2,-4)$
3. no solution
4. $(3,2)$
5. $(0,-3)$
6. $(2,1)$
7. $(-4,-1)$
8. $(1,0)$
9. $(0,9)$
10. infinite number of solutions
11. $(2,2)$
12. $(10,1)$
13. $(1,6)$
14. $(6,2)$
15. $(2,-3)$
16. $(1,-2)$
17. no solution
18. $(7,3)$
19. $(2,2)$
20. $(3,1)$
21. $(2,-2)$
22. no solution
23. $(2,4)$
24. $(0,3)$
25. $(6,-1)$
26. infinite number of solutions
27. $(0,-2)$
28. $(-3,4)$

Section 3.2 (Exercises on page 74.)

1. $(15,-6)$
2. $(2,-1)$
3. $\left(-\frac{8}{5},-\frac{11}{5}\right)$
4. no solution
5. infinite number of solutions
6. no solution
7. infinite number of solutions
8. $\left(\frac{38}{11},-\frac{7}{11}\right)$
9. $(3,-2)$
10. $(2,-1)$
11. no solution
12. $\left(\frac{1}{13},\frac{22}{13}\right)$
13. $(5,-3)$
14. $(1,4)$

15. $(7,0)$
16. no solution
17. $(3,-2)$
18. $(-4,1)$
19. $(-2,-6)$
20. $(2,-1)$
21. infinite number of solutions
22. $\left(-\frac{24}{7},\frac{3}{7}\right)$
23. $\left(-\frac{169}{29},-\frac{23}{29}\right)$
24. $\left(\frac{12}{5},-\frac{6}{5}\right)$
25. $\left(\frac{2}{7},-\frac{24}{7}\right)$
26. $\left(\frac{9}{5},\frac{23}{5}\right)$
27. $\left(\frac{17}{10},\frac{11}{10}\right)$
28. $\left(-\frac{8}{13},-\frac{41}{13}\right)$

Section 3.3 (Exercises on page 83.)

1. $(-7,-5)$
2. $(2,-1)$
3. $\left(\frac{12}{11},-\frac{4}{11}\right)$
4. $\left(\frac{19}{12},\frac{37}{12}\right)$
5. $\left(-\frac{1}{2},\frac{3}{2}\right)$
6. no solution
7. $\left(-4,-\frac{5}{2}\right)$
8. $\left(\frac{1}{3},-\frac{3}{2}\right)$
9. $\left(\frac{13}{12},-\frac{17}{12}\right)$
10. $(-5,-7)$
11. $\left(\frac{46}{7},-\frac{16}{7}\right)$
12. $(2,-1)$
13. $(102,-15)$
14. $\left(\frac{1}{4},1\right)$
15. infinite number of solutions
16. $(3,12)$
17. $(-4,3)$
18. $\left(\frac{6}{5},-\frac{3}{5}\right)$
19. $(3,-4)$
20. $(-2,-1)$
21. $\left(4,-\frac{1}{5}\right)$
22. $(-3,2)$
23. $\left(\frac{29}{11},\frac{1}{11}\right)$
24. $\left(\frac{1}{2},\frac{3}{2}\right)$
25. $\left(\frac{7}{22},\frac{5}{11}\right)$
26. $(20,9)$
27. $\left(\frac{14}{25},-\frac{23}{25}\right)$
28. $(2,-4)$
29. $(-1,-5)$
30. $(-2,3)$

31. No solution
32. Infinite number of solutions
33. $(0, -2)$
34. $(-3, 4)$
35. $(5, 1)$
36. $\left(\frac{1}{7}, -\frac{22}{7}\right)$
37. $\left(3, \frac{39}{2}\right)$
38. $\left(\frac{1}{10}, -\frac{103}{10}\right)$
39. $(-7, 1)$
40. $\left(-\frac{2}{3}, \frac{1}{2}\right)$
41. $\left(-2, \frac{1}{2}\right)$
42. $\left(-\frac{3}{14}, \frac{9}{14}\right)$
43. $\left(\frac{68}{31}, \frac{15}{31}\right)$
44. $\left(\frac{5}{31}, \frac{3}{31}\right)$
45. $\left(\frac{11}{39}, -\frac{40}{39}\right)$
46. $\left(-\frac{1}{10}, \frac{29}{15}\right)$
47. $\left(-\frac{1}{17}, -\frac{4}{17}\right)$

Section 3.4 (Exercises on page 93.)

1. 1.3 lbs of Almonds and 2.7 lbs of Pecans
2. 80
3. 44 letters and 12 postcards
4. 8 Nickels and 14 quarters
5. 4 and 1
6. Tuition $13,500; Room and Board $7,300
7. $2,800 at 8%; $3,200 at $7\frac{1}{2}$%
8. $6,180 at 6%; $2,820 at 10%
9. $750 at 5%; $1,500 at 6%
10. $800 at $4\frac{1}{2}$%; $700 at 3% loss
11. $4,444.44 at 5%; $5,555.56 at 4%
12. $909.09 at 3%; $1,090.91 at 5%
13. 410 mph
14. 6 miles
15. walking 4 mph; biking 10 mph
16. plane 620 mph; wind 20 mph
17. rowing 3.75 mph; water current 1.25 mph
18. $12,000 at 12%; $8,000 at 7%
19. 246 dimes; 54 nickels
20. 12.5 hours on day one
21. 1 hour
22. 2.5 hours
23. Jen's rate is 60 mph; Sarah's rate is 40 mph
24. 24 gal of 90%; 96 gal of 75%
25. 60 L of 40%; 30 L of 70%
26. 2 L of 40%; 6 L of 60%

27. 8 g of 25%; 12 g of 50%
28. $13\frac{1}{3}$ gal of 40%; $6\frac{2}{3}$ gal of 70%
29. $4\frac{1}{2}$ oz of 4%; $7\frac{1}{2}$ oz of 12%

Chapter 4

Section 4.1 (Exercises on page 104.)

1. 3^7
2. 10^3
3. 6^3
4. $\left(\frac{1}{2}\right)^6$
5. -1
6. 1
7. -32
8. 216
9. 81
10. -64
11. $\frac{1}{8}$
12. 1
13. 2.25
14. -49
15. 0
16. 16
17. -256
18. 1
19. -1
20. -1
21. -25
22. 0.000001
23. 81
24. $10^7 = 10,000,000$
25. 3^{11}
26. 8
27. x^{14}
28. 2^{19}
29. 2^{22}
30. 5^{13}
31. 6^7
32. 2^{53}
33. 15^{14}
34. 3^{38}
35. $8x^4$
36. $49x^{10}$
37. $64x^6$
38. $27x^{12}y^{15}$
39. $36x^6y^{18}$
40. $16x^8y^{20}z^{36}$

41. $-27x^3y^9z^6$

42. $8x^9$

43. $-24x^{15}$

44. $6b^9$

45. x^7y^4

46. $\dfrac{x^9y^8}{6}$

47. x^6y^9

48. $8x^{12}y^{23}$

49. $12x^{11}y^{10}$

50. $\dfrac{9x^8}{4y^{10}}$

51. $\dfrac{125y^{12}z^3}{8x^9}$

52. $\dfrac{3x^{10}}{9y^7}$

53. $\dfrac{x^{10}}{2y^5}$

54. $32x^{20}y^5$

55. $63x^9y^{11}$

56. $\dfrac{36x^4}{y^5}$

57. $-72x^{14}y^{17}$

58. $\dfrac{x^{14}y^{21}}{2}$

59. $\dfrac{x^9y^{14}}{4}$

60. $x^8y^{13}z^3$

61. $4^{28} = (2^2)^{28} = 2^{56}$ and $8^{18} = (2^3)^{18} = 2^{54}$. Since $56 > 54$, $4^{28} > 8^{18}$.

62. $3^9 + 3^9 + 3^9 = 3 \cdot 3^9 = 3^{10}$ and $9^6 = (3^2)^6 = 3^{12}$. Since $12 > 10$, $9^6 > 3^9 + 3^9 + 3^9$.

63. $27^9 = (3^3)^9 = 3^{27}$ and $9^{14} = (3^2)^{14} = 3^{28}$. Since $28 > 27$, $9^{14} > 27^9$.

64. $6^{18} = (2 \cdot 3)^{18} = 2^{18} \cdot 3^{18}$ and $3^{36} = 3^{18} \cdot 3^{18}$. Since $2^{18} < 3^{18}$, $6^{18} < 3^{36}$.

65. Not equal. $\left(3^4\right)^2 = 3^{4\cdot 2} = 3^8$; whereas, $3^{4^2} = 3^{16}$.

66. $(3+4)^3 \neq 3^3 + 4^3$ since $(3+4)^3 = 7^3 = 343$ and $3^3 + 4^3 = 27 + 64 = 91$. In order for $(a+b)^3 = a^3+b^3$ we need either a or b to be zero. Furthermore, the only value of n for which $(a+b)^n = a^n+b^n$ is when $n = 1$.

Section 4.2 (Exercises on page 114.)

1. Yes. One example: $\left(3x^3 + 7x^2 - 5\right)+\left(-3x^3 + 8x - 2\right) = 7x^2 + 8x - 7$.

2. Yes.

3. No. A monomial times a binomial will always give a binomial.

4. 5th degree; $m + n$ degree

5. 8
6. 7
7. 6
8. 3
9. 5
10. 3
11. -25
12. $-\frac{23}{2}$
13. 6
14. 35
15. -15
16. -6
17. 0
18. 27
19. 12
20. -97
21. -33
22. 1200
23. 108
24. $-x+16$
25. $5x-2$
26. $4x^2$
27. $-a^2-a-1$
28. $11x^2-3x-11$
29. $-2x^2-8x+6$
30. $x^3-10x^2+22x-5$
31. $7x^3-11x^2+6x-9$
32. $6a^3b^2+a^2b^2-2ab^2$
33. $19mn^2-16m^2n-5$
34. $12x^2-8x$
35. $30x^3+15x^2$
36. $-8x^3+6x^2-10x$
37. $15x^4-6x^3-21x^2$
38. $-15x^3y^3+21x^3y^2-9x^4y$
39. $-10x^4y^6+6x^3y^7-4x^5y^8+14x^2y^5$
40. $4x^2+4x-15$
41. $15x^2-22x+8$
42. $28x^2+13x-6$
43. $10x^2-x-3$
44. $24x^2+8x-2$
45. $4x^2-9$
46. $16x^2-1$
47. $25x^2-4$
48. $9x^2-16$
49. $\frac{9}{2}x^2-\frac{78}{5}x+2$
50. $6x^2-\frac{7}{12}x-\frac{1}{6}$
51. $49x^2-28x+4$
52. $16x^2+24x+9$
53. $4x^2-12x+9$

54. $x^3 - 6x^2 + 12x - 8$

55. $8x^3 + 12x^2 + 6x + 1$

56. $12x^3 - 17x^2 - 13x - 2$

57. $2x^3 - 11x^2 + 10x + 8$

58. $21x^3 - 19x^2 - 12x$

59. $16x^4 + 28x^3 - 30x^2$

60. $12x^3 + 2x^2 - 24x$

61. $10x^3 - 19x^2 + 26x - 8$

62. $6x^4 - 22x^3 + 41x^2 - 41x + 10$

63. $6x^4 - 28x^3 + 21x^2 + 21x - 10$

64. $2x^4 - x^3 - 24x^2 + 16x + 7$

65. $18x^4 - 15x^3 - 13x^2 - 2x - 12$

66. Let $x = 1$. Then $x+2(x-3) = 1 + 2(1-3) = 1 + 2(-2) = 1 + (-4) = -3$. However, $(x+2)(x-3) = (1+2)(1-3) = (3)(-2) = -6$.

Section 4.3 (Exercises on page 123.)

1. $\frac{1}{81}$
2. $\frac{1}{64}$
3. $-\frac{1}{625}$
4. 8
5. $\frac{1}{16}$
6. $-\frac{1}{4}$
7. $\frac{1}{4}$
8. 25
9. 49
10. $\frac{81}{256}$
11. 216
12. $\frac{81}{4}$
13. $\frac{5}{16}$
14. $\frac{-5}{36}$
15. $\frac{7}{12}$
16. $-\frac{7}{16}$
17. $\frac{9}{4}$
18. $\frac{106}{9}$
19. $\frac{7}{12}$
20. $-\frac{7}{8}$
21. $\frac{7}{2}$
22. $\frac{112}{9}$
23. $\frac{7}{3}$
24. $-\frac{9}{4}$

25. x^6

26. 2^3

27. 2^4

28. 5^7

29. 2^{41}

30. 3^{22}

31. xy^3

32. $\dfrac{y}{x^5}$

33. $x^{13}y^8$

34. $\dfrac{1}{x^2y^4}$

35. x^3y

36. $x^{10}y^8$

37. $\dfrac{6}{x}$

38. $\dfrac{2}{3x^2}$

39. $\dfrac{12}{x^2y}$

40. $\dfrac{10a}{b^4}$

41. $\dfrac{8y^9z^{15}}{x^6}$

42. $\dfrac{16x^6}{y^{10}z^2}$

43. $\dfrac{z^6}{9x^8y^4}$

44. $\dfrac{x^{12}z^{18}}{125y^9}$

45. $\dfrac{9y^6}{5x^3}$

46. $\dfrac{7x^5}{12y^5}$

47. $\dfrac{x^8}{12y^2}$

48. $\dfrac{3x^6}{y^3}$

49. $\dfrac{6x^{11}}{y^8z^7}$

50. $\dfrac{1}{x^4y^6}$

51. $\dfrac{y^{12}}{x^9}$

52. $\dfrac{x^2}{4y^{12}}$

53. $\dfrac{b^8}{a^4c^2}$

54. $\dfrac{27x^6}{y^9}$

55. $\dfrac{x^{20}z^{16}}{y^{28}}$

56. $\dfrac{8y^6}{x^9}$

57. $\dfrac{16}{5xy^9}$

58. $\dfrac{2}{y^5}$

59. $\dfrac{x^4}{9y^8}$

60. $\dfrac{4x^2y}{3}$

61. $\dfrac{2x^8}{5y^{10}}$

62. $\dfrac{1}{32x^2y^2}$

63. $\dfrac{3}{8x^2y^{11}}$

64. $\dfrac{8y^{14}}{9x^{13}}$

65. $\dfrac{4}{x^2y}$

66. $\dfrac{4y^6z^{10}}{x^6}$

67. $\dfrac{16x^{20}}{81y^8}$

68. $\dfrac{2x^{12}}{9y^{23}}$

69. $\dfrac{-8y^{24}}{x^{27}z^3}$

70. $\dfrac{3y^{12}}{4x^6}$

71. $\dfrac{27y^6z^8}{4x^{15}}$

72. Answers vary; $2^{13} \neq 2^6$

Section 4.4 (Exercises on page 130.)

1. $946,000,000,000,000,000$
2. 1,810,000
3. 0.0000000001
4. 0.0043
5. 0.000032
6. $578,900,000$
7. $693,000,000,000$
8. 0.000000034
9. $27,500,000$
10. $800,300$
11. 0.00000902
12. 0.0005486
13. 8.34×10^3
14. 7.9×10^{-4}
15. 4.2×10^{-5}
16. 4.36×10^6
17. 3.005×10^{-3}
18. 7.201×10^{-6}
19. 2.94×10^8
20. 3.45×10^{-7}
21. 7.03×10^{-8}
22. 6.6×10^9
23. 2.35×10^{12}
24. 5.32×10^{-10}
25. 7.2×10^{-12}
26. 2.236×10^5
27. 8.82×10^{10}
28. 8.7822×10^2

29. 3.0804×10^{-2}
30. 6.4×10^{6}
31. 1.6×10^{12}
32. 4×10^{-12}
33. 2×10^{12}
34. 2.3785×10^{3}
35. 3×10^{-7}

Chapter 5

Section 5.1 (Exercises on page 139.)

1. 26
2. 6
3. 20
4. 9
5. 3
6. 10
7. 1
8. 4
9. 4
10. 3
11. x^4
12. 1
13. xy^2
14. x^3y^2
15. x^2y^2z
16. 1
17. $2x^2y$
18. $3x^2yz^2$
19. $2x^3y^2z$
20. $5x^2$
21. $5(3x^2 - 2)$
22. $13x(2x^2 - 3)$
23. $5(2x^2 + 3x - 9)$
24. $x(x^2 + 6x - 17)$
25. $-18(x^6 - x^4 - x^2 + 1)$
26. $-y(x^3 - 2x^2 - 3x + 5)$
27. $2a^5(8a^3 - 9a - 15)$
28. $8x^2y^2(5x^2 + 3y^6 - 4xy^3)$
29. $2xy(8xy + 2x - 3y)$
30. $x^2y^2z^2(x + 3y - 2z)$
31. $(x + 2)(3 - x)$
32. $x^2(x - 1)(3x + 2)$
33. $(y - 3)(x + 5)$
34. $(x + y)(5x + 4y)$
35. $(x + 6)(3x^2 - 5x + 7)$
36. $(x^2 - 5x - 4)(7x + 8)$
37. $(xy + 1)(2y + 3x)$

38. $xy(x^2 - y^2)(x - y)$
39. $(5x^2 - 6)^2$
40. $(x^2 + 9x - 1)(3x - 1)$
41. $(x^2 + 2x + 3)(x^2 - 10x - 2)$
42. $(2x^2 + 3xy + y)(xy - 4x + 5y)$
43. $(x^2 + 4x + 5)(x^2 + 3x + 2)$
44. $(6x - 5y)(2x + 3y + 6)$
45. $13(x^2 - 2)(x^2 + xy + y)$
46. $5(3x^2 - 5)(3x + 14)$
47. $(2x + 3)(x^2 + 4)$
48. $(5x - 1)(x^2 + 9)$
49. $(3x - 2)(x^2 + 1)$
50. $(x + 3)(4x^2 + 9)$
51. $(2x - 1)(9x^2 + 1)$
52. $(3x - 4)(4x^2 + 1)$
53. $(5x - 1)(x^2 + 4)$
54. $(7x + 3)(x^2 + 9)$
55. $(4x - 3)(4x^2 + 1)$
56. $(6x - 1)(4x^2 - 3)$
57. $(4x - 1)(4x^2 - 5)$
58. $(2x - 3)(9x^2 + 4)$
59. $(x - 2)(16x^2 - 3)$
60. $(x - 1)(25x^2 - 7)$
61. $(2x + 1)(x^2 + 25)$
62. $(9x - 4)(x^2 + 1)$
63. $(2a + b)(a^2 - b)$
64. $(2x - 3)(y - 4)$
65. $(4x^2 + 1)(x - 5)$
66. $(7x - y)(x + 2)$
67. $(a + 4)(6b + 1)$
68. $(5x - 4y)(2x + 3)$
69. $(5x - 2z)(x + 3y)$
70. $(2x + 1)(3 + 2y)$

Section 5.2 (Exercises on page 148.)

1. $x^2 + 8x + 16$
2. $x^2 - 8x + 16$
3. $x^2 - 16$
4. $x^2 - 20x + 100$
5. $x^2 + 20x + 100$
6. $x^2 - 100$
7. $4x^2 - 4x + 1$
8. $4x^2 - 1$
9. $4x^2 + 4x + 1$
10. $16x^2 - 9$
11. $9x^2 + 42x + 49$
12. $16x^2 - 40x + 25$

13. $4x^2 + 12xy + 9y^2$

14. $9x^2 - 25y^2$

15. $16x^2 - 16xz + 4z^2$

16. $(x - 4)(x + 4)$

17. $(x - 9)(x + 9)$

18. $(3x - 2)(3x + 2)$

19. $5(x - 2)(x + 2)$

20. $9(2x - 1)(2x + 1)$

21. $7(2x - 3)(2x + 3)$

22. $(x + 3)^2$

23. $(x - 4)^2$

24. $(2x + 1)^2$

25. $2(x - 1)^2$

26. $3(2x + 3)^2$

27. $5(x + 2)^2$

28. $b = 36$

29. $b = 9$

30. $b = 49$

31. $b = 16$

32. $b = 1$

33. $b = 1$

34. $b = 4$

35. $b = 9$

36. $b = 1$

37. $b = 9$

38. $b = 1$

39. $b = 9$

40. $b = \pm 2$

41. $b = \pm 8$

42. $b = \pm 6$

43. $b = \pm 10$

44. $b = \pm 12$

45. $b = \pm 24$

46. $b = \pm 1$

47. $b = \pm 6$

48. $b = \pm 10$

49. Yes: $(x - 7)(x + 7)$

50. Yes: $(x - 11)(x + 11)$

51. No

52. Yes: $2(x - 2)(x + 2)$

53. Yes: $(4x - 5)(4x + 5)$

54. No

55. No

56. Yes: $(x - 7)^2$

57. No

58. Yes: $(x + 8)^2$

59. No

60. Yes: $(3x + 1)^2$

61. Yes: $(5x + 2)^2$

62. No

63. Yes: $(2x-3)^2$
64. $\left(x-\frac{1}{2}\right)\left(x+\frac{1}{2}\right)$
65. $(y-\frac{2}{3})(y+\frac{2}{3})$
66. $\left(\frac{3}{2}x-1\right)\left(\frac{3}{2}x+1\right)$
67. $(\frac{5}{4}y-1)(\frac{5}{4}y+1)$
68. $\left(\frac{1}{8}x-5\right)\left(\frac{1}{8}x+5\right)$
69. $(\frac{3}{7}x-4)(\frac{3}{7}x+4)$
70. $\left(\frac{4}{5}x-\frac{2}{7}\right)\left(\frac{4}{5}x+\frac{2}{7}\right)$
71. $(\frac{1}{11}x-\frac{7}{10})(\frac{1}{11}x+\frac{7}{10})$
72. $(x-13)(x+13)$
73. $(5x-2)(5x+2)$
74. $(x+10)^2$
75. $(y+7)^2$
76. $(x-11)^2$
77. $(2w-9)^2$
78. $(xy+2)^2$
79. $(xy+5)^2$
80. $7(2x+3)^2$
81. $(x-\frac{1}{2})^2$
82. $\left(x+\frac{1}{3}\right)^2$
83. $(y+\frac{3}{5})^2$
84. $\left(z-\frac{6}{7}\right)$
85. $(\frac{1}{9}x-1)^2$
86. $\left(\frac{1}{4}r-3\right)$
87. $\left(\frac{2}{3}x+\frac{1}{4}\right)^2$
88. $\left(\frac{5}{4}x+\frac{2}{3}\right)^2$
89. $(5x-1)(x-3)(x+3)$
90. $(3x-2)(x-1)(x+1)$
91. $(x+3)(2x-3)(2x+3)$
92. $(2x+1)(3x-1)(3x+1)$
93. $(3x-4)(2x-1)(2x+1)$
94. $(5x-1)(x-2)(x+2)$
95. $(7x+3)(x-3)(x+3)$
96. $(4x-3)(2x-1)(2x+1)$
97. $(6x-1)(2x-1)(2x+1)$
98. $(4x-1)(2x-1)(2x+1)$
99. $(2x-3)(3x-2)(3x+2)$
100. $(x+2)(4x-1)(4x+1)$
101. $(x-1)(5x-2)(5x+2)$
102. $(2x+1)(x-5)(x+5)$

Section 5.3 (Exercises on page 160.)

1. $(x+2)(x+1)$
2. $(x-2)(x+1)$
3. $(x+3)(x-1)$
4. $(x+7)(x-1)$
5. $(x-11)(x-1)$
6. $(x-11)(x+1)$

7. $(x+2)(x+3)$
8. $(x-4)(x+2)$
9. $(x-3)(x-4)$
10. $-1(x-7)(x-2)$
11. $(2x+1)(x-2)$
12. $(5x+1)(x+1)$
13. $2(x-1)^2$
14. $5(x+6)(x+1)$
15. $2(3x+1)(x+1)$
16. $(x+2y)^2$
17. $(x-3y)(x-4y)$
18. $(5x+y)(x+2y)$
19. $(x+2)^2$
20. $(x+7)(x+2)$
21. $(x+2)(x-1)$
22. prime over the integers
23. prime over the integers
24. prime over the integers
25. $(3x+1)(x+1)$
26. $3(x+4)(x-1)$
27. $(3x+2)(x+1)$
28. prime over the integers
29. $3(x^2+3x+1)$
30. $5(x^2+4x+2)$
31. prime over the integers
32. $(x-6y)(x-3y)$
33. $(7x-5y)(x+y)$
34. $b=\pm 4$
35. $b=\pm 2$
36. $b=\pm 6$
37. $b=\pm 8$
38. $b=\pm 5, \pm 7$
39. $b=\pm 1, \pm 5$
40. $b=\pm 3$
41. $b=\pm 1$
42. $b=\pm 9, \pm 11, \pm 19$
43. $b=\pm 6, \pm 10$
44. $b=\pm 6, \pm 9$
45. $b=\pm 11, \pm 13, \pm 17, \pm 31$
46. $b=\pm 6, \pm 10$
47. $b=\pm 2, \pm 7$
48. $b=\pm 5, \pm 13$
49. $4(3x-1)(x+2)$
50. $5x(x-3)(x+3)$
51. $8(2x-y)(x-2y)$
52. $8(2x^2-5)(2x^2+5)$
53. $-7x^2(2x-3)^2$
54. $\left(\frac{3}{2}y^2-5\right)\left(\frac{3}{2}y^2+5\right)$
55. $(9t^2+16)(3t+4)(3t-4)$
56. $2(y-3)(y+3)(y-2)(y+2)$

57. $2x^4(2x^3 - 7)(2x^3 + 7)$
58. $5(9 - a)(10 + a)$
59. $-15n(2n + 5)^2$
60. $3x(2x - 1)(x + 4)$
61. $3(7x + 5)(x - 3)$
62. $2(3x - 5)(2x + 1)$
63. $2(2x - 3)(3x + 1)$
64. $3x(4x + 3)(x - 3)$
65. $2x(6x - 1)(x - 5)$
66. $2(2x - 5)(x + 2)$
67. $2(4x - 1)(x + 6)$
68. $2(4x - 7)(x + 6)$
69. $2(3x - 10)(x + 3)$
70. $5(2x + 1)(x - 2)$
71. $2x(3x + 1)(x - 3)$
72. $2(7x - 1)(2x + 1)$
73. $4(2x - 3)(x - 1)$
74. $2x^2(5x + 2)(3x - 1)$
75. $3(4x - 3)(x + 2)$
76. $6(3x - 4)(x + 2)$

Section 5.4 (Exercises on page 164.)

1. $xy(2x - y)(2x + y)$
2. $5(x + 4y)(x - 2y)$
3. $(x^3 + 3)(x - 2)$
4. $(1 - a - b)(1 + a + b)$
5. $(3u - v)(2u + 3v)$
6. $x^2(3 - x^2y)(3 + x^2y)$
7. $(2y^2 + 3)(y - 2)(y + 2)$
8. $(3x - w)(y + 2z)$
9. $(x^2 - 7)(x - 3)(x + 3)$
10. $(7a - 4)(a + 3)$
11. $2t(2t + 15)(2t + 5)$
12. $5(x - 3)(x + 3)$
13. $a(a - 18)(a + 5)$
14. $9(3t^2 + 4)(t + 4)$
15. $[(x - 3)^5 - 1][(x - 3)^5 + 1]$
16. $3xy(x - y)^2$
17. $(2z - w)(x - 2y)$
18. $(b - 20)(b + 40)$
19. $(2 + x)(6 - 7x)$
20. $(2x - y)(3a + 2b)$
21. $6(x + 6)(x + 2)$
22. $(3y - 2x)(3y + 2x)(9y^2 + 4x^2)$
23. $(b - d)(b + d)(a^2 + c^2)$
24. $a(3a + 5)(a + 1)$
25. $(4y + 3z)(y - 8z)$
26. $2x(3x - 2)(3x + 2)$
27. $2xy(x - 4y)(x + y)$

28. $(5a + b)(3c - 4d)$
29. $(x + 2 - 3y)(x + 2 + 3y)$
30. $4y(3y - 1)(y + 4)$
31. $(3x^2 - 1)(6x - 1)$
32. $3a^2(a - 4)(a + 4)$
33. $(4x + 5)(10x - 9)$
34. $(3x + 2y + 4)(3x + 2y - 3)$
35. $(xy - 15)(xy + 15)$
36. $(8 - a)(a + 10)$
37. $(2x - 3y)(y^2 - 3x)$
38. $(x - 7)(2x - 15)(2x - 13)$
39. $x(2x - 5)^2$
40. $a(a + 1)(a - a^2 - 6)$
41. $(6x - 7y)(6x + 7y)$
42. $(x^2 + 4)(5x - 1)$
43. $2(3x - 5)(2x + 1)$
44. $(2x + 3)(x + 5)$
45. $(x - 4y)(x - 3y)$
46. $(3x + 2)(2x - 5)$
47. $2x^3y(x - xy + y)$
48. $3x(x + 3)^2$
49. $2x(3xy - 10)(xy + 3)$
50. $(9 - 5x)(1 + 2x)$
51. $(x - 1)(25x^2 - 4y)$
52. $4xy(3x - 2y)(3x + 2y)$
53. $2x^2(5x + 2)(3x - 1)$
54. $2(xy - 3)^2$
55. $2x^3(x - 4)(x - 3)$
56. $(8x - 9)(x - 1)$
57. $2(4x - 7)(x + 6)$
58. $(9x - 7)(9x + 7)$
59. $(7 - 2x)(2 + x)$
60. $(2x + 3)(x - 2)(x + 2)$
61. $(4x^2 + 9)(2x - 3)(2x + 3)$
62. $3(4x - 3)(x + 2)$
63. $4x^2y(x - 3y + 2)$
64. $4(x - 4)(x + 3)$
65. $2(4x - 1)(x + 6)$
66. $x(x - 18)(x + 5)$
67. $2x(2x + 5)(2x + 15)$
68. $3x^2y(x + 6)(x - 4)$
69. $(4x + 5y)(2x - y)$
70. $(1 - 10x)(1 + 2x)$

Section 5.5 (Exercises on page 174.)

1. $x = -2, 3$
2. $x = 2, 7$
3. $x = -4, -3$
4. $x = 0, 6$
5. $x = -4$
6. $x = -1, 4$
7. $x = -6, -\frac{1}{2}$
8. $x = \frac{1}{4}, \frac{2}{3}$
9. $x = -\frac{1}{10}, \frac{2}{7}$
10. $x = -4, 1$
11. $x = 3, 4$
12. $x = -5$
13. $x = -3$
14. $x = -1, -\frac{1}{5}$
15. $x = \frac{1}{2}, \frac{3}{2}$
16. $x = -1, 4$
17. $x = -5, 2$
18. $x = -5$
19. no rational solutions
20. $x = -8, 0$
21. no rational solutions
22. $x = -2, 0, 3$
23. $x = 0, 3, 6$
24. $x = \pm 3$
25. $x = \pm 1, \pm\sqrt{3}$
26. $x = 1, 11$
27. $x = -\frac{1}{2}, 2$
28. $x = \frac{3}{4}, 2$
29. $x = -\frac{1}{2}, 3$
30. $x = -3$
31. $x = \frac{2}{3}, 0$
32. $x = -\frac{3}{2}, 2$
33. $x = -5, 3$
34. $x = -2, \frac{5}{2}$
35. $x = \frac{1}{2}, 2$
36. $x = -\frac{1}{2}, \frac{2}{3}$
37. $x = -\frac{1}{2}, \frac{5}{3}$
38. $x = 0, \frac{7}{5}$
39. $x = -1, 9$
40. $x = \frac{4}{3}, 2$
41. $x = \pm\frac{5}{4}$
42. $x = -\frac{1}{2}, \frac{3}{2}$
43. $x = -\frac{1}{2}, 0, 5$
44. $x = -\frac{1}{2}, \pm\frac{1}{3}$
45. $x = \pm\frac{2}{3}, \pm 1$
46. $x = \frac{4}{9}, \pm 1$
47. $x = -\frac{3}{2}, -\frac{1}{5}, 0$
48. $x = \pm\frac{3}{2}, \pm 1$

49. $x = \frac{1}{6}, \pm\frac{1}{2}$
50. $x = \pm\frac{1}{4}, \pm 1$
51. $x = 0, 1, 6$
52. $x = -5, 0, 2$
53. $x = -3, \pm\frac{3}{2}$
54. $x = \pm\frac{1}{2}, \pm 2$
55. $x = 0, 1, \frac{3}{2}$
56. $b = \pm 6$
57. $b = \pm 4$
58. $b = \pm 6, \pm 10$
59. $b = 0, \pm 8$
60. $b = \pm 7, \pm 8, \pm 13$
61. $b = \pm 2, \pm 7$
62. $b = \pm 7, \pm 11$
63. $b = \pm 16$
64. $b = \pm 3, \pm 7, \pm 17$
65. $b = -10$
66. $b = -8$
67. $b = -15$
68. $b = 0$

Chapter 6

Section 6.1 (Exercises on page 184.)

1. 9
2. -11
3. $\dfrac{3}{4}$
4. -20
5. $-\dfrac{10}{7}$
6. not a real number
7. -2
8. -2
9. 10
10. $-\dfrac{5}{3}$
11. -4
12. not a real number
13. -2
14. $-\dfrac{1}{2}$
15. $-\dfrac{1}{2}$
16. 5
17. 4
18. $\dfrac{3}{2}$
19. 0.8
20. 0.6
21. 0.5
22. 0.2

23. -0.3
24. 5
25. 7
26. -8
27. 9
28. 10
29. 7
30. $\dfrac{3}{5}$
31. -3
32. $|x^7|$
33. x^{10}
34. $|x|$
35. x^4
36. $10x^2$
37. $7|y^3|$
38. $6x^3$
39. $8x^6$
40. $4x^4$
41. $3|x^3|$
42. $9x^6$
43. $4x^8$
44. $2x^4$
45. $4x^3$
46. $2x^2|y^3|$
47. $11|x^3y|$
48. $5x^5y^7$
49. $-3x^2y^5$
50. $6x^2|y^5|$

Section 6.2 (Exercises on page 192.)

1. $2x\sqrt{2x}$
2. $9x\sqrt{2}$
3. $-2x^2\sqrt[3]{3}$
4. $6x\sqrt{2x}$
5. $x^2\sqrt[4]{x}$
6. $\sqrt{x}$
7. $\sqrt{x}$
8. $\sqrt[3]{x}$
9. $y\sqrt[3]{x^2y}$
10. $y^2\sqrt{x}$
11. $x^3\sqrt[3]{y^2}$
12. $2y^2\sqrt{xy}$
13. $2x\sqrt{6}$
14. $3x^2y^3\sqrt{2xy}$
15. $4a^2b^7\sqrt{2ab}$
16. $2x^5y^3\sqrt{10xy}$
17. $8ab\sqrt{b}$
18. $8a^2b^3\sqrt{5b}$

19. $6x^2y^3\sqrt{3y}$
20. $12x^2y^3\sqrt{2x}$
21. $4x^3y\sqrt{2y}$
22. $6xy^3\sqrt{7x}$
23. $15y\sqrt{2xy}$
24. $8x^3y\sqrt{3x}$
25. $2xy^3\sqrt[3]{2x}$
26. $3xy^2\sqrt[3]{2x^2y}$
27. $2x^2y\sqrt[3]{y}$
28. $2xy^2\sqrt[3]{2x^2y}$
29. $2xy^2\sqrt[3]{5xy^2}$
30. $9x^4y^6z^9\sqrt{3xy}$
31. $3x^3y^4z^8\sqrt[3]{5y^2z}$
32. $3x^3y^2z^5\sqrt[4]{4xy^3z^2}$
33. $6x^5y^6z^{10}\sqrt{7yz}$
34. $3x^2y^3z^4\sqrt[3]{4xz^2}$
35. $7x^5y^3z\sqrt{3z}$
36. $8x^2y^4z^6\sqrt{5xy}$
37. $2x^2y^2z^3\sqrt[4]{2y^3}$
38. $8x^4y^7z^3\sqrt{3y}$
39. $-3x^5y^2z^3\sqrt[3]{4yz^2}$
40. $2x^2y^3z^2\sqrt[4]{3y^2z^3}$
41. $2x^3y^4z^2\sqrt[5]{4x^4y^3}$
42. $-3xy^3\sqrt[3]{3xy}$

Section 6.3 (Exercises on page 198.)

1. $7\sqrt{3}$
2. $5\sqrt[3]{2}$
3. $-5\sqrt{3}$
4. $21\sqrt{5}$
5. $6\sqrt[3]{2}+2\sqrt{2}-8$
6. $-\dfrac{29\sqrt{2}}{21}$
7. $10\sqrt{3x}$
8. $\dfrac{3\sqrt{3}}{5}$
9. $x\sqrt{2}$
10. $25x\sqrt{2}-2x$
11. $8y\sqrt{3y}$
12. $2x\sqrt[3]{7}-x\sqrt{7}$
13. $\sqrt{x}$
14. $-12x\sqrt{3}$
15. $16x\sqrt{2}$
16. $15\sqrt{2}-10\sqrt{3}$
17. $-9x\sqrt{3x}$
18. $-45\sqrt{2}$
19. $13\sqrt{3}-12\sqrt{5}$
20. $6\sqrt{x}$
21. $-34\sqrt{3x}$
22. $-2xy\sqrt{3}$

23. $\sqrt{2}$
24. $\sqrt{3}+6\sqrt{5}$
25. $7x\sqrt{2x}+5x\sqrt{2}$
26. $5+2\sqrt{3}$
27. $7-3\sqrt{10}$
28. $-7\sqrt{2}$
29. $-3\sqrt[3]{3}$
30. $-\dfrac{\sqrt{5}}{4}$
31. $46\sqrt[3]{2}$
32. $5\sqrt{2}$
33. $9x\sqrt{7x}$
34. $-4\sqrt{2}$
35. $30\sqrt[3]{2}$
36. $10\sqrt[4]{3}$
37. $29x^2\sqrt{3x}-3x\sqrt{3x}$
38. $7\sqrt{2x}+5x\sqrt{3x}$
39. $21\sqrt[3]{2}$
40. $-8\sqrt{3}$
41. $5\sqrt{7}$
42. $\dfrac{11\sqrt{3}}{4}$
43. $-9\sqrt{3}$
44. $-11\sqrt[3]{2}$
45. $36\sqrt{2}$
46. $\sqrt[3]{2}$
47. $11\sqrt{2x}$
48. $-\dfrac{10\sqrt{2}}{9}$
49. $-14\sqrt{3}$
50. $-7\sqrt{2}$

Section 6.4 (Exercises on page 204.)

1. $3\sqrt{10}$
2. 120
3. $-3\sqrt[3]{3}$
4. $-60\sqrt{3}$
5. $-126\sqrt{3}$
6. $3ab^6\sqrt{2a}$
7. $5x^2y\sqrt{2x}$
8. $2-\sqrt{6}$
9. $3\sqrt{3}$
10. $4-2\sqrt{10}$
11. $3a-3\sqrt{ab}$
12. $5x\sqrt{2}-x\sqrt{5}$
13. $15x-22y\sqrt{x}+8y^2$
14. $15x+\sqrt{xy}-2y$
15. $3x-y^2$
16. $x+10\sqrt{x}+25$

17. $4x - 12\sqrt{x} + 9$
18. $6 - 3\sqrt{3} + 2\sqrt{2} - \sqrt{6}$
19. $x - 16$
20. $x + 6\sqrt{x} + 9$
21. $x + 3\sqrt{x} - 4$
22. $16x + 8\sqrt{3x} + 3$
23. $18 - 17\sqrt{6}$
24. $24\sqrt{2} - 12\sqrt{3}$
25. $61 + 24\sqrt{5}$
26. $42 - 9\sqrt{14}$
27. -38
28. $126 - 34\sqrt{3}$
29. $-6 - 19\sqrt{5}$
30. $11 - 4\sqrt{6}$
31. 12
32. $-102 - 21\sqrt{2}$
33. $24x\sqrt{3} - 12x$
34. $-11 + \sqrt{2}$
35. $60x^4$
36. $34 - 24\sqrt{2}$
37. $57 - 40\sqrt{2}$
38. $44 - 25\sqrt{5}$
39. 29
40. $40\sqrt{3} - 24\sqrt{5}$
41. $-32\sqrt[4]{2}$
42. $42 + 18\sqrt{14}$
43. $138 + 69\sqrt{2}$
44. $62 - 20\sqrt{6}$
45. $130 - 50\sqrt{3}$
46. $40 - 20\sqrt{2}$
47. $12 - 3\sqrt{5}$
48. $16x + 8x\sqrt{7x} + 7x^2$
49. $-36\sqrt[3]{3}$
50. 40

Section 6.5 (Exercises on page 212.)

1. $\dfrac{4\sqrt{6}}{3}$
2. $\sqrt{3}$
3. $\dfrac{\sqrt{2}}{4}$
4. $\dfrac{\sqrt{3x}}{x}$
5. $\dfrac{\sqrt{11}}{2}$
6. $\dfrac{2\sqrt{2}}{3}$
7. $\dfrac{\sqrt{6}}{3}$
8. $\dfrac{\sqrt{55}}{5}$

9. $\sqrt{2}$

10. $\dfrac{2\sqrt{6}}{3}$

11. $3\sqrt{3}$

12. $\dfrac{\sqrt{3}}{2}$

13. $\dfrac{\sqrt{5}}{3}$

14. $\dfrac{\sqrt{2}}{2}$

15. $\dfrac{\sqrt{6}}{4}$

16. $\dfrac{\sqrt{14}}{7}$

17. $\dfrac{8\sqrt{7}}{7}$

18. $\dfrac{\sqrt{3}}{2}$

19. $\dfrac{2\sqrt{10}}{5}$

20. $\dfrac{2\sqrt{5}}{5}$

21. $\dfrac{2\sqrt{3}}{3}$

22. $\dfrac{7\sqrt{2}}{12}$

23. $3\sqrt{7}$

24. $\dfrac{5\sqrt[4]{x^3}}{x}$

25. $\dfrac{\sqrt{10x}}{4}$

26. $\dfrac{5\sqrt{3}}{4}$

27. $\dfrac{2\sqrt{6}}{5}$

28. $2\sqrt{2}$

29. $20\sqrt{3}$

30. $\dfrac{4\sqrt{3}}{3}$

31. $\dfrac{7\sqrt[3]{4}}{2}$

32. $\dfrac{\sqrt[3]{18}}{3}$

33. $\dfrac{3\sqrt[5]{x^3}}{x}$

34. $2\sqrt[4]{2}$

35. $\dfrac{8\sqrt[7]{x^4}}{x}$

36. $\dfrac{5\sqrt[3]{3}}{3}$

37. $-3(\sqrt{7}-3)$

38. $\dfrac{2(3+\sqrt{2})}{7}$

39. $-2(1-\sqrt{3})$

40. $5(\sqrt{7}-\sqrt{5})$

41. $\dfrac{3(\sqrt{5}-1)}{2}$

42. $-\dfrac{5(\sqrt{3}+6)}{11}$

43. $\dfrac{5+\sqrt{3}}{2}$

44. $-\dfrac{4(\sqrt{7}+4)}{3}$

45. $3(\sqrt{5}-\sqrt{3})$

46. $\dfrac{5(\sqrt{7}+\sqrt{3})}{4}$

47. $5(\sqrt{6}-\sqrt{3})$

48. $\dfrac{-4(\sqrt{7}-4)}{3}$

49. $-6(2+\sqrt{6})$

50. $\dfrac{7(\sqrt{3}-\sqrt{7})}{2}$

51. $-2(1+\sqrt{5})$

52. $\dfrac{\sqrt{7}+\sqrt{3}}{2}$

53. $\dfrac{-(\sqrt{5}+3)}{2}$

54. $-4(2-\sqrt{6})$

55. $7(2\sqrt{2}+5)$

56. $4(3+\sqrt{6})$

57. $\dfrac{9(\sqrt{10}-\sqrt{6})}{2}$

58. $\dfrac{-2(\sqrt{7}+4)}{3}$

59. $-3(\sqrt{11}+4)$

60. $3\sqrt{2}-4$

61. $\dfrac{4(\sqrt{3}+2\sqrt{5})}{17}$

62. $-2(\sqrt{11}+3\sqrt{2})$

63. $\dfrac{24(5-2\sqrt{3})}{13}$

64. $\dfrac{(2+\sqrt{2})(5+\sqrt{3})}{22}$

65. $-\dfrac{(1-\sqrt{7})(4-3\sqrt{2})}{2}$

Section 6.6 (Exercises on page 224.)

1. $4\sqrt[3]{x^2}+3\sqrt[3]{x}-2\sqrt{x}$

2. $5\sqrt{x}+2\sqrt[3]{x^2}+3\sqrt[4]{x^3}$

3. $\dfrac{4}{\sqrt[3]{x^2}}+\dfrac{2}{\sqrt[3]{x}}+6\sqrt{x}$

4. $\dfrac{3}{4\sqrt[3]{x^2}}+\dfrac{3}{8\sqrt{x}}+\dfrac{7\sqrt[3]{x}}{5}$

5. $7\sqrt[7]{x^4}-\dfrac{3}{\sqrt[5]{x^3}}+\dfrac{11\sqrt[4]{x}}{12}$

6. $\dfrac{5\sqrt[3]{x^2}}{6}-3\sqrt[5]{x}+6\sqrt[4]{x^3}$

7. $2\sqrt{x^3}-6\sqrt[3]{x^2}-\dfrac{4}{\sqrt{x}}$

8. $\dfrac{9}{\sqrt[4]{x}}+6\sqrt[5]{x^3}-\dfrac{2}{\sqrt[3]{x}}$

9. $x^{2/5}+x^{1/3}-2x^{1/2}$

10. $5x^{4/7}+8x^{1/2}+2x^{2/3}-5x^{1/4}$

11. $7x^{2/5}+3x^{2/3}-4x^{3/4}$

12. $9x^{1/3}-5x^{-3/4}-2x^{5/2}$

13. $\dfrac{6}{5}x^{-1/2}-2x^{-1/3}+8x^{1/4}$

14. $\frac{1}{2}x^{-1/3} + 3x^{3/2} - 4x^{-1/3}$

15. $4x^{4/5} - \frac{1}{3}x^{-1/2} + 5x^{3/4}$

16. $\frac{1}{4}x^{1/3} + 3x^{2/5} - 6x^{5/4}$

17. 6

18. 16

19. 27

20. $\frac{1}{5}$

21. $\frac{1}{8}$

22. $\frac{1}{2}$

23. $\frac{1}{9}$

24. 32

25. 64

26. $\frac{1}{27}$

27. 1

28. -2

29. 40

30. $\frac{1}{8}$

31. -4

32. $\frac{1}{125}$

33. $\frac{1}{27}$

34. $\frac{1}{25}$

35. -7

36. 27

37. 3

38. $-\frac{4}{3}$

39. $\frac{16}{81}$

40. $\frac{8}{27}$

41. $\frac{9}{4}$

42. $\frac{5}{13}$

43. 9

44. 38

45. -1

46. 3

47. $\frac{2}{5}$

48. 1

49. $-\frac{1}{2}$

50. $-\frac{1}{8}$

51. $-\frac{1}{3}$

52. 5

53. $\frac{62}{5}$

54. $\frac{9}{8}$

55. $64x^2y^3$

56. $\dfrac{1}{x^{3/4}}$

57. $x^{5/12}$

58. $\dfrac{2}{5}$

59. $\dfrac{1}{x^{5/12}}$

60. $x^{7/4}$

61. $\dfrac{1}{x^{17/10}}$

62. $16a^{1/2}b^4$

63. $9x^6y^8$

64. $64x^6y^9$

65. $x^{2/3}$

66. $\dfrac{2}{3x^{1/3}y^{1/2}}$

67. $\dfrac{x^{13/12}}{y^{2/3}}$

68. $\dfrac{x^{5/4}y}{4}$

Appendix A

Formulas

Area Formulas:

- Area of square:

$$\boxed{A_{square} = s^2 = \text{side}^2}$$

- Area of a rectangle:

$$\boxed{A_{rectangle} = l \times w = \text{length} \times \text{width}}$$

- Area of a triangle:

$$\boxed{A_{triangle} = \frac{1}{2} \times b \times h = \frac{1}{2} \times \text{base} \times \text{height}}$$

- Area of a parallelogram:

$$\boxed{A_{parallelogram} = b \times h = \text{base} \times \text{height}}$$

- Area of a trapezoid:

$$\boxed{A_{trapezoid} = \frac{1}{2} \times h\,(b_1 + b_2) = \frac{1}{2} \times \text{height} \cdot (\text{sum of the bases})}$$

- Area of a circle:

$$\boxed{A_{circle} = \pi r^2 = \pi(\text{radius})^2}$$

Perimeter and Circumference Formulas:

- Perimeter:

$$\boxed{P_{perimeter} = \text{distance around}}$$

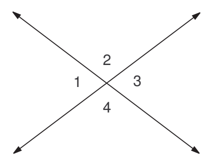
  In other words, add each side together.

- Circumference of a circle:

$$\boxed{C_{circle} = 2\pi r = 2\pi(\text{radius})}$$

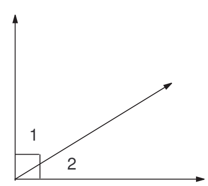
Angle Properties:

- **Vertical Angles** are opposite angles formed by intersecting lines. Vertical angles always have the same measurement.

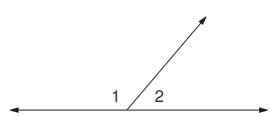


In the above figure, 1 and 3 are vertical angles; 2 and 4 are vertical angles.

- **Complementary Angles** are two angles whose sum is 90°.



- **Supplementary Angles** are two angles whose sum is 180°.



- **Corresponding Angles** have the same location relative to lines ℓ, m and transversal t.
(**IMPORTANT:** $\ell \parallel m$ if and only if corresponding angles formed by ℓ, m, and t are
congruent.) In Figure A-1, $\angle 1$ and $\angle 5$ are corresponding angles. The following pairs are also corresponding angles: $\angle 2$ and $\angle 6$; $\angle 3$ and $\angle 7$; $\angle 4$ and $\angle 8$.

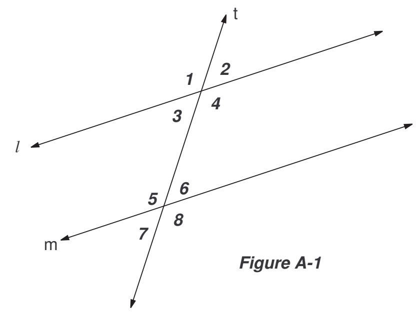


Figure A-1

- **Alternate Interior Angles** are nonadjacent angles formed by lines ℓ, m, and transversal t, the union of whose interiors contain the region between ℓ and m.
(**IMPORTANT:** $\ell \parallel m$ if and only if alternate interior angles formed by ℓ, m and t are congruent.) In Figure A-1, $\angle 3$ and $\angle 6$ are alternate interior angles. Likewise, $\angle 4$ and $\angle 5$ are also alternate interior angles.

- **Alternate Exterior Angles** are angles on the outer sides of two lines cut by a transversal, but on opposite sides of the transversal (**IMPORTANT:** $\ell \parallel m$ if and only if alternate exterior angles formed by ℓ, m and t are congruent.) In Figure A-1, $\angle 2$ and $\angle 7$ are alternate exterior angles. Similarly, $\angle 1$ and $\angle 8$ are alternate exterior angles.

- **Interior Angles on the same side of the transversal** are interior angles whose interiors are the same. (**IMPORTANT:** $\ell \parallel m$ if and only if the interior angles on the same side of the transversal are supplementary.) In Figure A-1, $\angle 3$ and $\angle 5$, as well as $\angle 4$ and $\angle 6$, are interior angles on the same side of the transversal.