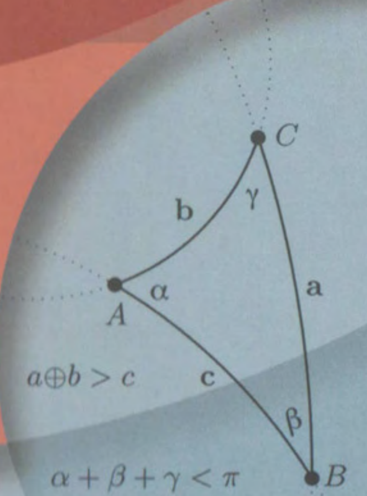
ANALYTIC HYPERBOLIC GEOMETRY

Mathematical Foundations and Applications



$$\mathbf{a} = \ominus B \oplus C$$

$$\mathbf{b} = \ominus C \oplus A$$

$$\mathbf{c} = \ominus A \oplus B$$

$$\delta = \pi - \alpha - \beta - \gamma > 0$$

$$\|\mathbf{a}\|^2 = \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)}$$

$$\|\mathbf{b}\|^2 = \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)}$$

$$\|\mathbf{c}\|^2 = \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)}$$

$$a \oplus b > c$$

$$\alpha + \beta + \gamma < \pi$$

$$\cos\alpha = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|}$$

$$\tan\frac{\delta}{2} = \frac{ab\sin\gamma}{1 - ab\cos\gamma}$$

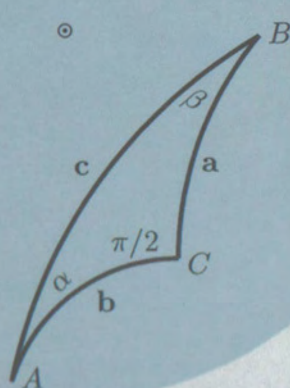
The Hyperbolic Pythagorean Theorem

$$\mathbf{a} = \ominus B \oplus C, \qquad a = \|\mathbf{a}\|$$

$$\mathbf{b} = \ominus C \oplus A, \qquad b = \|\mathbf{b}\|$$

$$\mathbf{c} = \ominus A \oplus B, \qquad c = \|\mathbf{c}\|$$

$$\boxed{a^2 \oplus b^2 = c^2}$$



Abraham A. Ungar

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Abraham A. Ungar

Department of Mathematics
North Dakota State University, USA

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ANALYTIC HYPERBOLIC GEOMETRY
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This book on Analytic Hyperbolic Geometry
and its application in Special Relativity
commemorates 2005 as the 100th anniversary of
Albert Einstein's (1879 – 1955) miraculous year, 1905,
that gave birth to the Special Theory of Relativity,
and the 50th anniversary of his death in April 18, 1955.
This book is dedicated
to the Centenary Celebration of
Einstein's Special Relativity and to the practice of
twenty-first century Special Relativity
by means of
Analytic Hyperbolic Geometry.

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Preface

This book is about the foundations and applications of analytic hyperbolic geometry from the viewpoint of hyperbolic vectors, called *gyrovectors*. The underlying mathematical tools, gyrogroups and gyrovector spaces, are developed along analogies they share with groups and vector spaces. As a result, a gyrovector space approach to hyperbolic geometry, fully analogous to the standard vector space approach to Euclidean geometry, emerges.

Owing to its strangeness, some regard themselves as excluded from the profound insights of hyperbolic geometry so that this enormous portion of human achievement is a closed door to them. But this book opens the door on its mission to make the hyperbolic geometry of Bolyai and Lobachevsky widely accessible by introducing a gyrovector space approach to hyperbolic geometry guided by analogies that it shares with the common vector space approach to Euclidean geometry.

Writing this first book on analytic hyperbolic geometry became possible following the successful adaption of vector algebra for use in hyperbolic geometry in the author's 2001 book *"Beyond the Einstein Addition Law and its Gyroscopic Thomas Precession: The Theory of Gyrogroups and Gyrovector Spaces"* (Kluwer Acad.). A most convincing way to describe the success of the author's adaption of vector algebra for use in hyperbolic geometry is found in Scott Walter's review of the author's 2001 book, part of which is therefore quoted below.

> Over the years, there have been a handful of attempts to promote the non-Euclidean style for use in problem solving in relativity and electrodynamics, the failure of which to attract any substantial following, compounded by the absence of any positive results must give pause to anyone considering a similar undertaking. Until recently, no one was in a position to offer an improvement on the tools available since 1912. In his [2001] book, Ungar furnishes

the crucial missing element from the panoply of the non-Euclidean style: an elegant nonassociative algebraic formalism that fully exploits the structure of Einstein's law of velocity composition. The formalism relies on what the author calls the "missing link" between Einstein's velocity addition formula and ordinary vector addition: Thomas precession ...

Ungar lays out for the reader a sort of vector algebra in hyperbolic space, based on the notion of a gyrovector. A gyrovector space differs in general from a vector space in virtue of inclusion of Thomas precession, and exclusion of the vector distributive law. As a result, when expressed in terms of gyrovectors, Einstein (noncommutative) velocity addition law becomes "gyrocommutative" .... One advantage of this approach is that hyperbolic geometry segues into Euclidean geometry, with notions such as group, vector, and line passing over to their hyperbolic gyro-counterparts (gyrogroup, etc.) ...

One might suppose that there is a price to pay in mathematical regularity when replacing ordinary vector addition with Einstein's addition, but Ungar shows that the latter supports gyrocommutative and gyroassociative binary operations, in full analogy to the former. Likewise, some gyrocommutative and gyroassociative binary operations support scalar multiplication, giving rise to gyrovector spaces, which provide the setting for various models of hyperbolic geometry, just as vector spaces form the setting for the common model of Euclidean geometry. In particular, Einstein gyrovector spaces provide the setting for the Beltrami ball model of hyperbolic geometry, while Möbius gyrovector spaces provide the setting for the Poincaré ball model of hyperbolic geometry.

Scott Walter
Foundations of Physics 32, pp. 327–330 (2002)

Analytic hyperbolic geometry, as presented in this book, is now performing better than ever, emphasizing the interdisciplinary collaborations required to further develop this extraordinary mathematical innovation and its applications. But, there have been some challenges during the initial phase of its development, challenging preconceived notions like the dogma of Einsteinian relativity vs. Minkowskian relativity, which was not struck down until the emergence of analytic hyperbolic geometry.

Armed with a gyrovector space structure, hyperbolic geometry is perfect for use in relativity physics. It is therefore fitting that the completion of this book, the first book on analytic hyperbolic geometry and on the central role it plays in special relativity, comes to fruition this year, the 100th anniversary of Einstein's miraculous year, 1905 [Einstein (1998)]. In this year Einstein submitted his doctoral dissertation (April 30) (i) on the determination of molecular dimensions; and he published four seminal papers (ii) on the photoelectric effect (published June 9), for which he was awarded the Nobel Prize in 1921; (iii) on the existence of atoms by measuring Brownian motion of particles in solution (published July 18); (iv) on the electrodynamics of moving bodies (published September 26), his first paper on special theory of relativity and a landmark in the development of modern physics; and a second, shorter paper (v) on the special theory of relativity, that contains the famous $E = mc^2$ [Einstein and Calaprice (2005), p. xxxi].

As a mathematical prerequisite for a fruitful reading of this book it is assumed familiarity with Euclidean geometry from the point of view of vectors and, occasionally, with differential calculus and functions of a complex variable. The book is aimed at a large audience. It includes both elementary and advanced topics, and is structured so that it can be enjoyed equally by undergraduates, graduate students, researchers and academics in geometry, algebra, mathematical physics, theoretical physics and astronomy.

Abraham A. Ungar

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Acknowledgements

The major undertaking of unleashing the power of hyperbolic geometry in the creation of Analytic Hyperbolic Geometry involves the mathematical abstraction of the relativistic effect known as "Thomas precession" and the introduction of hyperbolic vectors, called "gyrovectors". It has received the generous scientific support from Helmuth K. Urbantke of the Institute for Theoretical physics, Vienna, Austria, and Scott Walter of the University of Nancy 2, Nancy, France, who helped place Analytic Hyperbolic Geometry on the road to mainstream literature.

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Contents

Chapter 1

Introduction

This introductory chapter presents a few hyperbolic gems from the book to amaze both the uninitiated and the practicing expert. The actual study of analytic hyperbolic geometry, thus, begins in Chap. 2.

Geometry, according to Herodotus, and the Greek derivation of the word, had its origin in Egypt in the mensuration of land, and fixing of boundaries necessitated by the repeated inundations of the Nile. It consisted at first of isolated facts of observation and crude rules for calculation until it came under the influence of Greek thought. Following the introduction of geometry from Egypt to Greece by Thales of Miletus, 640 – 546 B.C., geometric objects were abstracted, thus paving the way for attempts to give geometry a connected and logical presentation. The most famous of these attempts is that of Euclid, about 300 B.C. [Sommerville (1914), p. 1].

According to the Euclid parallel postulate, given a line L and a point P not on L there is one and only one line L' which contains P and is parallel to L. Euclid's parallel postulate does not seem as intuitive as his other axioms. Hence, it was felt for many centuries that it ought to be possible to find a way of proving it from more intuitive axioms. The history of the study of parallels is full of reproaches against the lack of self-evidence of the Euclid parallel postulate. According to Sommerville [Sommerville (1914), p. 3], Sir Henry Savile referred to it as one of the great blemishes in the beautiful body of geometry [Praelectiones, Oxford, 1621, p. 140]. Following Bolyai and Lobachevsky, however, the parallel postulate became the property that distinguishes Euclidean geometry from non-Euclidean ones.

The Hungarian Geometer János Bolyai (1802 – 1860) and the Russian Mathematician Nikolai Ivanovich Lobachevsky (1793 – 1856) independently worked out a geometry that seemed consistent and yet negated the Euclidean parallel postulate, published in 1832 and 1829. Carl Friedrich Gauss (1777 – 1855), who was the dominant figure in the mathematical world at the time, was probably the first to understand clearly the possibility of a logically and sound geometry different from Euclid's. According to Harold E. Wolfe [Wolfe (1945), p. 45], it was Gauss who coined the term *non-Euclidean geometry.* The contributions of Gauss to the birth of hyperbolic geometry are described by Sonia Ursini in [Ursini (2001)]. According to Duncan M. Y. Sommerville [Sommerville (1914), p. 24], the ideas inaugurated by Bolyai and Lobachevsky did not attain any wide recognition for many years, and it was only after Baltzer had called attention to them in 1867 that non-Euclidean geometry began to be seriously accepted and studied. In 1871 Felix Klein suggested calling the non-Euclidean geometry of Bolyai and Lobachevsky *hyperbolic geometry* [Sommerville (1914), p. 25]. The discovery of hyperbolic geometry and its development is one of the great stories in the history of mathematics; see, for instance, the accounts of [Rosenfeld (1988)] and [Gray (1989)] for details.

1.1 The Vector and Gyrovector Approach to Euclidean and Hyperbolic Geometry

Commonly, three methods are used to study Euclidean geometry:

(1) *The Synthetic Method:* This method deals directly with geometric objects (figures). It derives some of their properties from other properties by logical reasoning.
(2) *The Analytic Method:* This method uses a coordinate system, expressing properties of geometric objects by numbers (coordinates). It derives properties from other properties by numerical expressions and equations, numerical results being interpreted in terms of geometric objects [Boyer (2004)].
(3) *The Vector Method:* The vector method occupies a middle position between the synthetic and the analytic method. It deals with geometric objects directly and derives properties from other properties by computation with vector expressions and equations [Hausner (1998)].

Euclid treated his Euclidean geometry synthetically. Also Bolyai and Lobachevsky treated their hyperbolic geometry synthetically. Because progress in geometry needs computational facilities, the invention of analytic geometry by Descartes (1596–1650) made simple approaches to more geometric problems possible. Later, further simplicity for geometric calculations became possible by the introduction of vectors and their addition by the parallelogram law. The parallelogram law for vector addition is so intuitive that its origin is unknown. It may have appeared in a now lost work of Aristotle (384–322). It was also the first corollary in Isaac Newton's (1642–1727) "Principia Mathematica" (1687), where Newton dealt extensively with what are now considered vectorial entities, like velocity and force, but never with the concept of a vector. The systematic study and use of vectors were a 19th and early 20th century phenomenon. Vectors were born in the first two decades of the 19th century with the geometric representations of complex numbers. The development of the algebra of vectors and of vector analysis as we know it today was first revealed in sets of notes made by J. Willard Gibbs (1839–1903) for his students at Yale University.

The synthetic and analytic methods for the study of Euclidean geometry are accessible to the study hyperbolic geometry as well. Hitherto, however, the vector method had been deemed inaccessible to that study.

In the years 1908 – 1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865 – 1942), professor and rector of Zagreb University, showed that this theory has a natural interpretation in hyperbolic geometry [Varičak (1910a)]. For his chagrin, however, Varičak had to admit in 1924 [Varičak (1924), p. 80] that the adaption of vector algebra for use in hyperbolic geometry was just not feasible, as Scott Walter notes in [Walter (1999b), p. 121].

Following Varičak's 1924 realization that, unlike Euclidean geometry, the hyperbolic geometry of Bolyai and Lobachevsky does not admit vectors, there are in the literature no attempts to treat hyperbolic geometry vectorially. There are, however, few attempts to treat hyperbolic geometry analytically [Jackson and Greenspan (1955); Patrick (1986)], dating back to Sommerville's 1914 book [Sommerville (1914); Sommerville (1919)]. Accordingly, following Bolyai and Lobachevsky, most books on hyperbolic geometry treat the geometry synthetically, some treat it analytically, but no book treats it vectorially.

Fortunately, some 80 years since Varičak's 1924 realization, the adaption of vectors for use in hyperbolic geometry, where they are called *gyrovectors*, has been accomplished in [Ungar (2000); Ungar (2001)], allowing Euclidean and hyperbolic geometry to be united [Ungar (2004c)]. Following the adaption of vector algebra for use in hyperbolic geometry, the hyperbolic geometry of Bolyai and Lobachevsky is now effectively regulated by gyrovector spaces just as Euclidean geometry is regulated by vector spaces. Accordingly, we develop in this book a gyrovector space approach to hyperbolic geometry that is fully analogous to the common vector space approach to Euclidean geometry [Hausner (1998)]. In particular, we find in this book that gyrovectors are equivalence classes of directed gyrosegments, Def. 5.4, p. 119, that add according to the gyroparallelogram law, Figs. 8.21 – 8.22, p. 291, just like vectors, which are equivalence classes of directed segments that add according to the common parallelogram law.

It should be remarked here that in applications to Einstein's special theory of relativity, Chap. 10, Einsteinian velocity gyrovectors are 3-dimensional gyrovectors fully analogous to Newtonian velocity vectors. Hence, in particular, relativistic gyrovectors are different from the common 4-vectors of relativity physics. In fact, the passage from n-gyrovectors to $(n+1)$-vectors is illustrated in Remark 4.20, p. 107, and employed in the study of the special relativistic Lorentz transformation group in Chap. 10. The 4-vectors are important in special relativity and in its extension to general relativity. Early attempts to employ 4-vectors in gravitation, 1905 – 1910, are described in [Walter (2005)].

In the same way that vector spaces are commutative groups of vectors that admit scalar multiplication, gyrovector spaces are gyrocommutative gyrogroups of gyrovectors that admit scalar multiplication. Accordingly, the nonassociative algebra of gyrovector spaces is our framework for analytic hyperbolic geometry just as the associative algebra of vector spaces is the framework for analytic Euclidean geometry. Moreover, gyrovector spaces include vector spaces as a special, degenerate case corresponding to trivial gyroautomorphisms. Hence, our gyrovector space approach forms the theoretical framework for uniting Euclidean and hyperbolic geometry.

1.2 Gyrolanguage

In order to elaborate a precise language for dealing with analytic hyperbolic geometry, which emphasizes analogies with classical notions, we extensively use the prefix "gyro", giving rise to *gyrolanguage*, the language that we use in this book. The resulting gyrolanguage rests on the unification of Euclidean and hyperbolic geometry in terms of analogies they share [Ungar (2004c)]. The prefix "gyro" stems from *Thomas gyration*. The latter, in turn, is the mathematical abstraction of the peculiar relativistic effect known as the *Thomas precession* into an operator, called a *gyrator* and denoted "gyr". The gyrator generates special automorphisms called *gyroautomorphisms*. The effects of the gyroautomorphisms are called gyrations in the same way that the effects of rotation automorphisms are called rotations.

The natural emergence of gyrolanguage is well described by a 1991 letter that the author received from Helmuth Urbantke of the Institute for Theoretical Physics, University of Vienna, sharing with him instructive experience [Ungar (1991), ft. 36]:

> "While giving a seminar about your work, the word *gyromorphism* instead of [Thomas] precession came over my lips. Since it ties in with the many morphisms the mathematicians love, it might appeal to you."
>
> Helmuth K. Urbantke, 1991

Indeed, we will find in this book that the translation of hyperbolic geometry into the gyrolanguage of gyrovector spaces is possible, and the pursuit of this translation entails no pain for unlimited profit.

Analytic Euclidean geometry in n dimensions models points by n-tuples of numbers that form an n-dimensional vector space with an inner product. Vector spaces thus algebraically regulate analytic Euclidean geometry, allowing the principles of (associative) algebra to manipulate Euclidean geometric objects. Contrastingly, synthetic Euclidean geometry is the kind of geometry for which Euclid is famous and that the reader learned in high school.

Analytic hyperbolic geometry in n dimensions is the subject of this book. It models points by n-tuples of numbers that form an n-dimensional gyrovector space with an inner product. Gyrovector spaces thus algebraically regulate analytic hyperbolic geometry, allowing the principles of (nonassociative) algebra to manipulate hyperbolic geometric objects. Contrastingly, synthetic hyperbolic geometry is the kind of geometry for which Bolyai and Lobachevsky are famous and that one learns from the literature on classical hyperbolic geometry.

With one exception, proofs are obtained in this book analytically. The exceptional case is the proof of the *gyrotriangle defect identity* which is the identity shown at the bottom of Fig. 1.2. Instructively, this identity is verified both analytically, Theorem 8.44, and synthetically, Theorem 8.47.

It is the gyrotriangle defect identity at the bottom of Fig. 1.2 that gives rise to the elegant values of the squared hyperbolic length (gyrolength) of the sides of a hyperbolic triangle (gyrotriangle) in terms of its hyperbolic angles (gyroangles), also shown in Fig. 1.2 as well as in Theorem 8.48 on p. 280.

While Euclidean geometry has a single standard model, hyperbolic geometry is studied in the literature by several standard models. In this book, analytic hyperbolic geometry appears in three mutually isomorphic models. These are:

(*I*) The Poincaré ball (or disc, in two dimensions) model.
(*II*) The Beltrami (also known as the Klein) ball (or disc, in two dimensions) model.
(*III*) The Proper Velocity (PV, in short) space (or plane, in two dimensions) model.

The PV space model of hyperbolic geometry is also known as Ungar space model [Ungar (2001)]. The terms "Ungar gyrogroups" and "Ungar gyrovector spaces" were coined by Jing-Ling Chen in [Chen and Ungar (2001)] following the emergence of gyrolanguage in [Ungar (1991)]. Ungar gyrogroups and gyrovector spaces may be used to describe algebraic structures of relativistic proper velocities. Hence, in this book these are called *PV gyrogroups* and *PV vector spaces.*

Before the emergence of gyrolanguage the author coined the term "K-loop" in [Ungar (1989b)] to honor related pioneering work of Karzel in the 1960s, and to emphasize relations with loops that have later been studied in [Krammer (1998); Sabinin, Sabinina and Sbitneva (1998); Issa (1999); Issa (2001)]. With the emergence of gyrolanguage, however, since 1991 the author's K-loops became "gyrocommutative gyrogroups" following the need to accommodate "non-gyrocommutative gyrogroups" and to emphasize analogies with groups. The ultimate fate of mathematical terms depends on their users. Thus, for instance, some like the term "K-loop" that the author coined in 1989 (as recorded in [Kiechle (2002), pp. 169 – 170] and, in more detail, in [Sexl and Urbantke (2001), pp. 141 – 142]), and some prefer using the alternative term "Bruck-loop" (as evidenced, for instance, from MR:2000j:20129 in *Math. Rev.*).

A new term, "dyadic symset", which has recently emerged from an interesting work of Lawson and Lim in [Lawson and Lim (2004)], turns out to be identical to a two-divisible, torsion-free, gyrocommutative gyrogroup according to [Lawson and Lim (2004), Theorem 8.8]. It thus seems that, as

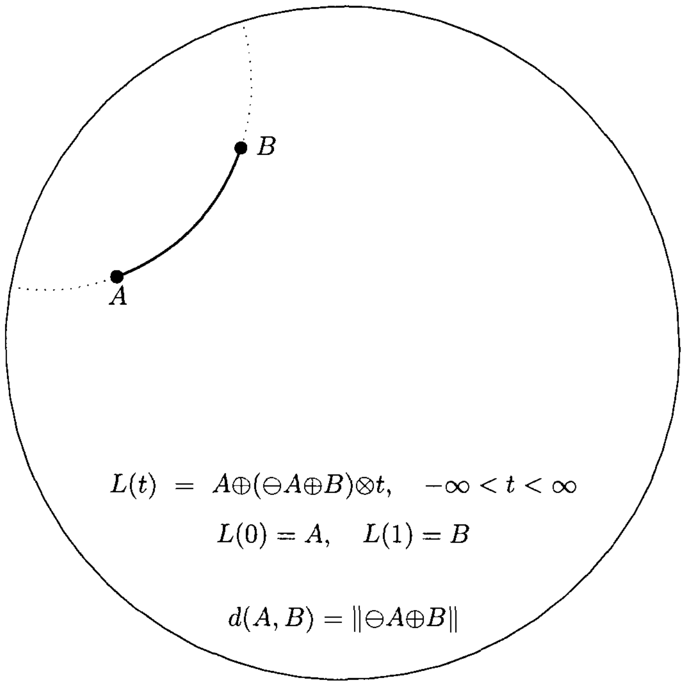

Fig. 1.1 Analytic generation of gyrolines (hyperbolic lines) and gyrosegments in the Poincaré disc (and ball) model of hyperbolic geometry in terms of Möbius addition $\oplus$ and scalar multiplication $\otimes$. The analogies between analytic lines and analytic gyrolines are obvious. Thus, for instance, the gyrodistance of B from A is $d(A,B) = \|\ominus A\oplus B\|$. Furthermore, $\mathbf{v} = \ominus A\oplus B$ is a gyrovector with tail A and head B. Two gyrovectors are shown in Fig. 1.5. As in Euclidean geometry, gyrovectors are equivalence classes of directed gyrosegments that add according to the gyroparallelogram law, as shown in Figs. 8.21 – 8.22 on p. 291.

Michael Kinyon notes in his MR:2003d:20109 review in *Math. Rev.* of Hubert Kiechle's nice introductory book on the "Theory of K-loops" [Kiechle (2002)], "It is unlikely that there will be any convergence of terminology in the near future."

Since the models of hyperbolic geometry are regulated algebraically by gyrovector spaces just as the standard model of Euclidean geometry is regulated algebraically by vector spaces, the theory of gyrogroups and gyrovector spaces develops in this book an internal ecology. It includes the special gyrolanguage, key examples, definitions and theorems, central themes, and a few gems, like those illustrated in Figs. 1.1 – 1.8 to amaze both the uninitiated and the practicing expert on hyperbolic geometry.

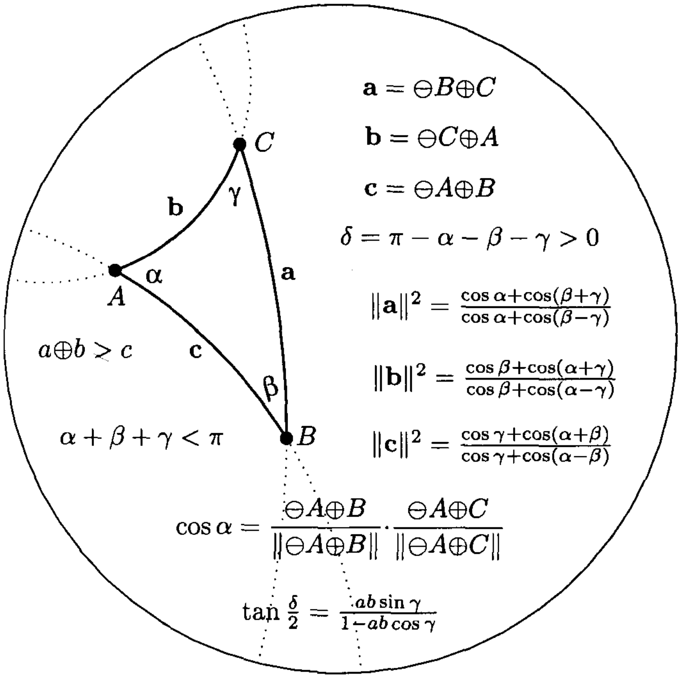

Fig. 1.2 Elegance and Beauty. A Möbius hyperbolic triangle ABC (that is, a gyrotriangle) in the Poincaré disc model of hyperbolic geometry (that is, in the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$, with Möbius addition $\oplus$ and scalar multiplication $\otimes$ in the open unit disc $\mathbb{R}^2_{s=1}$ of the Euclidean plane $\mathbb{R}^2$). Its vertices are the points A, B, C, and its sides are formed by corresponding gyrovectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, that link its vertices, in full analogy with Euclidean triangles. Its hyperbolic side lengths (that is, side gyrolengths), $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{c}\|$, are uniquely determined by its hyperbolic angles (gyroangles). Its gyrotriangle defect, δ, is determined by any two sides and their included gyroangle by an elegant identity.

1.3 Analytic Hyperbolic Geometry

One of the tasks of the geometer who is interested in analytic hyperbolic geometry is to construct mathematical models and a theory that correspond to elements of the relativistic and quantum physical world. The criteria for judging the success of our analytic hyperbolic geometry are generality, simplicity, and beauty. These are illustrated in Figs. 1.1 – 1.8 of this introductory chapter.

Figures 1.1 – 1.6 present hyperbolic geometric objects along with related hyperbolic geometric formulas. No knowledge of any of the formulas in this introductory chapter is assumed. These will be introduced and explained

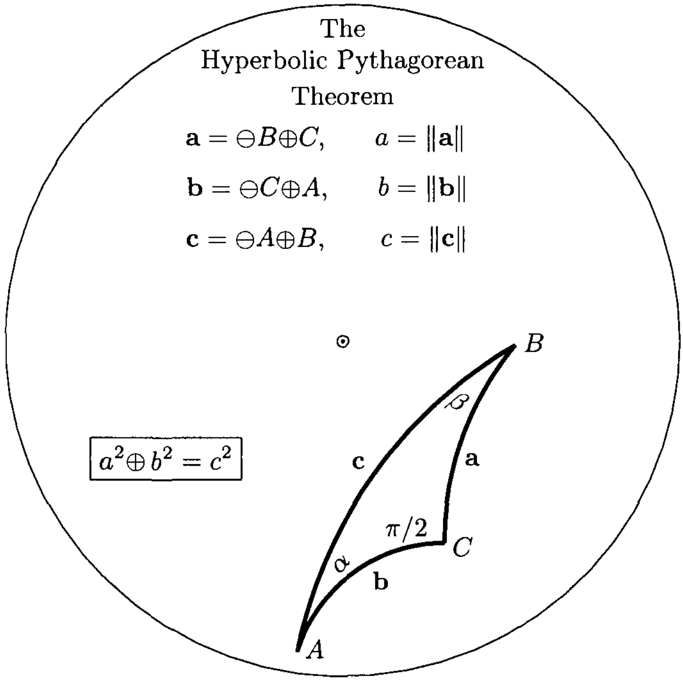

Fig. 1.3 The Hyperbolic Pythagorean Theorem in the Poincaré disc model of hyperbolic geometry (that is, in the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$ with Möbius addition $\oplus$ and scalar multiplication $\otimes$). Both graphically and symbolically the hyperbolic Pythagorean theorem shares visual analogies with its Euclidean counterpart. Classically, a different hyperbolic Pythagorean theorem appears in the literature in a form that shares no analogies with its Euclidean counterpart, leading authors [Wallace and West (1998)] to assert that "the Pythagorean theorem is strictly Euclidean" since "in the hyperbolic model the Pythagorean theorem is not valid." The ability of analytic hyperbolic geometry to capture a hyperbolic Pythagorean theorem which is fully analogous to its Euclidean counterpart is remarkable, allowing us to embark on gyrotrigonometry, a hyperbolic trigonometry fully analogous to the standard Euclidean trigonometry, shown in Fig. 1.4.

in the following chapters. The formulas involve the Möbius addition $\oplus$ and scalar multiplication $\otimes$ in the disc

$$\mathbb{R}^2_s = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < s\} \tag{1.1}$$

of the Euclidean plane $\mathbb{R}^2$, with radius $s > 0$ and center at the origin. Möbius addition $\oplus$ stems from the well known Möbius transformation without rotation of the complex open disc of radius $s > 0$, presented in Sec. 3.4, p. 72. Recalling that a groupoid is a nonempty set with a binary operation,

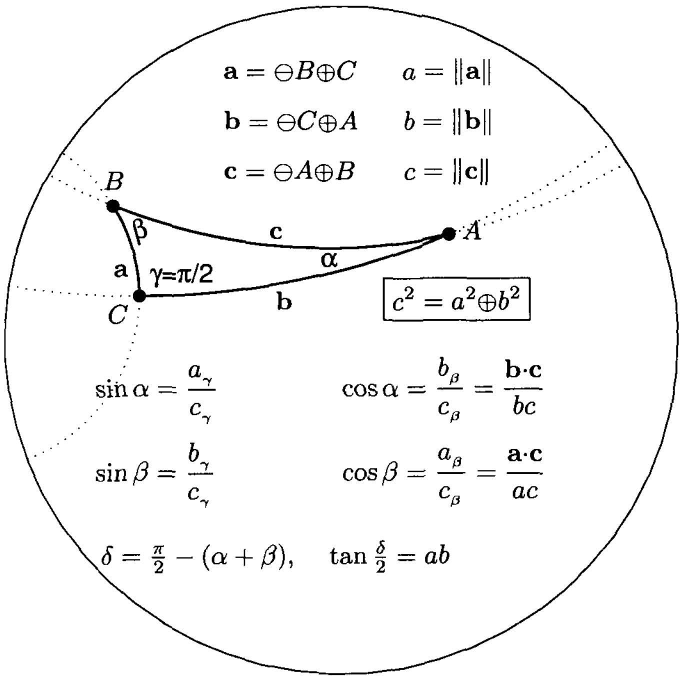

Fig. 1.4 Hyperbolic trigonometry (that is, gyrotrigonometry) in the Poincaré disc model of hyperbolic geometry (that is, in the Möbius gyrovector plane; or, more generally, in any gyrovector space with gyrogroup operation $\oplus$), fully analogous to standard trigonometry in vector spaces, is illustrated. It is particularly convenient to illustrate gyrotrigonometry in the Poincaré disc model since this model is conformal: the measure of the hyperbolic angle (gyroangle) included by two intersecting hyperbolic lines (gyrolines) in the Poincaré disc model equals the measure of the Euclidean angle included by two corresponding intersecting Euclidean tangent lines. Employing gyrotrigonometric identities, we verify in this book analytically the standard congruence theorems of hyperbolic geometry, known as the AAA, AAS, ASA, SAS, SsA, and SSS Gyrotriangle Congruence Theorems. All these congruence theorems are valid in both Euclidean and hyperbolic geometry with one exception. It is only the AAA congruency that is valid in hyperbolic geometry but invalid in Euclidean geometry.

Möbius addition $\oplus$ gives rise to the Möbius groupoid $(\mathbb{R}_s^2, \oplus)$. In the limit of large s the disc expands to the whole of its Euclidean plane $\mathbb{R}^2$, and its Möbius addition reduces to ordinary vector addition in the plane $\mathbb{R}^2$; see (3.127) on p. 75.

Möbius addition is neither commutative nor associative, but it gives rise to automorphisms of the Möbius groupoid $(\mathbb{R}_s^2, \oplus)$ that repair the two deficiencies as we will see in (1.2) – (1.7) below.

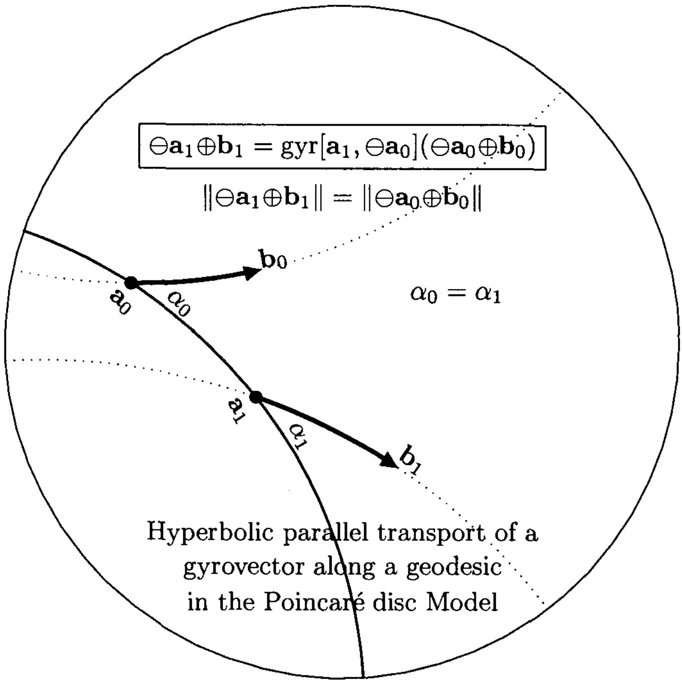

Fig. 1.5 A gyrovector is a curved vector. The hyperbolic parallel transport of a gyrovector involves a gyroautomorphism. The hyperbolic parallel transport of the gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, rooted at $\mathbf{a}_0$, to the gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$, rooted at $\mathbf{a}_1$, along the geodesic that links $\mathbf{a}_0$ and $\mathbf{a}_1$ in the Poincaré disc model of hyperbolic geometry (that is, in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ with Möbius addition $\oplus$ and scalar multiplication $\otimes$) is shown. Both graphically and symbolically the hyperbolic parallel transport shares visual analogies with its Euclidean counterpart. For instance, the geodesic passing through $\mathbf{a}_0$ and $\mathbf{a}_1$ is generated by the formula $\mathbf{a}_0\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t$, $t \in \mathbb{R}$, illustrated in Fig. 1.1.

For any $\mathbf{a}, \mathbf{b} \in \mathbb{R}_s^2$, let us consider the map $\mathrm{gyr}[\mathbf{a}, \mathbf{b}] : (\mathbb{R}_s^2, \oplus) \to (\mathbb{R}_s^2, \oplus)$ given by the equation

$$\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{v} = \ominus(\mathbf{a}\oplus\mathbf{b})\oplus(\mathbf{a}\oplus(\mathbf{b}\oplus\mathbf{v})) \tag{1.2}$$

The map $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ of the Möbius groupoid $(\mathbb{R}_s^2, \oplus)$ measures the nonassociativity of Möbius addition $\oplus$ in the disc $\mathbb{R}_s^2$. It becomes trivial, $\mathrm{gyr}[\mathbf{a}, \mathbf{b}] = I$, when $\oplus$ is associative.

Surprisingly, the map $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ turns out to be an *automorphism* of the Möbius groupoid. We recall that a map $\phi : \mathbb{R}_s^2 \to \mathbb{R}_s^2$ of the groupoid $(\mathbb{R}_s^2, \oplus)$ is an automorphism if it is bijective (that is, one-to-one) and preserves the groupoid binary operation, that is, $\phi(\mathbf{a}\oplus\mathbf{b}) = \phi(\mathbf{a})\oplus\phi(\mathbf{b})$.

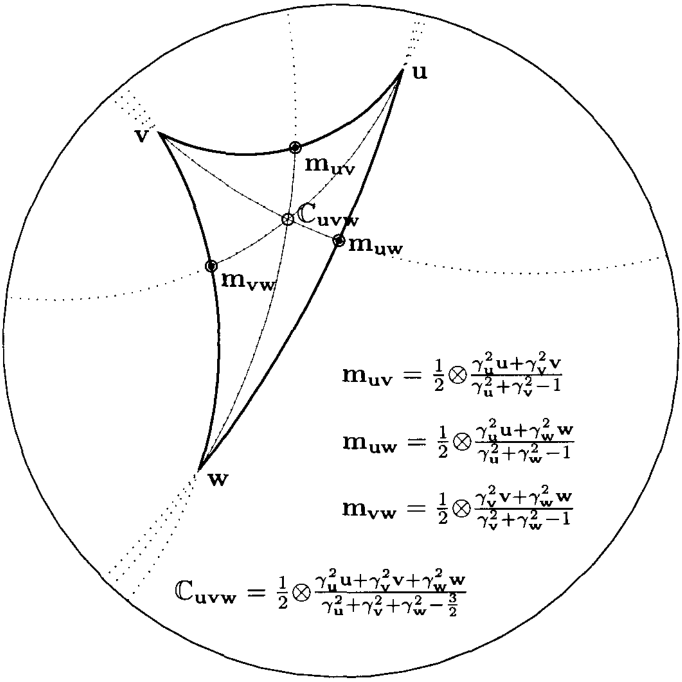

Fig. 1.6 A gyrotriangle $\mathbf{uvw}$ in the Poincaré disc model of hyperbolic geometry (that is, in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$) is shown with the gyromidpoints $\mathbf{m_{uv}}$, $\mathbf{m_{uw}}$ and $\mathbf{m_{vw}}$ of its sides, its gyromedians $\mathbf{um_{vw}}$, $\mathbf{vm_{uw}}$ and $\mathbf{wm_{uu}}$, and its gyrocentroid $\mathbb{C}_{\mathbf{uvw}}$. The gyrotriangle gyrocentroid, that is, the hyperbolic triangle centroid, is expressed in terms of the three gyrovectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ that form the gyrotriangle vertices, and their *gamma factors* $\gamma_{\mathbf{v}} = (1 - \mathbf{v}^2/s^2)^{-1/2}$, etc. Note that in the limit of large s, $s \to \infty$, the gyro-operations $\oplus$ and $\otimes$ reduce to their classical counterparts, vector addition and scalar multiplication, so that gyromidpoints reduce to corresponding midpoints, $\mathbf{m_{uv}} \to (\mathbf{u} + \mathbf{v})/2$, etc., and the gyrotriangle gyrocentroid reduces to a corresponding triangle centroid, $\mathbb{C}_{\mathbf{uvw}} \to (\mathbf{u} + \mathbf{v} + \mathbf{w})/3$; see Sec. 6.20. A translation of this figure from its Poincaré disc model into a corresponding one in the Beltrami (also known as the Klein) disc model gives the gyrotriangle gyrocentroid in Einstein gyrovector spaces and reveals remarkable analogies between classical and relativistic mechanics. In particular, the analogies that gyromidpoints and gyrocentroids capture reveal that the Einstein relativistic mass (which is velocity dependent) is nothing else but the gyro-Newtonian mass; see Chap. 10.

Furthermore, the resulting *gyroautomorphisms* $\text{gyr}[\mathbf{a}, \mathbf{b}]$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}_s^2$, "repair" the breakdown of commutativity and associativity in Möbius addition, giving rise to their gyro-counterparts, the *gyrocommutative law*,

$$\mathbf{a}\oplus\mathbf{b} = \text{gyr}[\mathbf{a}, \mathbf{b}](\mathbf{b}\oplus\mathbf{a})) \tag{1.3}$$

and the *gyroassociative law* (left and right),

$$\begin{aligned}\mathbf{a}\oplus(\mathbf{b}\oplus\mathbf{v}) &= (\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\\ (\mathbf{a}\oplus\mathbf{b})\oplus\mathbf{v} &= \mathbf{a}\oplus(\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{v})\end{aligned}\tag{1.4}$$

for all $\mathbf{a},\mathbf{b},\mathbf{v}\in\mathbb{R}_s^2$.

As in the gyroassociative and gyrocommutative laws, (1.3) – (1.4), and in the hyperbolic parallel transport, Fig. 1.5, the gyroautomorphisms capture remarkable analogies with classical results, allowing Euclidean and hyperbolic geometry to be united. In addition, the gyroautomorphisms have their own rich structure as we see, for instance, from the gyroautomorphism inversion property

$$(\mathrm{gyr}[\mathbf{a},\mathbf{b}])^{-1} = \mathrm{gyr}[\mathbf{b},\mathbf{a}]\tag{1.5}$$

from the loop property (left and right)

$$\begin{aligned}\mathrm{gyr}[\mathbf{a},\mathbf{b}] &= \mathrm{gyr}[\mathbf{a}\oplus\mathbf{b},\mathbf{b}]\\ \mathrm{gyr}[\mathbf{a},\mathbf{b}] &= \mathrm{gyr}[\mathbf{a},\mathbf{b}\oplus\mathbf{a}]\end{aligned}\tag{1.6}$$

and from the elegant nested gyroautomorphism identity

$$\mathrm{gyr}[\mathbf{a},\mathbf{b}] = \mathrm{gyr}[\ominus\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{b},\mathbf{a}]\tag{1.7}$$

$\mathbf{a},\mathbf{b}\in\mathbb{R}_s^2$, that they possess.

Owing to its gyroassociative and gyrocommutative laws, (1.3) – (1.4), Möbius addition is a grouplike operation. Suggestively, the key features of Möbius addition $\oplus$ give rise to the definition of the abstract gyrogroup (both gyrocommutative and non-gyrocommutative) in Defs. 2.5, p. 23, and 2.6, p. 24. Moreover, Möbius addition admits scalar multiplication, $\otimes$, Def. 6.80, p. 185, turning the Möbius gyrocommutative gyrogroup $(\mathbb{R}_s^2,\oplus)$ into a Möbius gyrovector plane $(\mathbb{R}_s^2,\oplus,\otimes)$. The use of Möbius addition and scalar multiplication to generate gyrolines analytically in a Möbius gyrovector plane is shown in Fig. 1.1. Möbius gyrolines are identical to the well known geodesics of the Poincaré disc model of hyperbolic geometry as we see, visually, in Fig. 1.1 and, analytically, in Sec. 7.3. Remarkably, the analytic generation of gyrolines in hyperbolic geometry is fully analogous to the analytic generation of lines in Euclidean geometry.

In gyrolanguage we prefix a gyro to any term that describes a concept in Euclidean geometry and in associative algebra to mean the analogous concept in hyperbolic geometry and nonassociative algebra. The prefix gyro stems from Thomas gyration. Thomas gyration, in turn, is a special automorphism abstracted from the relativistic effect known as Thomas precession. The destiny of Thomas precession in the foundations of hyperbolic geometry thus began to unfold following its extension by abstraction in [Ungar (1988a); Ungar (1988b); Ungar (1989b); Ungar (1989a)] since 1988.

Following the extension of groups and vector spaces of associative algebra and Euclidean geometry to nonassociative counterparts, gyrolanguage gives rise to gyroterms like gyrogroups and gyrovector spaces, gyrolines and gyroangles, of nonassociative algebra and hyperbolic geometry. Similarly, commutativity and associativity in associative algebra and Euclidean geometry are extended in gyrolanguage to gyrocommutativity and gyroassociativity in nonassociative algebra and hyperbolic geometry.

We sometimes abuse gyrolanguage a bit and drop the prefix gyro when it coincides with a classical term. Thus, for instance, elements of a gyrovector space are called points rather than gyropoints, as they should be called in gyrolanguage. But, "vectors" of a gyrovector space are called gyrovectors since they do not exist classically. Furthermore, we use the terms gyrogeodesics and (hyperbolic) geodesics interchangeably since, for instance, the gyrogeodesics (also called *gyrolines*) of Möbius gyrovector spaces are nothing else but the familiar geodesics of the Poincaré model of hyperbolic geometry.

The most impressive examples of the need to abuse gyrolanguage a bit come (i) from the gyro-Euclidean geometry, which is nothing else but the hyperbolic geometry of Bolyai and Lobachevsky and (ii) from the gyromass, which is nothing else but the Einstein relativistic mass. We certainly do not recommend to abandon the classical term "hyperbolic geometry" in favor of its gyrolanguage equivalent term "gyro-Euclidean geometry" and, similarly, we do not recommend to abandon the term "relativistic mass" in favor of its gyrolanguage equivalent term "gyro-mass".

In contrast, we find it useful to adopt the term "gyrotrigonometry". It is, in fact, hyperbolic trigonometry, but it is more similar, in terms of analogies, to Euclidean trigonometry than to traditional hyperbolic trigonometry, which is expressed in terms of the familiar hyperbolic functions cosh and sinh [McCleary (2002), p. 52].

Three other examples come from gyrolines, gyroangles, and gyrotriangles, which coincide with hyperbolic lines, hyperbolic angles, and hyperbolic triangles respectively. Thus, when a gyroterm in gyrolanguage coincides with a classical term, abuse of gyrolanguage may occur. Some gyroterms that coincide with classical terms cannot be abandoned since they come with dual counterparts that, classically, are not recognized as duals since their duality symmetries can only be captured by gyrotheoretic techniques. Thus, for instance, gyrolines, gyroangles, and gyrotriangles are associated with their corresponding dual counterparts, cogyrolines, cogyroangles, and cogyrotriangles.

Gyrolanguage abuse must be done with care, as the example of the gyrocosine function in Fig. 1.2 indicates. The definition of the gyrocosine of a gyroangle is presented in Fig. 1.2. We cannot view it as the "hyperbolic" cosine of a hyperbolic angle since the term "hyperbolic cosine" is already in use in a different sense. Abusing notation, we use the same notation for the trigonometric functions and their gyro-counterparts. Thus, for instance, the gyrocosine function in Fig. 1.2 is denoted by cos. This notation for the elementary gyrotrigonometric functions cos, sin, tan, etc. is justified since the gyrotrigonometric functions are interrelated by the same identities that interrelate the trigonometric functions. Thus, for instance, the trigonometric identity $\cos^2\alpha + \sin^2\alpha = 1$ (along with all other trigonometric identities between elementary trigonometric functions) remains valid in gyrotrigonometry as well. Furthermore, in the conformal model of the Poincaré ball, corresponding gyroangles and angles have the same measure, so that the elementary trigonometric functions are identical with their gyro-counterpart in all the hyperbolic models that are isomorphic (in the sense of gyro-algebra) to the Poincaré ball model, as verified in Theorem 8.3, p. 238.

1.4 The Three Models

There are infinitely many models of hyperbolic geometry. The three models that we study in this book are particularly interesting, as we describe below.

(I) The Poincaré ball model of hyperbolic geometry is algebraically regulated by Möbius gyrovector spaces where Möbius addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean circular arcs (with finite or infinite radius, the latter being diameters of the ball.) that intersect the boundary of the ball orthogonally, shown in Figs. 1.1 − 1.6 for the two-dimensional ball, that is, the disc. The model is conformal to the Euclidean model in the sense that the measure of the hyperbolic angle between two intersecting gyrolines is equal to the measure of the Euclidean angle between corresponding intersecting tangent lines, Figs. 8.1 − 8.3, pp. 240 − 242.

Möbius addition is a natural generalization of the Möbius transformation without rotation of the complex open unit disc from the theory of functions of a complex variable, as we will see in Sec. 3.5. Thus, although more than 150 years have passed since August Ferdinand Möbius first studied the transformations that now bear his name [Ahlfors (1984)], this book demonstrates that the rich structure he thereby exposed is still far from being exhausted.

(II) The Beltrami ball model of hyperbolic geometry is algebraically regulated by Einstein gyrovector spaces where Einstein addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean straight lines in the ball, Fig. 6.8, p. 196. Einstein addition, in turn, is the standard velocity addition of relativistically admissible velocities that Einstein introduced in his 1905 paper that founded the special theory of relativity. In this book, accordingly, the presentation of Einstein's special theory of relativity is solely based on Einstein velocity addition law, taking the reader to the immensity of the underlying hyperbolic geometry. Thus, 100 years after Einstein introduced the relativistic velocity addition law that now bears his name, this book demonstrates that placing Einstein velocity addition centrally in special relativity theory is an old idea whose time has come back.

The approach to special relativity from Einstein velocity addition fills a noticeable gap in the relativity physics arena. Thus, for instance,

(1) the seemingly notorious Thomas precession, which is either ignored or studied as an isolated phenomenon in most relativity physics books; and
(2) the seemingly confusing relativistic mass, which does not mesh up with Minkowskian relativity

mesh extraordinarily well with the analytic hyperbolic geometric approach to Einsteinian relativity [Ungar (2005)]. The term "Minkowskian relativity", as opposed to Einsteinian relativity, was coined by L. Pyenson in [Pyenson (1982), p. 146]. The historical struggle between Einsteinian relativity and Minkowskian relativity is skillfully described by S. Walter in [Walter (1999b)] where, for the first time, the term "Minkowskian relativity" appears in a title.

Rather than being notorious and confusing, Thomas precession and Einstein's relativistic mass provide unexpected insights that are not easy to come by, by means other than analytic hyperbolic geometric techniques, as Figs. 1.7 and 1.8 indicate. Hence, this 2005 book on analytic hyperbolic

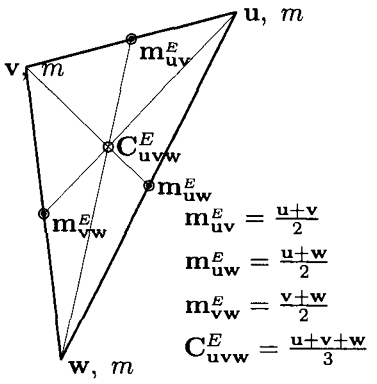

Fig. 1.7 Euclidean triangle centroid in a vector space, and its classical mechanics interpretation as a barycenter. The barycenter is the center of momentum in a Newtonian velocity space of three equal masses m located at the triangle vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}$. These masses have, accordingly, Newtonian velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ relative to some inertial rest frame. Following [Hausner (1998)], many Euclidean geometric facts may be made quite vivid and intuitive with the help of the center of momentum notion, as this Fig. 1.7 indicates. Fig. 1.8 indicates the natural extension to the hyperbolic triangle centroid.

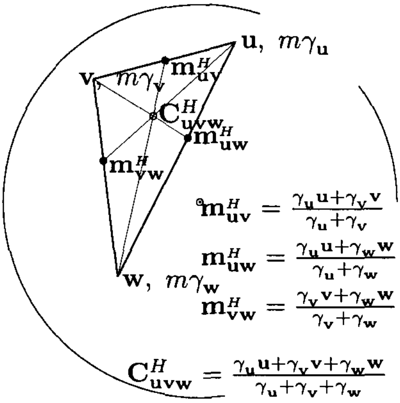

Fig. 1.8 Gyrotriangle gyrocentroid in an Einstein gyrovector space, and its relativistic mechanics interpretation as a gyrobarycenter. The gyrobarycenter is the relativistic center of momentum in an Einsteinian velocity space of three equal rest masses m located at the gyrotriangle vertices $\mathbf{u}, \mathbf{v}, \mathbf{w}$. These masses have Einsteinian velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ relative to some inertial rest frame, and they are, accordingly, relativistically corrected by corresponding Lorentz factors $\gamma_{\mathbf{u}}, \gamma_{\mathbf{v}}, \gamma_{\mathbf{w}}$. The gyrovector space approach to hyperbolic geometry has, thus, much to show in terms of creative power of discovery.

geometry is dedicated to the centenary of the birth of Einstein's special theory of relativity, 1905 – 2005.

The remarkable fit between geometry and physics that Figs. 1.7 and 1.8 exhibit is not fortuitous. It demonstrates that the relativistic mass plays in relativistic mechanics and its underlying hyperbolic geometry the same important role that the Newtonian mass plays in classical mechanics and its underlying Euclidean geometry. The relativistic mass is thus an asset rather than a liability. The relativistic center of momentum and gyrobarycentric coordinates associated with the relativistic mass are studied in Chap. 10.

(III) The PV space model of hyperbolic geometry (also called the Ungar model, a term coined by Jing-Ling Chen in 2001 [Chen and Ungar (2001)]) is governed by PV gyrovector spaces where PV addition plays a role analogous to the role that vector addition plays in vector spaces. The geodesics of this model (gyrolines) are Euclidean hyperbolas with asymptotes that intersect at the space origin, Fig. 6.12. PV addition turns out to be the "proper velocity" addition of proper velocities in special relativity. As opposed to (i) *coordinate velocity* in special relativity, measured by observer's time and composed by Einstein addition, (ii) *proper velocity* in special relativity is measured by traveler's time and composed by PV addition.

The power and elegance of the gyrovector space approach to hyperbolic geometry is convincingly illustrated by the analytic expressions that represent the (i) hyperbolic angle (gyroangle), Fig. 1.2; (ii) the hyperbolic Pythagorean theorem, Fig. 1.3; (iii) the hyperbolic trigonometry (gyrotrigonometry), Fig. 1.4; (iv) the hyperbolic parallel transport, Fig. 1.5; and (v) the hyperbolic triangle centroid (gyrotriangle gyrocentroid), side hyperbolic midpoints (gyromidpoints) and hyperbolic medians (gyromedians), Fig. 1.6, where in their analytic hyperbolic form they share symbolic and visual analogies with their Euclidean counterparts.

Along with remarkable analogies, a striking disanalogy is presented in Fig. 1.2, indicating quantitatively that unlike the Euclidean side lengths of a Euclidean triangle, the hyperbolic side lengths of a hyperbolic triangle are uniquely determined by its hyperbolic angles. In gyrolanguage we say that the side gyrolengths of a gyrotriangle in a gyrovector space are uniquely determined by the gyrotriangle gyroangles.

Aesthetic criteria are fundamental to the development of mathematical ideas [Penrose (2005), p. 22]. The conversion law from gyrotriangle gyroangles α, β, γ to their corresponding gyrotriangle side gyrolengths $\|\mathbf{a}\|, \|\mathbf{b}\|, \|\mathbf{c}\|$ in a gyrotriangle ABC is shown in Fig. 1.2, and in the AAA to SSS Conversion Theorem 8.48, p. 280. It presents an extraordinary unexpected hidden beauty that analytic hyperbolic geometry reveals. We thus encounter here one of the remarkable interrelations between truth and beauty, which are abound in the area of analytic hyperbolic geometry.

1.5 Applications in Quantum and Special Relativity Theory

The applicability in physics of the gyrovector space approach to hyperbolic geometry is demonstrated in Chaps. 9 and 10.

Chapter 9 demonstrates that Bloch vector of quantum computation theory is, in fact, a gyrovector rather than a vector. This discovery of the relationship between "Bloch vector" and the Poincaré model of hyperbolic geometry led Péter Lévay to realize in [Lévay (2004a)] and [Lévay (2004b)] that the so called *bures metric* in quantum computation is equivalent to the metric that results from the distance function $d(A, B)$ presented in Fig. 1.1.

Like Möbius addition, Einstein velocity addition is neither commutative nor associative. Hence, the study of special relativity in the literature follows the lines laid down by Minkowski, in which the role of Einstein velocity addition and its interpretation in the hyperbolic geometry of Bolyai and Lobachevsky are ignored [Barrett (1998)]. The breakdown of commutativity and associativity in Einstein velocity addition, thus, poses a significant problem. Einstein's opinion about significant problems in science is well known:

> The significant problems we have cannot be solved at the same level of thinking with which we created them.
>
> Albert Einstein (attributed)

Indeed, it is the gyrovector space approach to Einstein's special relativity and to hyperbolic geometry that resolves the significant problem of commutativity and associativity breakdown in Einstein velocity addition. In this novel approach,

(1) Einstein velocity addition emerges triumphant as a gyrocommutative, gyroassociative binary operation between gyrovectors in hyperbolic geometry; fully analogous to
(2) Newton velocity addition, which is a commutative, associative binary operation between vectors in Euclidean geometry.

Chapter 10 demonstrates that the gyrovector space approach, which unifies Euclidean and hyperbolic geometry, unifies some aspects of classical and relativistic mechanics as well. The way to unite the geometry and the physics of the concept of the center of momentum (CM), for instance, is indicated in Figs. 1.7 and 1.8.

In classical mechanics the CM of three equal masses with velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in a Newtonian velocity space $(\mathbb{R}^3, +, \cdot)$ is the centroid of triangle $\mathbf{uvw}$, Fig. 1.7. In full analogy, in relativistic mechanics the CM of three equal rest masses with velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in an Einsteinian velocity gyrospace $(\mathbb{R}^3_s, \oplus, \otimes)$ is the gyrocentroid of triangle $\mathbf{uvw}$, Fig. 1.8, where each of the three rest masses is relativistically corrected according to its individual velocity.

Accordingly, it is Einstein's relativistic mass correction that comes to the rescue of the analogies that triangle centroids and gyrotriangle gyrocentroids share in Figs. 1.7 and 1.8. These analogies are by no means restricted to equal rest masses. They are extended in Chap. 10 to analogies between barycentric and gyrobarycentric coordinates that correspond to arbitrary non-negative masses, thus seeing analytic hyperbolic geometry at work. By listening to the sounds of relativistic velocities and their Einstein velocity addition, analytic hyperbolic geometry significantly extends Einstein's unfinished symphony.

Chapter 2

Gyrogroups

The reason for starting a book on analytic hyperbolic geometry with chapters on gyrogroups and gyrovector spaces is that some gyrocommutative gyrogroups give rise to gyrovector spaces just as some commutative groups give rise to vector spaces. Gyrovector spaces, in turn, algebraically regulate analytic hyperbolic geometry just as vector spaces regulate algebraically analytic Euclidean geometry. To elaborate a precise language we prefix a gyro to any term that describes a concept in Euclidean geometry to mean the analogous concept in hyperbolic geometry. The prefix gyro stems from Thomas gyration which is, in turn, the mathematical abstraction of a special relativistic effect known as Thomas precession.

Developing gyrogroup and gyrovector space theoretic concepts and techniques, we will find that the hyperbolic geometry of Bolyai [Gray (2004)] and Lobachevsky is just the gyro-counterpart of Euclidean geometry. We start with the presentation of the concepts of gyroassociativity and gyrocommutativity of gyrogroup operations, that strikingly preserve the flavor of their classical counterparts. The extension of gyrocommutative gyrogroups into gyrovector spaces will be studied in Chap. 6, thus paving the way to our gyrovector space approach to analytic hyperbolic geometry, Chap. 8, and its applications, Chaps. 9–10. In gyrolanguage analytic hyperbolic geometry is a branch of *gyrogeometry*, and its trigonometry is called *gyrotrigonometry*. The link between gyrogeometry and the hyperbolic geometry of Bloyai and Lobachevsky is uncovered in Chap. 7 by elementary methods of differential geometry.

2.1 Definitions

Definition 2.1 (Binary Operations, Groupoids, and Groupoid Automorphisms). *A binary operation $+$ in a set S is a function $+ : S \times S \to S$. We use the notation $a + b$ to denote $+(a, b)$ for any $a, b \in S$. A groupoid $(S, +)$ is a nonempty set, S, with a binary operation, $+$. An automorphism ϕ of a groupoid $(S, +)$ is a bijective (that is, one-to-one) self-map of S, $\phi : S \to S$, which preserves its groupoid operation, that is, $\phi(a + b) = \phi(a) + \phi(b)$ for all $a, b \in S$.*

Groupoids may have identity elements. An identity element of a groupoid $(S, +)$ is an element $0 \in S$ such that $0 + s = s + 0 = s$ for all $s \in S$.

Definition 2.2 (Loops). *A loop is a groupoid $(S, +)$ with an identity element in which each of the two equations $a + x = b$ and $y + a = b$ for the unknowns x and y possesses a unique solution.*

Definition 2.3 (Groups). *A group is a groupoid $(G, +)$ whose binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $\qquad 0+a=a$

for all $a \in G$. There is an element $0 \in G$ satisfying axion (G1) such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying

(G2) $\qquad -a + a = 0$

Moreover, the binary operation obeys the associative law

(G3) $\qquad (a + b) + c = a + (b + c)$

for all $a, b, c \in G$.

The binary operation in a given set is known as the set operation. The set of all automorphisms of a groupoid $(S, \oplus)$, denoted $Aut(S, \oplus)$, forms a group with group operation given by bijection composition. The identity automorphism is denoted by I. We say that an automorphism τ is *trivial* if $\tau = I$.

Groups are classified into commutative and noncommutative groups.

Definition 2.4 (Commutative Groups). *A group $(G, +)$ is commutative if its binary operation obeys the commutative law*

(G6) $\qquad a + b = b + a$

for all $a, b \in G$.

A most natural, but hardly known, generalization of the group concept is the concept of the gyrogroup, the formal definition of which follows. Readers who, instructively, wish to see a good intuitive motivation for the gyro-extension of groups before embarking on the formal Def. 2.5 of gyrogroups may find it in the Möbius transformation of the disc, as presented in Sec. 3.4, p. 72.

Definition 2.5 (Gyrogroups). *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* G *there is at least one element,* 0, *called a left identity, satisfying*

(G1) $\qquad 0\oplus a = a$

for all $a \in G$. *There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* $a \in G$ *there is an element* $\ominus a \in G$, *called a left inverse of* a, *satisfying*

(G2) $\qquad \ominus a\oplus a = 0$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\qquad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c$

The map $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$,

(G4) $\qquad \mathrm{gyr}[a, b] \in Aut(G, \oplus)$

and the automorphism $\mathrm{gyr}[a, b]$ *of* G *is called the gyroautomorphism of* G *generated by* $a, b \in G$. *The operation* $\mathrm{gyr} : G \times G \to Aut(G, \oplus)$ *is called the gyrator of* G. *Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left loop property*

(G5) $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]$

The gyrogroup axioms in Def. 2.5 are classified into three classes.

(1) The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms;
(2) The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms; and
(3) The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation

$$a\ominus b = a\oplus(\ominus b) \tag{2.1}$$

in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 2.6 (Gyrocommutative Gyrogroups). *A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G6) $\qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

for all $a, b \in G$.

Definition 2.7 (The Gyrogroup Cooperation (Coaddition)). *Let $(G, \oplus)$ be a gyrogroup. The gyrogroup cooperation (or, coaddition) is a second binary operation, $\boxplus$, in G given by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{2.2}$$

for all $a, b \in G$.

We will find that the gyrogroup cooperation captures useful analogies between gyrogroups and groups, and uncovers duality symmetries.

The gyrogroup gyroautomorphisms are uniquely determined by the gyrogroup axioms, as we will see in Theorem 2.8. In the special case when all the gyrations of a (gyrocommutative) gyrogroup are trivial, the (gyrocommutative) gyrogroup reduces to a (commutative) group, where the gyrogroup operation and cooperation coincide, being jointly reduced to the group operation.

2.2 First Gyrogroup Theorems

While it is clear how to define right identity and right inverse, the existence of such elements is not presumed. Indeed, the existence of unique identity and unique inverse, both left and right, is a consequence of the gyrogroup axioms, as the following theorem shows.

Theorem 2.8 *Let $(G, +)$ be a gyrogroup. For any elements $a, b, c, x \in G$ we have:*

(1) If $a + b = a + c$, then $b = c$ (general left cancellation law; see (9)).
(2) $\mathrm{gyr}[0, a] = I$ for any left identity 0 in G.
(3) $\mathrm{gyr}[x, a] = I$ for any left inverse x of a in G.
(4) $\mathrm{gyr}[a, a] = I$
(5) There is a left identity which is a right identity.

(6) *There is only one left identity.*
(7) *Every left inverse is a right inverse.*
(8) *There is only one left inverse of* a.
(9) $-a + (a + b) = b$ *(Left Cancellation Law).*
(10) $\text{gyr}[a, b]x = -(a + b) + \{a + (b + x)\}$ *(The Gyrator Identity).*
(11) $\text{gyr}[a, b]0 = 0$
(12) $\text{gyr}[a, b](-x) = -\text{gyr}[a, b]x$
(13) $\text{gyr}[a, 0] = I$

Proof.

(1) Let x be a left inverse of a corresponding to a left identity, 0, in G. We have $x + (a + b) = x + (a + c)$. By left gyroassociativity, $(x + a) + \text{gyr}[x, a]b = (x + a) + \text{gyr}[x, a]c$. Since 0 is a left identity, $\text{gyr}[x, a]b = \text{gyr}[x, a]c$. Since automorphisms are bijective, $b = c$.
(2) By left gyroassociativity we have for any left identity 0 of G, $a + x = 0 + (a + x) = (0 + a) + \text{gyr}[0, a]x = a + \text{gyr}[0, a]x$. By (1) we then have $x = \text{gyr}[0, a]x$ for all $x \in G$ so that $\text{gyr}[0, a] = I$.
(3) By the left loop property and by (2) above we have $\text{gyr}[x, a] = \text{gyr}[x + a, a] = \text{gyr}[0, a] = I$.
(4) Follows from an application of the left loop property and (2) above.
(5) Let x be a left inverse of a corresponding to a left identity, 0, of G. Then by left gyroassociativity and (3) above, $x + (a + 0) = (x + a) + \text{gyr}[x, a]0 = 0 + 0 = 0 = x + a$. Hence, by (1), $a + 0 = a$ for all $a \in G$ so that 0 is a right identity.
(6) Suppose 0 and 0^* are two left identities, one of which, say 0, is also a right identity. Then $0 = 0^* + 0 = 0^*$.
(7) Let x be a left inverse of a. Then $x + (a + x) = (x + a) + \text{gyr}[x, a]x = 0 + x = x = x + 0$, by left gyroassociativity, (G2), (3), (5), and (6) above. By (1) we have $a + x = 0$ so that x is a right inverse of a.
(8) Suppose x and y are left inverses of a. By (7) above, they are also right inverses, so $a + x = 0 = a + y$. By (1), $x = y$.
(9) By left gyroassociativity and by (3) above, $-a + (a + b) = (-a + a) + \text{gyr}[-a, a]b = b$.
(10) Follows from an application of the left cancellation law (9) to the left gyroassociative law (G3).
(11) Follows from (10) with $x = 0$.
(12) Since $\text{gyr}[a, b]$ is an automorphism of $(G, +)$ we have from (11)

$\mathrm{gyr}[a,b](-x) + \mathrm{gyr}[a,b]x = \mathrm{gyr}[a,b](-x+x) = \mathrm{gyr}[a,b]0 = 0$, and hence the result.

(13) Follows from (10) with $b = 0$ and a left cancellation, (9).

□

Following items (2) and (13) of Theorem 2.8, the cooperation $\boxplus$ in a gyrogroup $(G, \oplus)$ satisfies

$$a \boxplus 0 = 0 \boxplus a = a \tag{2.3}$$

Using the abbreviations $a \boxminus b = a \boxplus (\ominus b)$ and $\boxminus a = 0 \boxminus a$, it follows from Def. 2.7 of $\boxplus$ that the cosubtraction takes the form

$$\begin{aligned} a \boxminus b &= a \boxplus (\ominus b) \\ &= a \oplus \mathrm{gyr}[a, \ominus(\ominus b)](\ominus b) \\ &= a \ominus \mathrm{gyr}[a, b]b \end{aligned} \tag{2.4}$$

and

$$\boxminus a = 0 \boxminus a = \ominus a \tag{2.5}$$

By (2.5) and Theorem 2.8 (4),

$$\begin{aligned} \boxminus a \boxplus a &= \ominus a \boxplus a \\ &= \ominus a \oplus \mathrm{gyr}[\ominus a, \ominus a]a \\ &= \ominus a \oplus a \\ &= 0 \end{aligned} \tag{2.6}$$

Similarly, by (2.5) and Theorem 2.8 (4),

$$\begin{aligned} a \boxplus (\boxminus a) &= a \boxplus (\ominus a) \\ &= a \oplus \mathrm{gyr}[a, a](\ominus a) \\ &= a \ominus a \\ &= 0 \end{aligned} \tag{2.7}$$

Hence, the operation $\oplus$ and the cooperation $\boxplus$ of a gyrogroup $(G, \oplus)$ share a common identity element, 0, and a common inversion,

$$\ominus a = \boxminus a \tag{2.8}$$

for all $a \in G$. These results about the gyrogroup cooperation will be formalized in Theorem 2.34.

The cooperation of a gyrogroup $(G, \oplus)$ or $(G, +)$ is denoted by $\boxplus$. The cooperation of a gyrogroup $(G, \oplus_{\mathrm{M}})$, for instance, will be denoted by $\boxplus_{\mathrm{M}}$, etc.

Theorem 2.9 *Any three elements a, b, c of a gyrogroup $(G, +)$ satisfy the nested gyroautomorphism identities*

$$\mathrm{gyr}[a, b + c]\mathrm{gyr}[b, c] = \mathrm{gyr}[a + b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b] \tag{2.9}$$

$$\mathrm{gyr}[a, -\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] = I \tag{2.10}$$

and the gyroautomorphism identities

$$\mathrm{gyr}[-a, a + b]\mathrm{gyr}[a, b] = I \tag{2.11}$$

$$\mathrm{gyr}[b, a + b]\mathrm{gyr}[a, b] = I \tag{2.12}$$

Proof. By two successive applications of the left gyroassociative law in two different ways, we obtain the following two chains of equations for all $a, b, c, x \in G$,

$$\begin{aligned} a + (b + (c + x)) &= a + ((b + c) + \mathrm{gyr}[b, c]x) \\ &= (a + (b + c)) + \mathrm{gyr}[a, b + c]\mathrm{gyr}[b, c]x \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} a + (b + (c + x)) &= (a + b) + \mathrm{gyr}[a, b](c + x) \\ &= (a + b) + (\mathrm{gyr}[a, b]c + \mathrm{gyr}[a, b]x) \\ &= ((a + b) + \mathrm{gyr}[a, b]c) + \mathrm{gyr}[a + b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b]x \\ &= (a + (b + c)) + \mathrm{gyr}[a + b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b]x \end{aligned} \tag{2.14}$$

By comparing the extreme right hand sides of these two chains of equations, and by employing the left cancellation law, Theorem 2.8 (1), we obtain (2.9).

In the special case when $c = -b$, (2.9) reduces to

$$I = \mathrm{gyr}[a + b, -\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \tag{2.15}$$

from which (2.10) follows by a left loop (that is, by applying the left loop property) and the left gyroassociative law,

$$\begin{aligned} I &= \mathrm{gyr}[a+b,\ -\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] \\ &= \mathrm{gyr}[(a+b)-\mathrm{gyr}[a,b]b, -\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] \\ &= \mathrm{gyr}[a+(b-b), -\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] \\ &= \mathrm{gyr}[a, -\mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b] \end{aligned} \tag{2.16}$$

To verify (2.11) we consider the special case of (2.9) when $b = -a$,

$$\mathrm{gyr}[a, -a+c]\mathrm{gyr}[-a, c] = \mathrm{gyr}[0, \mathrm{gyr}[a,-a]c]\mathrm{gyr}[a,-a] = I$$

Replacing a by $-a$ and c by b we obtain (2.11).

Finally, (2.12) is derived from (2.11) by left looping the first gyroautomorphism in (2.11) followed by a left cancellation, Theorem 2.8 (9),

$$\begin{aligned} I &= \mathrm{gyr}[-a, a+b]\mathrm{gyr}[a,b] \\ &= \mathrm{gyr}[-a+(a+b), a+b]\mathrm{gyr}[a,b] \\ &= \mathrm{gyr}[b, a+b]\mathrm{gyr}[a,b] \end{aligned} \tag{2.17}$$

□

The nested gyroautomorphism identity (2.10) in Theorem 2.9 allows the equation that defines the coaddition $\boxplus$ to be dualized as we see from the following

Theorem 2.10 *Let $(G, \oplus)$ be a gyrogroup with cooperation $\boxplus$ given by Def. 2.7,*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \tag{2.18}$$

Then

$$a \oplus b = a \boxplus \mathrm{gyr}[a, b]b \tag{2.19}$$

Proof. Let a and b be any two elements of G. By (2.10) we have

$$\begin{aligned} a \boxplus \mathrm{gyr}[a,b]b &= a \oplus \mathrm{gyr}[a, \ominus \mathrm{gyr}[a,b]b]\mathrm{gyr}[a,b]b \\ &= a \oplus b \end{aligned} \tag{2.20}$$

thus verifying (2.19). □

In view of the duality symmetry that Identities (2.18) and (2.19) share, the gyroautomorphism $\mathrm{gyr}[a, \ominus b]$ is called the *cogyroautomorphism* associated with the gyroautomorphism $\mathrm{gyr}[a, b]$.

2.3 The Associative Gyropolygonal Gyroaddition

As an application of the nested gyroautomorphism identity (2.15) we present in the next theorem the gyrogroup counterpart (2.21) of the group identity $(-a+b)+(-b+c) = -a+c$.

Theorem 2.11 *Let $(G,+)$ be a gyrogroup. Then*

$$(-a+b)+\mathrm{gyr}[-a,b](-b+c) = -a+c \tag{2.21}$$

for all $a,b,c \in G$.

Proof. By left gyroassociativity and (2.15) we have

$$\begin{aligned}(-a+b)+\mathrm{gyr}[-a,b](-b+c) &= (-a+b)+(-\mathrm{gyr}[-a,b]b+\mathrm{gyr}[-a,b]c)\\ &= \{(-a+b)-\mathrm{gyr}[-a,b]b\}+\mathrm{gyr}[-a+b,-\mathrm{gyr}[-a,b]b]\mathrm{gyr}[-a,b]c\\ &= \{-a+(b-b)\}+c\\ &= -a+c\end{aligned} \tag{2.22}$$

□

Theorem 2.12 **(The Gyrotranslation Theorem, I).** *Let $(G,+)$ be a gyrogroup. Then*

$$-(-a+b)+(-a+c) = \mathrm{gyr}[-a,b](-b+c) \tag{2.23}$$

for all $a,b,c \in G$.

Proof. The proof follows from Identity (2.21) and a left cancellation. □

The identity of Theorem 2.11 can readily be generalized to any number of terms, for instance,

$$(-a+b)+\mathrm{gyr}[-a,b]\{(-b+c)+\mathrm{gyr}[-b,c](-c+d)\} = -a+d \tag{2.24}$$

Theorem 2.11 suggests the following definition:

Definition 2.13 **(Gyropolygonal Gyroaddition of Adjacent Sides).** *Let $(G,+)$ be a gyrogroup, and let (a,b), $a,b \in G$ be a pair of two elements of G.*

(*i*) *The value of the pair (a,b) is $-a+b \in G$.*
(*ii*) *a and b are called the tail and the head of the pair (a,b), respectively.*
(*iii*) *Two pairs, (a,b) and (c,d), are adjacent if $b=c$.*

(iv) *A gyropolygonal path* $P(a_0, \ldots, a_n)$ *from a point* a_0 *to a point* a_n *in* G *is a finite sequence of successive adjacent pairs*

$$(a_0, a_1), (a_1, a_2), \ldots, (a_{n-2}, a_{n-1}), (a_{n-1}, a_n)$$

in G. *The pairs* (a_{k-1}, a_k), $k = 1, \ldots, n$, *are the sides of the gyropolygonal path* $P(a_0, \ldots, a_n)$, *and the points* $a_0, \ldots, a_n$ *are the vertices of the gyropolygonal path* $P(a_0, \ldots, a_n)$.

(v) *The gyropolygonal gyroaddition,* $\oplus$, *of two adjacent sides*

$$(a, b) = -a + b \quad \text{and} \quad (c, d) = -c + d$$

of a gyropolygonal path is given by the equation

$$(-a + b) \oplus (-b + c) = (-a + b) + \operatorname{gyr}[-a, b](-b + c)$$

We may note that two pairs with, algebraically, equal values need not be equal geometrically. Indeed, geometrically they are not equal if they have different tails (or, equivalently, different heads). To reconcile this seemingly conflict between algebra and geometry we will introduce in Chap. 5 equivalence classes of pairs in our way to convert pairs of points in a gyrocommutative gyrogroup to *gyrovectors* and, similarly, to *cogyrovectors*.

Following Def. 2.13, the identity of Theorem 2.11 can be written as the identity

$$(\ominus a \oplus b) \oplus (\ominus b \oplus c) = \ominus a \oplus c \tag{2.25}$$

in a gyrogroup $(G, \oplus)$.

Theorem 2.14 *The gyropolygonal gyroaddition is associative in any gyrogroup* $(G, +)$.

Proof. On the one hand

$$(-a + b) \oplus \{(-b + c) \oplus (-c + d)\} = (-a + b) \oplus (-b + d) = -a + d$$

and on the other hand

$$\{(-a + b) \oplus (-b + c)\} \oplus (-c + d) = (-a + c) \oplus (-c + d) = -a + d$$ □

The gyropolygonal gyrosubtraction is just the gyropolygonal gyroaddition in the reversed direction along the gyropolygonal path. Thus, for instance, left gyrosubtracting gyropolygonally $\ominus a \oplus b$ from both sides of (2.25) amounts to left adding $\ominus b \oplus a$ to both sides of (2.25),

$$(\ominus b \oplus a) \oplus (\ominus a \oplus b) \oplus (\ominus b \oplus c) = (\ominus b \oplus a) \oplus (\ominus a \oplus c) \tag{2.26}$$

Identity (2.26), in turn, is equivalent to

$$\ominus b \oplus c = (\ominus b \oplus a) \diamondplus (\ominus a \oplus c) \tag{2.27}$$

owing to the associativity of the gyropolygonal gyroaddition and to the identity

$$(\ominus b \oplus a) \diamondplus (\ominus a \oplus b) = \ominus b \oplus b = 0 \tag{2.28}$$

Interestingly, Theorem 2.14 uncovers an associative addition, the gyropolygonal gyroaddition, defined under special circumstances in the nonassociative environment of the gyrogroup.

2.4 Two Basic Gyrogroup Equations and Cancellation Laws

We wish to solve the equation

$$a \oplus x = b \tag{2.29}$$

in a gyrogroup $(G, \oplus)$ for the unknown x. Assuming that a solution x exists, we have by the left cancellation law, Theorem 2.8 (9),

$$\begin{aligned} \ominus a \oplus b &= \ominus a \oplus (a \oplus x) \\ &= x \end{aligned} \tag{2.30}$$

Conversely, if $x = \ominus a \oplus b$ then it is a solution of (2.29) as we see by substitution followed by a left cancellation.

We now wish to solve the slightly different equation

$$x \oplus a = b \tag{2.31}$$

in the gyrogroup $(G, \oplus)$ for the unknown x. Assuming that a solution x exists, we have the following chain of equations

$$\begin{aligned} x &= x \oplus 0 \\ &= x \oplus (a \ominus a) \\ &= (x \oplus a) \oplus \mathrm{gyr}[x, a](\ominus a) \\ &= (x \oplus a) \ominus \mathrm{gyr}[x, a]a \\ &= (x \oplus a) \ominus \mathrm{gyr}[x \oplus a, a]a \\ &= b \ominus \mathrm{gyr}[b, a]a \\ &= b \boxminus a \end{aligned} \tag{2.32}$$

where we employ (i) the identity element 0 of G; (ii) the left gyroassociative law; (iii) property (12) in Theorem 2.8; (iv) the left loop property; and finally (v) we eliminate x by means of its equation (2.31), and use the notation in (2.4). Hence, if (2.31) possesses a solution, it must be the one given by (2.32).

Conversely, substituting x from (2.32) in its equation (2.31), we have by the nested gyroautomorphism identity (2.10) and the left gyroassociative law

$$\begin{aligned} x \oplus a &= (b \boxminus a) \oplus a \\ &= (b \ominus \mathrm{gyr}[b, a]a) \oplus a \\ &= (b \ominus \mathrm{gyr}[b, a]a) \oplus \mathrm{gyr}[b, \ominus \mathrm{gyr}[b, a]a]\mathrm{gyr}[b, a]a \\ &= b \oplus (\ominus \mathrm{gyr}[b, a]a \oplus \mathrm{gyr}[b, a]a) \\ &= b \oplus 0 \\ &= b \end{aligned} \tag{2.33}$$

as desired.

Formalizing the results in (2.29) – (2.33) we have the following theorem:

Theorem 2.15 *Let $(G, \oplus)$ be a gyrogroup, and let $a, b \in G$. The unique solution of the equation*

$$a \oplus x = b \tag{2.34}$$

in G for the unknown x is

$$x = \ominus a \oplus b \tag{2.35}$$

and the unique solution of the equation

$$x \oplus a = b \tag{2.36}$$

in G for the unknown x is

$$x = b \boxminus a \tag{2.37}$$

Having established the unique solution of each of the gyrogroup equations (2.29) and (2.31), we see that gyrogroups are loops. Indeed, gyrogroups are special loops that share remarkable analogies with groups. This, in turn, explains the origin of the term "loop property". It is owing to that property that gyrogroups are loops, as we see from (2.32). Indeed, it is clear from (2.32) that it is the loop property that makes the left gyroassociative law effective in solving a basic gyrogroup equation.

Substituting the solution x from (2.30) in its equation (2.29) we obtain the left cancellation law

$$a\oplus(\ominus a\oplus b) = b \tag{2.38}$$

already established in Theorem 2.8(9).

Similarly, substituting the solution x from (2.32) in its equation (2.31) we obtain a right cancellation law

$$(b \boxminus a)\oplus a = b \tag{2.39}$$

The right cancellation law (2.39) can be dualized,

$$(b\ominus a) \boxplus a = b \tag{2.40}$$

as we see from the chain of equations

$$\begin{aligned} b &= b\oplus 0 \\ &= b\oplus(\ominus a\oplus a) \\ &= (b\ominus a)\oplus \mathrm{gyr}[b, \ominus a]a \\ &= (b\ominus a)\oplus \mathrm{gyr}[b\ominus a, \ominus a]a \\ &= (b\ominus a) \boxplus a \end{aligned} \tag{2.41}$$

where we employ the left gyroassociative law and the left loop property.

The cancellation laws in (2.38), (2.39) and (2.40) demonstrate that in order to capture analogies with classical results, both the gyrogroup operation and its associated cooperation are necessary. The various cancellation laws are shown in Table 2.1.

Table 2.1 Main Cancellation Laws. Unlike groups, there are various cancellation laws in gyrogroups $(G, \oplus)$. To capture analogies with groups, both the gyrogroup operation and cooperation are needed.

Formula	Terminology	Source
$a\oplus(\ominus a\oplus b) = b$	Left Cancellation Law	Eq. (2.38)
$(b\ominus a) \boxplus a = b$	(First) Right Cancellation Law	Eq. (2.40)
$(b \boxminus a)\oplus a = b$	(Second) Right Cancellation Law	Eq. (2.39)

The use of the right cancellation law is exemplified in the proof of the following theorem:

Theorem 2.16 **(The Cogyrotranslation Theorem).** *Let $(G, \oplus)$ be a gyrogroup. Then for all $a, b, x \in G$,*

$$(a \oplus \text{gyr}[a, b]x) \boxminus (b \oplus x) = a \boxminus b \tag{2.42}$$

In particular, if the condition

$$\text{gyr}[a, b] = I \tag{2.43}$$

holds, then

$$(a \oplus x) \boxminus (b \oplus x) = a \boxminus b = a \ominus b \tag{2.44}$$

Proof. By the left gyroassociative law, a right cancellation, and a left loop followed by a right cancellation we have

$$\begin{aligned}(a \boxminus b) \oplus (b \oplus x) &= ((a \boxminus b) \oplus b) \oplus \text{gyr}[a \boxminus b, b]x \\ &= a \oplus \text{gyr}[a, b]x\end{aligned} \tag{2.45}$$

from which (2.42) follows by a right cancellation.

The special case (2.44) follows from the condition $\text{gyr}[a, b] = I$ and from (2.42) and (2.4). □

Theorem 2.17 *Let $(G, +)$ be a gyrogroup. Then*

$$(a \boxminus b) + (b \boxminus c) = a - \text{gyr}[a, b]\text{gyr}[b, c]c \tag{2.46}$$

for all $a, b, c \in G$.

Proof. It follows from the definition of the gyrogroup cooperation and (2.45) that

$$\begin{aligned}(a \boxminus b) + (b \boxminus c) &= (a \boxminus b) + (b - \text{gyr}[b, c]c) \\ &= a + \text{gyr}[a, b](-\text{gyr}[b, c]c) \\ &= a - \text{gyr}[a, b]\text{gyr}[b, c]c\end{aligned} \tag{2.47}$$

□

Definition 2.18 **(Left and Right Gyrotranslations).** *Let $(G, \oplus)$ be a gyrogroup. Gyrotranslations in G are the self-maps of G given by*

$L_x :$	$a \mapsto x \oplus a$	Left Gyrotranslation of a by x
$R_x :$	$a \mapsto a \oplus x$	Right Gyrotranslation of a by x

By Theorem 2.15, gyrotranslations are bijective.

2.5 Commuting Automorphisms with Gyroautomorphisms

Theorem 2.19 *For any two elements a, b of a gyrogroup $(G, +)$ and any automorphism A of $(G, +)$, $A \in Aut(G, +)$,*

$$A\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab]A \tag{2.48}$$

Proof. For any three elements $a, b, x \in (G, +)$ and any automorphism $A \in Aut(G, +)$ we have by the left gyroassociative law,

$$\begin{aligned}(Aa + Ab) + A\mathrm{gyr}[a, b]x &= A((a + b) + \mathrm{gyr}[a, b]x) \\ &= A(a + (b + x)) \\ &= Aa + (Ab + Ax) \\ &= (Aa + Ab) + \mathrm{gyr}[Aa, Ab]Ax\end{aligned} \tag{2.49}$$

Hence, by a left cancellation, Theorem 2.8(1),

$$A\mathrm{gyr}[a, b]x = \mathrm{gyr}[Aa, Ab]Ax$$

for all $x \in G$, implying (2.48). □

As an application of Theorem 2.19 we have the following

Theorem 2.20 *Let a, b be any two elements of a gyrogroup $(G, +)$ and let $A \in Aut(G)$ be an automorphism of G. Then*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[Aa, Ab]$$

if and only if the automorphisms A and $\mathrm{gyr}[a, b]$ commute.

Proof. If $\mathrm{gyr}[Aa, Ab] = \mathrm{gyr}[a, b]$ then by Theorem 2.19 the automorphisms $\mathrm{gyr}[a, b]$ and A commute. Conversely, if $\mathrm{gyr}[a, b]$ and A commute then by Theorem 2.19 $\mathrm{gyr}[Aa, Ab] = A\mathrm{gyr}[a, b]A^{-1} = \mathrm{gyr}[a, b]$. □

As a simple, but useful, consequence of Theorem 2.20 we note the identity

$$\mathrm{gyr}[\mathrm{gyr}[a, b]a, \mathrm{gyr}[a, b]b] = \mathrm{gyr}[a, b] \tag{2.50}$$

As another application of Theorem 2.19 we have the following

Theorem 2.21 *A gyrogroup $(G, \oplus)$ and the groupoid $(G, \boxplus)$ of its gyrogroup cooperation possess the same automorphism group,*

$$Aut(G, \boxplus) = Aut(G, \oplus) \tag{2.51}$$

Proof. Let $\tau \in Aut(G, \oplus)$. Then by Theorem 2.19

$$\begin{aligned}\tau(a \boxplus b) &= \tau(a \oplus \mathrm{gyr}[a, \ominus b]b) \\ &= \tau a \oplus \tau \mathrm{gyr}[a, \ominus b]b \\ &= \tau a \oplus \mathrm{gyr}[\tau a, \ominus \tau b]\tau b \\ &= \tau a \boxplus \tau b\end{aligned} \tag{2.52}$$

so that $\tau \in Aut(G, \boxplus)$, implying

$$Aut(G, \boxplus) \supseteqq Aut(G, \oplus) \tag{2.53}$$

Conversely, let $\tau \in Aut(G, \boxplus)$. Then, owing to the first right cancellation law, (2.40), and (2.52), we have

$$\begin{aligned}\tau a &= \tau((a \oplus b) \boxminus b) \\ &= \tau(a \oplus b) \boxminus \tau b\end{aligned} \tag{2.54}$$

so that by the second right cancellation law, (2.39),

$$\tau(a \oplus b) = \tau a \oplus \tau b \tag{2.55}$$

Hence, $\tau \in Aut(G, \oplus)$, implying

$$Aut(G, \boxplus) \subseteqq Aut(G, \oplus) \tag{2.56}$$

so that by (2.53) and (2.56), we have the the desired equality

$$Aut(G, \boxplus) = Aut(G, \oplus) \tag{2.57}$$

□

Theorem 2.21 enhances the duality symmetry in Theorem 2.10.

2.6 The Gyrosemidirect Product Group

Definition 2.22 (Gyroautomorphism Groups, Gyrosemidirect Product Groups). *Let $G = (G, +)$ be a gyrogroup, and let $Aut(G) = Aut(G, +)$ be the automorphism group of G. A gyroautomorphism group, $Aut_0(G)$, of G is any subgroup of $Aut(G)$ containing all the gyroautomorphisms* $\mathrm{gyr}[a, b]$ *of G, $a, b \in G$. The gyrosemidirect product group*

$$G \times Aut_0(G) \tag{2.58}$$

of a gyrogroup G and any gyroautomorphism group, $Aut_0(G)$, is a group of pairs (x, X), where $x \in G$ and $X \in Aut_0(G)$, with operation given by the gyrosemidirect product

$$(x, X)(y, Y) = (x + Xy, \operatorname{gyr}[x, Xy]XY) \tag{2.59}$$

In analogy with the notion of the semidirect product in group theory, the gyrosemidirect product group

$$G \times Aut(G) \tag{2.60}$$

is called the gyroholomorph of G.

It is anticipated in Def. 2.22 that the gyrosemidirect product (2.58) of a gyrogroup and any one of its gyroautomorphism groups is a set that forms a group. In the following theorem we show that this is indeed the case.

Theorem 2.23 *Let $(G, +)$ be a gyrogroup, and let $Aut_0(G, +)$ be a gyroautomorphism group of G. Then the gyrosemidirect product $G \times Aut_0(G)$ is a group, with group operation given by the gyrosemidirect product* (2.59).

Proof. We will show that the set $G \times Aut_0(G)$ with its binary operation (2.59) satisfies the group axioms.
(i) Existence of a left identity: A left identity element of $G \times Aut_0(G)$ is the pair $(0, I)$, where 0 is the identity element of G, and I is the identity automorphism of G,

$$\begin{aligned}(0, I)(a, A) &= (0 + Ia, \operatorname{gyr}[0, Ia]IA) \\ &= (a, A)\end{aligned} \tag{2.61}$$

(ii) Existence of a left inverse: A left inverse of $(a, A) \in G \times Aut_0(G)$ is the pair $(-A^{-1}a, A^{-1})$, where A^{-1} is the inverse automorphism of A,

$$\begin{aligned}(-A^{-1}a, A^{-1})(a, A) &= (-A^{-1}a + A^{-1}a, \operatorname{gyr}[-A^{-1}a, A^{-1}a]A^{-1}A) \\ &= (0, I)\end{aligned} \tag{2.62}$$

(iii) Validity of the associative law: We have to show that

(1) the two successive products

$$\begin{aligned}
&(a_1, A_1)((a_2, A_2)(a_3, A_3)) \\
&= (a_1, A_1)(a_2 + A_2a_3, \text{gyr}[a_2, A_2a_3]A_2A_3) \\
&= (a_1 + A_1(a_2 + A_2a_3), \text{ gyr}[a_1, A_1(a_2 + A_2a_3)]A_1\text{gyr}[a_2, A_2a_3]A_2A_3) \\
&= (a_1 + (A_1a_2 + A_1A_2a_3), \\
&\qquad\qquad \text{gyr}[a_1,\ A_1a_2 + A_1A_2a_3]\text{gyr}[A_1a_2,\ A_1A_2a_3]A_1A_2A_3)
\end{aligned} \tag{2.63}$$

where we employ (2.59) and (2.48), and
(2) the two successive products

$$\begin{aligned}
&((a_1, A_1)(a_2, A_2))(a_3, A_3) \\
&\quad= (a_1 + A_1a_2,\ \text{gyr}[a_1,\ A_1a_2]A_1A_2)(a_3, A_3) \\
&\quad= ((a_1 + A_1a_2) + \text{gyr}[a_1,\ A_1a_2]A_1A_2a_3, \\
&\qquad \text{gyr}[a_1 + A_1a_2, \text{gyr}[a_1,\ A_1a_2]A_1A_2a_3]\text{gyr}[a_1,\ A_1a_2]A_1A_2A_3)
\end{aligned} \tag{2.64}$$

where we employ (2.59), are identically equal. Hence, using the notation

$$\begin{aligned}
a_1 &= a \\
A_1a_2 &= b \\
A_1A_2A_3 &= c
\end{aligned} \tag{2.65}$$

we have to establish the identity

$$\begin{aligned}
&(a + (b + c),\ \text{gyr}[a, b + c]\text{gyr}[b, c]A_1A_2A_3) \\
&\quad= ((a + b) + \text{gyr}[a, b]c,\ \text{gyr}[a + b,\ \text{gyr}[a, b]c]\text{gyr}[a, b]A_1A_2A_3)
\end{aligned} \tag{2.66}$$

This identity between two pairs is equivalent to the two identities between their corresponding entries,

$$\begin{aligned}
a + (b + c) &= (a + b) + \text{gyr}[a, b]c \\
\text{gyr}[a, b + c]\text{gyr}[b, c] &= \text{gyr}[a + b,\ \text{gyr}[a, b]c]\text{gyr}[a, b]
\end{aligned} \tag{2.67}$$

The first identity is valid, being the left gyroassociative law, and the second identity is valid by (2.9). □

The gyrosemidirect product group enables problems in gyrogroups to be converted to the group setting thus gaining access to the powerful group theoretic techniques. An illustrative example is provided by the following

Theorem 2.24 *Let $(G, +)$ be a gyrogroup, let $a, b \in G$ be any two elements of G, and let $Y \in Aut(G)$ be any automorphism of $(G, +)$. Then, the unique solution of the automorphism equation*

$$Y = -\mathrm{gyr}[b, Xa]X \tag{2.68}$$

for the unknown automorphism $X \in Aut(G)$ is

$$X = -\mathrm{gyr}[b, Ya]Y \tag{2.69}$$

Proof. Let X be a solution of (2.68), and let $x \in G$ be given by the equation

$$x = b \boxminus Xa \tag{2.70}$$

so that, by a right cancellation, (2.39), $b = x + Xa$.

Then we have the following gyrosemidirect product

$$\begin{aligned}(x, X)(a, I) &= (x + Xa, \mathrm{gyr}[x, Xa]X) \\ &= (x + Xa, \mathrm{gyr}[x + Xa, Xa]X) \\ &= (b, \mathrm{gyr}[b, Xa]X) \\ &= (b, -Y)\end{aligned} \tag{2.71}$$

so that

$$\begin{aligned}(x, X) &= (b, -Y)(a, I)^{-1} \\ &= (b, -Y)(-a, I) \\ &= (b + Ya, -\mathrm{gyr}[b, Ya]Y)\end{aligned} \tag{2.72}$$

Comparing the second entries of the extreme sides of (2.72) we have

$$X = -\mathrm{gyr}[b, Ya]Y \tag{2.73}$$

Hence, if a solution X of (2.68) exists, then it must be given by (2.69).

Conversely, the automorphism X in (2.73) is, indeed, a solution of (2.68) as we see by substituting (2.73) in (2.68) and employing the nested gyration identity (2.10),

$$-\mathrm{gyr}[b, Xa]X = \mathrm{gyr}[b, -\mathrm{gyr}[b, Ya]Ya]\mathrm{gyr}[b, Ya]Y = Y \tag{2.74}$$

$\square$

2.7 Basic Gyration Properties

We use the notation

$$(\mathrm{gyr}[a,b])^{-1} = \mathrm{gyr}^{-1}[a,b] \tag{2.75}$$

for the inverse gyroautomorphism.

Theorem 2.25 **(Gyrosum Inversion, Gyroautomorphism Inversion).** *For any two elements a, b of a gyrogroup $(G, +)$ we have the gyrosum inversion law*

$$-(a+b) = \mathrm{gyr}[a,b](-b-a) \tag{2.76}$$

and the gyroautomorphism inversion law

$$\mathrm{gyr}^{-1}[a;b] = \mathrm{gyr}[-b,-a] \tag{2.77}$$

Proof. Being a group, the product of two elements of the gyrosemidirect product group $G \times Aut_0(G)$ has a unique inverse. It can be calculated in two different ways.

On the one hand, the inverse of the left hand side of the product

$$(a,I)(b,I) = (a+b, \mathrm{gyr}[a,b]) \tag{2.78}$$

in $G \times Aut_0(G)$ is

$$\begin{aligned}(b,I)^{-1}(a,I)^{-1} &= (-b,I)(-a,I) \\ &= (-b-a, \mathrm{gyr}[-b,-a])\end{aligned} \tag{2.79}$$

On the other hand, the inverse of the right hand side of the product (2.78) is

$$(-\mathrm{gyr}^{-1}[a,b](a+b),\ \mathrm{gyr}^{-1}[a,b]) \tag{2.80}$$

for all $a, b \in G$. Comparing corresponding entries in (2.79) and (2.80) we have

$$-b-a = -\mathrm{gyr}^{-1}[a,b](a+b) \tag{2.81}$$

$$\mathrm{gyr}[-b,-a] = \mathrm{gyr}^{-1}[a,b] \tag{2.82}$$

Eliminating $\mathrm{gyr}^{-1}[a,b]$ between (2.81) and (2.82) yields

$$-b-a = -\mathrm{gyr}[-b,-a](a+b) \tag{2.83}$$

Replacing (a, b) by $(-b, -a)$, (2.83) becomes

$$a + b = -\text{gyr}[a, b](-b - a) \tag{2.84}$$

Identities (2.84) and (2.82) complete the proof. □

Instructively, the gyrosum inversion law is verified in Theorem 2.25 in terms of the gyrosemidirect product group. A direct proof is however simpler. By the gyroautomorphism identity in Theorem 2.8 (10) we have

$$\begin{aligned}\text{gyr}[a, b](-b - a) &= -(a \oplus b) \oplus (a \oplus (b \oplus (-b \ominus a))) \\ &= -(a \oplus b) \oplus (a \ominus a) \\ &= -(a \oplus b)\end{aligned} \tag{2.85}$$

Lemma 2.26 *Let $(G, +)$ be a gyrogroup. Then for all $a, b \in G$*

$$\text{gyr}^{-1}[a, b] = \text{gyr}[a, -\text{gyr}[a, b]b] \tag{2.86}$$

$$\text{gyr}^{-1}[a, b] = \text{gyr}[-a, a + b] \tag{2.87}$$

$$\text{gyr}^{-1}[a, b] = \text{gyr}[b, a + b] \tag{2.88}$$

$$\text{gyr}[a, b] = \text{gyr}[b, -b - a] \tag{2.89}$$

$$\text{gyr}[a, b] = \text{gyr}[-a, -b - a] \tag{2.90}$$

$$\text{gyr}[a, b] = \text{gyr}[-(a + b), a] \tag{2.91}$$

Proof. Identity (2.86) follows from (2.10). Identity (2.87) follows from (2.11). Identity (2.88) results from an application to (2.87) of the left loop property followed by a left cancellation. Identity (2.89) follows from (2.77) and (2.87),

$$\text{gyr}[a, b] = \text{gyr}^{-1}[-b, -a] = \text{gyr}[b, -b - a]$$

Identity (2.90) follows from an application, to the right hand side of (2.89), of the left loop property followed by a left cancellation. Identity (2.91) follows by inverting (2.87) by means of (2.77). □

Theorem 2.27 *The gyroautomorphisms of any gyrogroup $(G, +)$ are even,*

$$\text{gyr}[-a, -b] = \text{gyr}[a, b] \tag{2.92}$$

and inversive symmetric,

$$\text{gyr}^{-1}[a, b] = \text{gyr}[b, a] \tag{2.93}$$

satisfying the four mutually equivalent nested gyroautomorphism

$$\begin{aligned} \operatorname{gyr}[b, -\operatorname{gyr}[b, a]a] &= \operatorname{gyr}[a, b] \\ \operatorname{gyr}[b, \operatorname{gyr}[b, -a]a] &= \operatorname{gyr}[a, -b] \\ \operatorname{gyr}[-\operatorname{gyr}[a, b]b, a] &= \operatorname{gyr}[a, b] \\ \operatorname{gyr}[\operatorname{gyr}[a, -b]b, a] &= \operatorname{gyr}[a, -b] \end{aligned} \tag{2.94}$$

for all $a, b \in G$.

Proof. By (2.77), by the left loop property, and by (2.77) again we have

$$\begin{aligned} \operatorname{gyr}^{-1}[a, b] &= \operatorname{gyr}[-b, -a] \\ &= \operatorname{gyr}[-b - a, -a] \\ &= \operatorname{gyr}^{-1}[a, -(-b - a)] \end{aligned} \tag{2.95}$$

implying

$$\operatorname{gyr}[a, b] = \operatorname{gyr}[a, -(-b - a)] \tag{2.96}$$

Hence we have, by (2.90),

$$\operatorname{gyr}[a, -(-b - a)] = \operatorname{gyr}[-a, -b - a] \tag{2.97}$$

for all $a, b \in G$. If for any given $c \in G$ we select b to be the unique solution of the equation $-b - a = -c$, by Theorem 2.15, then the resulting equation can be written as

$$\operatorname{gyr}[a; c] = \operatorname{gyr}[-a, -c] \tag{2.98}$$

for all $a, c \in G$, thus verifying (2.92). Identity (2.93) follows from (2.77) and (2.92). Finally, the first identity in (2.94) follows from (2.86) and (2.93). By means of the gyroautomorphism inversion (2.93), the third identity in (2.94) is equivalent to the first one. The second (fourth) identity in (2.94) follows from the first (third) by replacing a by $-a$ (or, alternatively, by replacing b by $-b$). □

We are now in a position to find that the left gyroassociative law and the left loop property of gyrogroups have right counterparts.

Theorem 2.28 *Let* $(G, \oplus)$ *be a gyrogroup. Then, for any* $a, b, c \in G$ *we have*

(i)	$(a \oplus b) \oplus c = a \oplus (b \oplus \operatorname{gyr}[b, a]c)$	*Right Gyroassociative Law*
(ii)	$\operatorname{gyr}[a, b] = \operatorname{gyr}[a, b \oplus a]$	*Right Loop Property*

Proof. In a gyrogroup $(G, +)$: (i) The right gyroassociative law follows from the left gyroassociative law and the inversive symmetry (2.93) of gyroautomorphisms,

$$\begin{aligned} a + (b + \mathrm{gyr}[b, a]c) &= (a + b) + \mathrm{gyr}[a, b]\mathrm{gyr}[b, a]c \\ &= (a + b) + c \end{aligned} \tag{2.99}$$

(ii) The right loop property follows from (2.88) and (2.93). □

The right cancellation law allows the loop property to be dualized in the following

Theorem 2.29 (The Coloop Property - Left and Right). *Let $(G, \oplus)$ be a gyrogroup. Then*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \boxminus b, b] \qquad \textit{Left Coloop Property}$$

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \boxminus a] \qquad \textit{Right Coloop Property}$$

for all $a, b \in G$.

Proof. The proof follows from an application of the left and the right loop property followed by a right cancellation,

$$\begin{aligned} \mathrm{gyr}[a \boxminus b, b] &= \mathrm{gyr}[(a \boxminus b)\oplus b, b] = \mathrm{gyr}[a, b] \\ \mathrm{gyr}[a, b \boxminus a] &= \mathrm{gyr}[a, (b \boxminus a)\oplus a] = \mathrm{gyr}[a, b] \end{aligned} \tag{2.100}$$

□

A right and a left loop give rise to the identities in the following

Theorem 2.30 *Let $(G, \oplus)$ be a gyrogroup. Then*

$$\begin{aligned} \mathrm{gyr}[a\oplus b, \ominus a] &= \mathrm{gyr}[a, b] \\ \mathrm{gyr}[\ominus a, a\oplus b] &= \mathrm{gyr}[b, a] \end{aligned} \tag{2.101}$$

for all $a, b \in G$.

Proof. By a right loop, a left cancellation and a left loop we have

$$\begin{aligned} \mathrm{gyr}[a\oplus b, \ominus a] &= \mathrm{gyr}[a\oplus b, \ominus a\oplus(a\oplus b)] \\ &= \mathrm{gyr}[a\oplus b, b] \\ &= \mathrm{gyr}[a, b] \end{aligned} \tag{2.102}$$

thus verifying the first identity in (2.101). The second identity in (2.101) follows from the first one by gyroautomorphism inversion, (2.93). □

Theorem 2.31 **(The Cogyroautomorphic Inverse Property).** *Any gyrogroup $(G, +)$ possesses the cogyroautomorphic inverse property, that is,*

$$-(a \boxplus b) = (-b) \boxplus (-a) \tag{2.103}$$

for any $a, b \in G$.

Proof. To verify (2.103) we note that by Def. 2.7 of the cooperation $\boxplus$, by gyrosum inversion (2.76), by (2.86), and by gyroautomorphism inversion, Theorem 2.27, we have,

$$\begin{aligned} a \boxplus b &= a + \mathrm{gyr}[a, -b]b \\ &= -\mathrm{gyr}[a, \mathrm{gyr}[a, -b]b]\{-\mathrm{gyr}[a, -b]b - a\} \\ &= \mathrm{gyr}[a, \mathrm{gyr}[a, -b]b]\{-(-\mathrm{gyr}[a, -b]b - a)\} \\ &= \mathrm{gyr}[a, -\mathrm{gyr}[a, -b](-b)]\{-(-\mathrm{gyr}[a, -b]b - a)\} \\ &= \mathrm{gyr}^{-1}[a, -b]\{-(-\mathrm{gyr}[a, -b]b - a)\} \\ &= -(-b - \mathrm{gyr}^{-1}[a, -b]a) \\ &= -(-b - \mathrm{gyr}[b, -a]a) \\ &= -\{(-b) \boxplus (-a)\} \end{aligned} \tag{2.104}$$

Inverting both extreme sides we obtain the desired identity. □

Lemma 2.32 *For any two elements a and b of a gyrogroup $(G, +)$ we have*

$$\begin{aligned} \mathrm{gyr}[a, b]b &= -\{-(a + b) + a\} \\ \mathrm{gyr}[a, -b]b &= -(a - b) + a \end{aligned} \tag{2.105}$$

Proof. The first identity in (2.105) follows from Theorem 2.8(10) with $x = -b$, and the second identity in (2.105) follows from the first one by replacing b by $-b$. □

Theorem 2.33 *The two equations*

$$\begin{aligned} y &= -\mathrm{gyr}[a, x]x \\ x &= -\mathrm{gyr}[a, y]y \end{aligned} \tag{2.106}$$

in a gyrogroup $(G, +)$ are equivalent for any $a, x, y \in G$.

Proof. The two equations in (2.106) are symmetric so that it is enough to show that the first equation implies the second. By the first equation in (2.106) and Lemma 2.32 we have

$$\begin{aligned} y &= -\mathrm{gyr}[a, x]x \\ &= -(a + x) + a \end{aligned} \tag{2.107}$$

implying, by a right cancellation and by (2.4),

$$\begin{aligned} -(a + x) &= y \boxminus a \\ &= y - \mathrm{gyr}[y, a]a \end{aligned} \tag{2.108}$$

so that

$$a + x = -(y - \mathrm{gyr}[y, a]a) \tag{2.109}$$

Hence, it follows from (2.109), (i) by a left cancellation, (ii) by the gyrosum inversion law (2.76) in Theorem 2.25, (iii) by the nested gyroautomorphism identity (2.94), (iv) by the gyroautomorphism inversion law (2.93), and (v) by a left cancellation again, that

$$\begin{aligned} x &= -a - (y - \mathrm{gyr}[y, a]a) \\ &= -a + \mathrm{gyr}[y, -\mathrm{gyr}[y, a]a](\mathrm{gyr}[y, a]a - y) \\ &= -a + \mathrm{gyr}[a, y](\mathrm{gyr}[y, a]a - y) \\ &= -a + (a - \mathrm{gyr}[a, y]y) \\ &= -\mathrm{gyr}[a, y]y \end{aligned} \tag{2.110}$$

as desired. $\square$

Theorem 2.34 *Let $(G, \oplus)$ be a gyrogroup. Then the groupoid $(G, \boxplus)$ of the gyrogroup cooperation is a loop.*

Proof. The identity element of the groupoid $(G, \boxplus)$ is the identity element, 0, of the gyrogroup $(G, \oplus)$ since, by Theorem 2.8 (2) and (13) we have

$$a \boxplus 0 = 0 \boxplus a = a \tag{2.111}$$

If $x \boxplus a = 0$ then, by the right cancellation law we have

$$x = (x \boxplus a) \ominus a = 0 \ominus a = \ominus a \tag{2.112}$$

so that $x = \ominus a$ is a left inverse of a in $(G, \boxplus)$. Furthermore, $\ominus a$ is also a right inverse of a in $(G, \boxplus)$ since

$$a \boxplus (\ominus a) = a \ominus \mathrm{gyr}[a, a]a = a \ominus a = 0 \tag{2.113}$$

The unique solution of the equation

$$x \boxplus a = b \tag{2.114}$$

is, by a right cancellation,

$$x = b \ominus a \tag{2.115}$$

The unique solution of the equation

$$a \boxplus x = b \tag{2.116}$$

is

$$x = \mathrm{gyr}[b, \ominus a](\ominus a \oplus b) \tag{2.117}$$

as we show below. The equation in (2.116),

$$b = a \boxplus x = a \oplus \mathrm{gyr}[a, \ominus x]x \tag{2.118}$$

implies, by a left cancellation, the equation

$$\ominus a \oplus b = \mathrm{gyr}[a, \ominus x]x \tag{2.119}$$

or, equivalently,

$$\ominus \mathrm{gyr}[a, z]z = \ominus a \oplus b \tag{2.120}$$

where we use the notation $z = \ominus x$.

Solving (2.120) for the unknown z by means of Theorem 2.33 we have

$$\begin{aligned} z &= \ominus\mathrm{gyr}[a, \ominus a\oplus b](\ominus a\oplus b) \\ &= \ominus\mathrm{gyr}[b, \ominus a](\ominus a\oplus b) \end{aligned} \tag{2.121}$$

where we employ the second identity in (2.101).

Replacing z by $\ominus x$ in (2.121) we finally have

$$x = \mathrm{gyr}[b, \ominus a](\ominus a\oplus b) \tag{2.122}$$

as desired. Hence, by definition 2.2, the groupoid $(G, \boxplus)$ is a loop. □

We may remark that the solution (2.122) of (2.116) can be simplified by means of the even property (2.92) of gyroautomorphisms and a gyrosum inversion (2.76) as follows:

$$\begin{aligned} x &= \mathrm{gyr}[b, \ominus a](\ominus a\oplus b) \\ &= \mathrm{gyr}[\ominus b, a](\ominus a\oplus b) \\ &= \ominus(\ominus b\oplus a) \end{aligned} \tag{2.123}$$

Furthermore, in the gyrocommutative case the solution (2.122) of (2.116) reduces to $x = b\ominus a$.

Theorem 2.35 **(A Mixed Gyroassociative Law).** *Let $(G, \oplus)$ be a gyrogroup. Then*

$$(a \boxplus b)\oplus c = a\oplus\mathrm{gyr}[a, \ominus b](b\oplus c) \tag{2.124}$$

for all $a, b, c \in G$.

Proof. Whenever convenient we use the notation $g_{a,b} = \mathrm{gyr}[a, b]$, etc. By the definition of the gyrogroup cooperation, by the right gyroassociative law, and by the second nested gyroautomorphism identity in (2.94), we have

$$\begin{aligned} (a \boxminus b)\oplus c &= (a\ominus g_{a,b}b)\oplus c \\ &= a\oplus(\ominus g_{a,b}b\oplus\mathrm{gyr}[\ominus g_{a,b}b, a]c) \\ &= a\oplus(\ominus g_{a,b}b\oplus\mathrm{gyr}[a, b]c) \\ &= a\oplus\mathrm{gyr}[a, b](\ominus b\oplus c) \end{aligned} \tag{2.125}$$

which gives the desired identity when b is replaced by $\ominus b$. □

Useful identities in gyrogroups that need not be gyrocommutative and pointers to their proofs are listed in Table 2.2.

Table 2.2 List of identities in gyrogroups $(G, \oplus)$ that need not be gyrocommutative.

	Formula	Source
1	$a\oplus(\ominus a\oplus b) = b$ (Left cancellation)	Eq. (2.38)
2	$(b\ominus a) \boxplus a = b$ (Right cancellation)	Eq. (2.40)
3	$(b \boxminus a)\oplus a = b$ (Right cancellation)	Eq. (2.39)
4	$a\oplus(b\oplus c) = (a\oplus b)\oplus \mathrm{gyr}[a,b]c)$ (Left gyroassoc. ...)	Axm (G3), Def 2.5
5	$(a\oplus b)\oplus c = a\oplus(b\oplus \mathrm{gyr}[b,a]c)$ (Right gyroassoc. ...)	Thm 2.28
6	$a \boxplus b = a\oplus \mathrm{gyr}[a, \ominus b]b$	Eq. (2.18), Thm 2.10
7	$a\oplus b = a \boxplus \mathrm{gyr}[a,b]b$	Eq. (2.19), Thm 2.10
8	$\ominus(a \boxplus b) = (\ominus b) \boxplus (\ominus a)$ (Cogyroautomorphic ...)	Eq. (2.103), Thm 2.31
9	$(a \boxplus b)\oplus c = a\oplus \mathrm{gyr}[a, \ominus b](b\oplus c)$	Eq. (2.124), Thm 2.35
10	$\mathrm{gyr}[a,b]c = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus c)\}$	Thm 2.8
11	$\mathrm{gyr}[\ominus a, \ominus b] = \mathrm{gyr}[a,b]$ (Even symmetry)	Eq. (2.92), Thm 2.27
12	$gyr^{-1}[a,b] = \mathrm{gyr}[b,a]$ (Inversive symmetry)	Eq. (2.93), Thm 2.27
13	$\mathrm{gyr}[a\oplus b, b] = \mathrm{gyr}[a,b]$ (Left loop property)	Axm (G5), Def 2.5
14	$\mathrm{gyr}[a, b\oplus a] = \mathrm{gyr}[a,b]$ (Right loop property)	Thm 2.28
15	$\mathrm{gyr}[a \boxminus b, b] = \mathrm{gyr}[a,b]$ (Left coloop property)	Thm 2.29
16	$\mathrm{gyr}[a, b \boxminus a] = \mathrm{gyr}[a,b]$ (Right coloop property)	Thm 2.29
17	$\mathrm{gyr}[a\oplus b, \ominus a] = \mathrm{gyr}[a,b]$	Eq. (2.101), Thm 2.30
18	$\mathrm{gyr}[\ominus a, a\oplus b] = \mathrm{gyr}[b,a]$	Eq. (2.101), Thm 2.30
19	$\ominus(a\oplus b) = \mathrm{gyr}[a,b](\ominus b\ominus a)$	Eq. (2.76), Thm 2.25
20	$(a\oplus \mathrm{gyr}[a,b]x) \boxminus (b\oplus x) = a \boxminus b$	Eq. (2.42), Thm 2.16
21	$(\ominus a\oplus b)\oplus \mathrm{gyr}[\ominus a, b](\ominus b\oplus c) = \ominus a\oplus c$	Eq. (2.21), Thm 2.11
22	$(a \boxminus b)\oplus(b \boxminus c) = a\ominus \mathrm{gyr}[a,b]\mathrm{gyr}[b,c]c$	Eq. (2.46), Thm 2.17
23	$A\mathrm{gyr}[a,b] = \mathrm{gyr}[Aa, Ab]A$	Eq. (2.48), Thm 2.19
24	$\mathrm{gyr}[b, \ominus \mathrm{gyr}[b,a]a] = \mathrm{gyr}[a,b]$ (Nested ...)	Eq. (2.94), Thm 2.27
25	$\mathrm{gyr}[\ominus \mathrm{gyr}[a,b]b, a] = \mathrm{gyr}[a,b]$ (Nested ...)	Eq. (2.94), Thm 2.27
26	$y = \ominus \mathrm{gyr}[a,x]x \iff x = \ominus \mathrm{gyr}[a,y]y$	Eq. (2.106), Thm 2.33
27	$Y = \ominus \mathrm{gyr}[b, Xa]X \iff X = \ominus \mathrm{gyr}[b, Ya]Y$	Eq. (2.68), Thm 2.24

Chapter 3

Gyrocommutative Gyrogroups

In this book we are interested in gyrocommutative gyrogroups since some of these give rise to gyrovector spaces, which are our framework for analytic hyperbolic geometry.

3.1 Gyrocommutative Gyrogroups

Definition 3.1 (Gyroautomorphic Inverse Property). *A gyrogroup $(G, +)$ possesses the gyroautomorphic inverse property if for all $a, b \in G$,*

$$-(a + b) = -a - b \tag{3.1}$$

Theorem 3.2 (The Gyroautomorphic Inverse Theorem). *A gyrogroup is gyrocommutative if and only if it possesses the gyroautomorphic inverse property.*

Proof. Let $(G, +)$ be a gyrogroup possessing the gyroautomorphic inverse property. Then the gyrosum inversion law (2.76) specializes, by means of (2.92), to the gyrocommutative law $(G6)$ in Def. 2.6, p. 24. Conversely, if the gyrocommutative law is valid then by the gyrosum inversion law,

$$\mathrm{gyr}[a, b]\{-(-b - a)\} = a + b = \mathrm{gyr}[a, b](b + a) \tag{3.2}$$

so that by eliminating the gyroautomorphism on both extreme sides and inverting the gyro-sign we have

$$-(b + a) = -b - a \tag{3.3}$$

thus validating the gyroautomorphic inverse property. $\square$

Lemma 3.3 *For any given $b \in G$, the self-map*

$$a \mapsto c = \mathrm{gyr}[b, -a]a \tag{3.4}$$

of a gyrogroup $(G, +)$ is surjective (that is, it maps G onto itself).

Proof. By the even property (2.92) of gyroautomorphisms, and by Lemma 2.32 we have

$$\begin{aligned} c &= \mathrm{gyr}[b, -a]a \\ &= \mathrm{gyr}[-b, a]a \\ &= -\{-(-b + a) - b\} \end{aligned} \tag{3.5}$$

Hence, by inversion and by a right and a left cancellation, we have the following successive equivalent equations:

$$\begin{aligned} -c &= -(-b + a) - b \\ -c \boxplus b &= -(-b + a) \\ -(-c \boxplus b) &= -b + a \\ b - (-c \boxplus b) &= a \end{aligned} \tag{3.6}$$

so that for any given $b \in G$ and for all $c \in G$ we have an element a_{bc},

$$a_{bc} = b - (-c \boxplus b) \in G \tag{3.7}$$

satisfying

$$c = \mathrm{gyr}[b, -a_{bc}]a_{bc} \tag{3.8}$$

□

Lemma 3.3 enables us to verify an interesting necessary and sufficient condition that a gyrogroup cooperation $\boxplus$ is commutative.

Theorem 3.4 *Let $(G, +)$ be a gyrogroup. The gyrogroup cooperation $\boxplus$ is commutative if and only if the gyrogroup $(G, +)$ is gyrocommutative.*

Proof. For any $a, b \in G$ we have, by the chain of equations (2.104) in the proof of Theorem 2.31,

$$a \boxplus b = -(-b - \mathrm{gyr}[b, -a]a) \tag{3.9}$$

But by definition,

$$b \boxplus a = b + \mathrm{gyr}[b, -a]a \tag{3.10}$$

Hence

$$a \boxplus b = b \boxplus a \tag{3.11}$$

if and only if

$$-(-b-c) = b+c \tag{3.12}$$

for all $a, b \in G$, where

$$c = \mathrm{gyr}[b, -a]a \tag{3.13}$$

as we see from (3.9) and (3.10). But the self-map of G that takes a to c in (3.13),

$$a \mapsto \mathrm{gyr}[b, -a]a = c \tag{3.14}$$

is surjective, by Lemma 3.3, for any fixed $b \in G$. Hence, the commutative relation (3.11) for $\boxplus$ holds for all $a, b \in G$ if and only if (3.12) holds for all $b, c \in G$. The latter, in turn, is the gyroautomorphic inverse property that, by Theorem 3.2, is equivalent to the gyrocommutativity of the gyrogroup $(G, +)$. □

Theorem 3.5 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[a, b]\mathrm{gyr}[b + a, c] = \mathrm{gyr}[a, b + c]\mathrm{gyr}[b, c] \tag{3.15}$$

for all $a, b, c \in G$.

Proof. Using the notation $g_{a,b} = \mathrm{gyr}[a, b]$ whenever convenient, we have by Theorem 2.19, gyrocommutativity, and (2.9),

$$\begin{aligned} \mathrm{gyr}[a, b]\mathrm{gyr}[b + a, c] &= \mathrm{gyr}[g_{a,b}(b + a), g_{a,b}c]\mathrm{gyr}[a, b] \\ &= \mathrm{gyr}[a + b, \mathrm{gyr}[a, b]c]\mathrm{gyr}[a, b] \\ &= \mathrm{gyr}[a, b + c]\mathrm{gyr}[b, c] \end{aligned} \tag{3.16}$$

■

Theorem 3.6 *Let $a, b, c \in G$ be any three elements of a gyrocommutative gyrogroup $(G, +)$, and let $d \in G$ be determined by the "gyroparallelogram condition"*

$$d = (b \boxplus c) - a \tag{3.17}$$

Then,

$$\mathrm{gyr}[a, -b]\mathrm{gyr}[b, -c]\mathrm{gyr}[c, -d] = \mathrm{gyr}[a, -d] \tag{3.18}$$

for all $a, b, c \in G$.

Proof. By (3.15) and the right and left loop property we have

$$\mathrm{gyr}[a', b', +a']\mathrm{gyr}[b' + a', c'] = \mathrm{gyr}[a', b', +c']\mathrm{gyr}[b' + c', c'] \tag{3.19}$$

Let

$$\begin{aligned} a &= -c' \\ c &= -a' \\ b &= b' + a' \end{aligned} \tag{3.20}$$

so that, by the third equation in (3.20),

$$\begin{aligned} b' + c' &= (b \boxminus a') + c' \\ &= (b \boxplus c) - a \\ &= d \end{aligned} \tag{3.21}$$

Then (3.19), expressed in terms of (3.20) – (3.21), takes the form

$$\mathrm{gyr}[-c, b]\mathrm{gyr}[b, -a] = \mathrm{gyr}[-c, d]\mathrm{gyr}[d, -a] \tag{3.22}$$

Finally, (3.22) implies (3.18) by gyroautomorphism inversion, (2.77), and by the gyroautomorphism even property (2.92). $\square$

Remark 3.7 *Naming* (3.17) *a "gyroparallelogram condition" will be justified in Def. 6.40. The gyroparallelogram condition* (3.17) *is sufficient but not necessary for the validity of* (3.18), *a counterexample being Theorem 6.29.*

Theorem 3.8 *The gyroparallelogram condition* (3.17),

$$d = (b \boxplus c) - a \tag{3.23}$$

is equivalent to the identity

$$-c + d = \mathrm{gyr}[c, -b](b - a) \tag{3.24}$$

Proof. In a gyrocommutative gyrogroup $(G, +)$ the gyroparallelogram condition implies, by the commutativity of the gyrogroup cooperation, Theorem 3.4, by (2.92) – (2.94), by the definition of the gyrogroup cooperation and the left gyroassociative law,

$$\begin{aligned}
d &= (b \boxplus c) - a \\
&= (c \boxplus b) - \mathrm{gyr}[b, -c]\mathrm{gyr}[c, -b]a \\
&= (c \boxplus b) - \mathrm{gyr}[c, \mathrm{gyr}[c, -b]b]\mathrm{gyr}[c, -b]a \\
&= (c + \mathrm{gyr}[c, -b]b) - \mathrm{gyr}[c, \mathrm{gyr}[c, -b]b]\mathrm{gyr}[c, -b]a \\
&= c + (\mathrm{gyr}[c, -b]b - \mathrm{gyr}[c, -b]a) \\
&= c + \mathrm{gyr}[c, -b](b - a)
\end{aligned} \tag{3.25}$$

for all $a, b, c \in G$.

Finally, (3.24) follows from (3.25) by a left cancellation. □

Theorem 3.9 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[a, b]\{b + (a + c)\} = (a + b) + c \tag{3.26}$$

for all $a, b, c \in G$.

Proof. By gyroassociativity and gyrocommutativity we have the chain of equations

$$\begin{aligned}
b + (a + c) &= (b + a) + \mathrm{gyr}[b, a]c \\
&= \mathrm{gyr}[b, a](a + b) + \mathrm{gyr}[b, a]c \\
&= \mathrm{gyr}[b, a]\{(a + b) + c\}
\end{aligned} \tag{3.27}$$

from which (3.26) is derived by gyroautomorphism inversion. □

The special case of Theorem 3.9 corresponding to $c = -a$ provides us with a new cancellation law in gyrocommutative gyrogroups, called the left-right cancellation law.

Theorem 3.10 (The Left-Right Cancellation Law). *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$(a + b) - a = \mathrm{gyr}[a, b]b \tag{3.28}$$

for all $a, b, c \in G$.

Proof. Identity (3.28) follows from (2.105) and the gyroautomorphic inverse property (3.1). □

The left-right cancellation law (3.28) is not a complete cancellation since the echo of the "canceled" a remains in the argument of the involved gyroautomorphism.

Theorem 3.11 *Let $(G,+)$ be a gyrocommutative gyrogroup. Then*

$$a \boxplus b = a + \{(-a+b)+a\} \tag{3.29}$$

for all $a,b \in G$.

Proof. The equalities in the following chain are numbered for subsequent explanation.

$$\begin{aligned} a + \{(-a+b)+a\} &\overset{(1)}{=\!=\!=} \{a+(-a+b)\} + \mathrm{gyr}[a,-a+b]a \\ &\overset{(2)}{=\!=\!=} b + \mathrm{gyr}[b,-a+b]a \\ &\overset{(3)}{=\!=\!=} b + \mathrm{gyr}[b,-a]a \\ &\overset{(4)}{=\!=\!=} b + \mathrm{gyr}[-b,a]a \\ &\overset{(5)}{=\!=\!=} b + \{(-b+a)+b\} \\ &\overset{(6)}{=\!=\!=} \{b+(-b+a)\} + \mathrm{gyr}[b,-b+a]b \\ &\overset{(7)}{=\!=\!=} a + \mathrm{gyr}[b,-b+a]b \\ &\overset{(8)}{=\!=\!=} a + \mathrm{gyr}[a,-b]b \\ &\overset{(9)}{=\!=\!=} a \boxplus b \end{aligned} \tag{3.30}$$

The derivation of the equalities in (3.30) follows.

(1) Follows from the left gyroassociative law.
(2) Follows by a left cancellation.
(3) Follows by a right loop.
(4) Follows from the even property, (2.92), of gyroautomorphisms
(5) Follows from Theorem 3.10 (in which gyrocommutativity is assumed).
(6) Follows by left gyroassociativity.
(7) Follows by a left cancellation.
(8) Follows by a right loop.
(9) Follows from the cooperation definition in Def. 2.7.

□

Theorem 3.12 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$a \boxplus (a + b) = a + (b + a) \tag{3.31}$$

for all $a, b \in G$.

Proof. By a left cancellation and Theorem 3.11 we have

$$\begin{aligned} a + (b + a) &= a + (\{-a + (a + b)\} + a) \\ &= a \boxplus (a + b) \end{aligned} \tag{3.32}$$

□

Theorem 3.13 **(The Gyrotranslation Theorem, II).** *Let $(G, +)$ be a gyrocommutative gyrogroup. For all $a, b, c \in G$,*

$$\begin{aligned} -(a + b) + (a + c) &= \operatorname{gyr}[a, b](-b + c) \\ (a + b) - (a + c) &= \operatorname{gyr}[a, b](b - c) \end{aligned} \tag{3.33}$$

Proof. The first identity in (3.33) follows from the Gyrotranslation Theorem 2.12 with **a** replaced by −**a**. Hence, it is valid in nongyrocommutative gyrogroups as well. The second identity in (3.33) follows from the first by employing the gyroautomorphic inverse property, Theorem 3.2. Hence, it is valid in gyrocommutative gyrogroups. □

Theorem 3.14 *Let $a, b, c \in G$ be any three elements of a gyrocommutative gyrogroup $(G, +)$. Then,*

$$\operatorname{gyr}[-a + b, a - c] = \operatorname{gyr}[a, -b]\operatorname{gyr}[b, -c]\operatorname{gyr}[c, -a] \tag{3.34}$$

Proof. By Theorem 2.19 and by the gyrocommutative law we have

$$\begin{aligned} \operatorname{gyr}[a, b]\operatorname{gyr}[b + a, c] &= \operatorname{gyr}[\operatorname{gyr}[a, b](b + a), \operatorname{gyr}[a, b]c]\operatorname{gyr}[a, b] \\ &= \operatorname{gyr}[a + b, \operatorname{gyr}[a, b]c]\operatorname{gyr}[a, b] \end{aligned} \tag{3.35}$$

Hence, Identity (2.9) in Theorem 2.9 can be written as

$$\operatorname{gyr}[a, b + c]\operatorname{gyr}[b, c] = \operatorname{gyr}[a, b]\operatorname{gyr}[b + a, c] \tag{3.36}$$

By gyroautomorphism inversion, the latter can be written as

$$\operatorname{gyr}[a, b + c] = \operatorname{gyr}[a, b]\operatorname{gyr}[b + a, c]\operatorname{gyr}[c, b] \tag{3.37}$$

Using the notation $b + a = d$, which implies $a = -b + d$, Identity (3.37) becomes, by means of Theorem 2.30,

$$\begin{aligned}\text{gyr}[-b+d, b+c] &= \text{gyr}[-b+d, b]\text{gyr}[d, c]\text{gyr}[c, b] \\ &= \text{gyr}[-b, d]\text{gyr}[d, c]\text{gyr}[c, b]\end{aligned} \tag{3.38}$$

Renaming the elements $b, c, d \in G$, $(b, c, d) \to (-a, c, -b)$, (3.38) becomes

$$\text{gyr}[a-b, -a+c] = \text{gyr}[a, -b]\text{gyr}[-b, c]\text{gyr}[c, -a] \tag{3.39}$$

By means of the gyroautomorphic inverse property, Theorem 3.2, and the even property (2.92) of gyroautomorphisms in Theorem 2.27, Identity (3.39) can be written, finally, in the desired form (3.34). □

The special case of Theorem 3.14 when $b = -c$ is interesting, giving rise to the following

Lemma 3.15 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$\text{gyr}[a, -b] = \text{gyr}[-a+b, a+b]\text{gyr}[a, b] \tag{3.40}$$

Theorem 3.16 *Let $a, b, c \in G$ be any three elements of a gyrocommutative gyrogroup $(G, +)$. Then,*

$$a - \text{gyr}[a, b]\text{gyr}[b, c]c = a \boxminus \text{gyr}[a \boxminus b, b \boxminus c]c \tag{3.41}$$

$$(a \boxminus b) + (b \boxminus c) = a \boxminus \text{gyr}[a \boxminus b, b \boxminus c]c \tag{3.42}$$

and

$$\begin{aligned}\text{gyr}[a, b]\text{gyr}[b, c]c &= a - \{(a \boxminus b) + (b \boxminus c)\} \\ &= a + \{(b \boxminus a) + (c \boxminus b)\}\end{aligned} \tag{3.43}$$

Proof. By left cancellations, gyroassociativity, gyrocommutativity and the loop properties, we have the following chain of equations

$$\begin{aligned}u + b &= u + (v + (-v + b)) \\ &= \{u + v\} + \text{gyr}[u, v](-v + b) \\ &= \{u + (b - (b - v))\} + \text{gyr}[u, v](-v + b) \\ &= \{(u + b) - \text{gyr}[u, b](b - v)\} + \text{gyr}[u, v](-v + b) \\ &= \{(u + b) - \text{gyr}[u, b]\text{gyr}[b, -v](-v + b)\} + \text{gyr}[u, v](-v + b) \\ &= \{(u + b) - \text{gyr}[u + b, b]\text{gyr}[b, -v + b](-v + b)\} + \text{gyr}[u, v](-v + b)\end{aligned} \tag{3.44}$$

so that, by a right cancellation,

$$(u+b)\boxminus\mathrm{gyr}[u,v](-v+b) = (u+b)-\mathrm{gyr}[u+b,b]\mathrm{gyr}[b,-v+b](-v+b) \quad (3.45)$$

for all $u, b, v \in G$. Hence, if we use the notation

$$\begin{aligned} a &= u + b \\ c &= -v + b \end{aligned} \quad (3.46)$$

then

$$\begin{aligned} u &= a \boxminus b \\ v &= b \boxminus c \end{aligned} \quad (3.47)$$

and Identity (3.45) takes the desired form (3.41).

Identity (3.42) follows from (3.41) and Theorem 2.17. Finally, (3.43) follows from (3.41), (3.42), the gyroautomorphic inverse property, the commutativity of the cooperation, and a left cancellation. □

Gyrogroup theory encompasses a repertoire of identities that allow remarkable algebraic manipulations from which rich geometry is uncovered. The following theorem is interesting, as well as its proof, which exemplifies the use of several gyrogroup algebraic manipulations.

Theorem 3.17 *Let $(G, +)$ be a gyrocommutative gyrogroup. The composite gyration J of G,*

$$J = \mathrm{gyr}[a, x]\mathrm{gyr}[-(x + a), x + b]\mathrm{gyr}[x, b] \quad (3.48)$$

$x, a, b \in G$, is independent of x.

Proof. By the gyrator identity, Theorem 2.8(10), and the gyroautomorphic inverse property, Theorem 3.2, we have for all $c \in G$

$$\begin{aligned} \mathrm{gyr}[-a, b]c &= -(-a + b) + (-a + (b + c)) \\ &= (a - b) + (-a + (b + c)) \end{aligned} \quad (3.49)$$

Applying the composite gyration J in (3.48) to any $c \in G$, the proof is provided by the following chain of equalities, which are numbered for later reference in the proof.

$$
\begin{aligned}
Jc &\overset{(1)}{=\!=\!=} \mathrm{gyr}[a,x]\mathrm{gyr}[-(x+a),x+b]\mathrm{gyr}[x,b]c \\
&\overset{(2)}{=\!=\!=} \mathrm{gyr}[a,x]\mathrm{gyr}[-(x+a),x+b]\{-(x+b)+(x+(b+c))\} \\
&\overset{(3)}{=\!=\!=} \mathrm{gyr}[a,x]\{-[-(x+a)+(x+b)] \\
&\qquad\qquad + [-(x+a)+((x+b)+\{-(x+b)+(x+(b+c))\})]\} \\
&\overset{(4)}{=\!=\!=} \mathrm{gyr}[a,x]\{[(x+a)-(x+b)]+[-(x+a)+(x+(b+c))]\} \\
&\overset{(5)}{=\!=\!=} \mathrm{gyr}[a,x]\{\mathrm{gyr}[x,a](a-b)+[-(x+a)+(x+(b+c))]\} \qquad (3.50)\\
&\overset{(6)}{=\!=\!=} (a-b)+\{-\mathrm{gyr}[a,x](x+a)+\mathrm{gyr}[a,x](x+(b+c))\} \\
&\overset{(7)}{=\!=\!=} (a-b)+\{-(a+x)+\mathrm{gyr}[a,x](x+(b+c))\} \\
&\overset{(8)}{=\!=\!=} (a-b)+\{-a+[-x+(x+(b+c))]\} \\
&\overset{(9)}{=\!=\!=} (a-b)+\{-a+(b+c)\} \\
&\overset{(10)}{=\!=\!=} \mathrm{gyr}[-a,b]c
\end{aligned}
$$

The derivation of (3.50) follows.

(*i*) Equality (2) follows from (1) by applying the gyrator identity, Theorem 2.8(10).
(*ii*) Similarly, Equality (3) follows from (2) by applying the gyrator identity.
(*iii*) Equality (4) follows from (3) by applying the gyroautomorphic inverse property of gyrocommutative gyrogroups, Theorem 3.2; and the left cancellation law, Theorem 2.8(9).
(*iv*) Equality (5) follows from (4) by Theorem 3.13.
(*v*) Equality (6) follows from (5) by applying the automorphism $\mathrm{gyr}[a, x]$ termwise, and using the gyration inversive symmetry (2.93).
(*vi*) Equality (7) follows from (6) by the gyrocommutative law.
(*vii*) Equality (8) follows from (7) by the left gyroassociative law noting

that gyrations are even, (2.92).

(*viii*) Equality (9) follows from (8) by a left cancellation.

(*ix*) Equality (10) follows from (9) by (3.49).

It follows from (3.50) that $Jc = \mathrm{gyr}[-a, b]c$ for all $c \in G$, implying $J = \mathrm{gyr}[-a, b]$, so that J is independent of x, as desired. □

Theorem 3.17 implies

$$\mathrm{gyr}[a, x]\mathrm{gyr}[x + a, -(x + b)]\mathrm{gyr}[x, b] = \mathrm{gyr}[-a, b] \tag{3.51}$$

resulting in a "master" gyrocommutative gyrogroup identity. According to the gyrocommutative protection principle [Ungar (2003)], it remains valid in nongyrocommutative gyrogroups as well. It is a master identity in the sense that it is a source of other identities obtained by the substitution of various gyrocommutative gyrogroup expressions for x. Thus, for instance, the substitutions $x = a$ and $x = b$ in (3.51) give, respectively, the following two equivalent connections between $\mathrm{gyr}[-a, b]$ and $\mathrm{gyr}[a, b]$,

$$\begin{aligned} \mathrm{gyr}[-a, b] &= \mathrm{gyr}[-2a, a + b]\mathrm{gyr}[a, b] \\ \mathrm{gyr}[-a, b] &= \mathrm{gyr}[a, b]\mathrm{gyr}[b + a, -2b] \end{aligned} \tag{3.52}$$

where we use the notation $2a = a + a$ for a in $(G, +)$.

Noting that $\mathrm{gyr}[-2a, a + b] = \mathrm{gyr}[-a + b, a + b]$, the first identity in (3.52) can be written as

$$\begin{aligned} \mathrm{gyr}[-a + b, a + b] &= \mathrm{gyr}[-a, b]\mathrm{gyr}[b, a] \\ &= \mathrm{gyr}[-a + b, b]\mathrm{gyr}[b, a + b] \end{aligned} \tag{3.53}$$

The second identity in (3.52) can be manipulated by (2.48) of Theorem 2.19 and by the gyrocommutative law into

$$\begin{aligned} \mathrm{gyr}[-a, b] &= \mathrm{gyr}[a, b]\mathrm{gyr}[b + a, -2b] \\ &= \mathrm{gyr}[\mathrm{gyr}[a, b](b + a), -2\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \\ &= \mathrm{gyr}[a + b, -2\mathrm{gyr}[a, b]b]\mathrm{gyr}[a, b] \end{aligned} \tag{3.54}$$

Comparing (3.54) with the first identity in (3.52) we have, by the left loop property and (2.93),

$$\begin{aligned} \mathrm{gyr}[a + b, -2\mathrm{gyr}[a, b]b] &= \mathrm{gyr}[-2a, a + b] \\ &= \mathrm{gyr}[-a + b, a + b] \\ &= \mathrm{gyr}^{-1}[a + b, -a + b] \end{aligned} \tag{3.55}$$

The following theorem is similar to Theorem 3.17.

Theorem 3.18 *Let $(G, +)$ be a gyrocommutative gyrogroup. The composite gyration J of G,*

$$J = \mathrm{gyr}[a, x]\mathrm{gyr}[-\mathrm{gyr}[x, a](a - b), x + b]\mathrm{gyr}[x, b] \tag{3.56}$$

$x, a, b \in G$, is independent of x.

Proof. By gyration algebra we have, on the one hand,

$$\begin{aligned}
&\{(x + a) - (x + b)\} - \{(x + a) - (x + c)\}\\
&= \mathrm{gyr}[x + a, -(x + b)]\{-(x + b) + (x + c)\}\\
&= \mathrm{gyr}[x + a, -(x + b)]\mathrm{gyr}[x, b](-b + c)\\
&= \mathrm{gyr}[(x + a) - (x + b), -(x + b)]\mathrm{gyr}[x, b](-b + c)\\
&= \mathrm{gyr}[\mathrm{gyr}[x, a](a - b), -(x + b)]\mathrm{gyr}[x, b](-b + c)
\end{aligned} \tag{3.57}$$

On the other hand, however, we have

$$\begin{aligned}
&\{(x + a) - (x + b)\} - \{(x + a) - (x + c)\}\\
&= \{[(x + a) - x] - \mathrm{gyr}[x + a, -x]b\} - \{[(x + a) - x] - \mathrm{gyr}[x + a, -x]c\}\\
&= -\mathrm{gyr}[(x + a) - x, -\mathrm{gyr}[x + a, -x]b](\mathrm{gyr}[x + a, -x]b - \mathrm{gyr}[x + a, -x]c)\\
&= -\mathrm{gyr}[(x + a) - x, -\mathrm{gyr}[x + a, -x]b]\mathrm{gyr}[x + a, -x](b - c)\\
&= -\mathrm{gyr}[x + a, -x]\mathrm{gyr}[\mathrm{gyr}[-x, x + a]((x + a) - x), -b](b - c)\\
&= -\mathrm{gyr}[x, a]\mathrm{gyr}[-x + (x + a), -b](b - c)\\
&= \mathrm{gyr}[x, a]\mathrm{gyr}[a, -b](-b + c)
\end{aligned} \tag{3.58}$$

Comparing the right hand sides of (3.57) and (3.58) we have for $d = -b + c$,

$$\mathrm{gyr}[x, a]\mathrm{gyr}[a, -b]d = \mathrm{gyr}[\mathrm{gyr}[x, a](a - b), -(x + b)]\mathrm{gyr}[x, b]d \tag{3.59}$$

for all $x, a, b, d \in G$.

Hence,

$$\mathrm{gyr}[a, -b] = \mathrm{gyr}[a, x]\mathrm{gyr}[\mathrm{gyr}[x, a](a - b), -(x + b)]\mathrm{gyr}[x, b] \tag{3.60}$$

In particular, the right hand side of (3.60) is independent of x, and the proof is complete. □

Theorem 3.19 *Let $a, b, c \in G$ be any three elements of a gyrocommutative gyrogroup $(G, +)$, and let $T : G^3 \to G$ be a map given by the equation*

$$T(a, b, c) = (b \boxplus c) - a \tag{3.61}$$

Then,

$$x + T(a, b, c) = T(x + a, x + b, x + c) \tag{3.62}$$

for all $x \in G$, *and*

$$\tau T(a, b, c) = T(\tau a, \tau b, \tau c) \tag{3.63}$$

for all $\tau \in Aut(G, +)$.

Proof. The proof of (3.63) follows from Identities (2.52) and (2.55).
To prove (3.62) we have to verify the identity

$$x + \{(b \boxplus c) - a\} = \{(x + b) \boxplus (x + c)\} - (x + a) \tag{3.64}$$

The proof of (3.64) is presented in the following chain of equalities, which are numbered for subsequent explanation.

$$\begin{aligned}
&\{(x+b) \boxplus (x + c)\} - (x + a) \\
&\overset{(1)}{=\!=\!=} (x + b) + \text{gyr}[x + b, -(x + c)]\{(x + c) - (x + a)\} \\
&\overset{(2)}{=\!=\!=} (x + b) + \text{gyr}[x + b, -(x + c)]\text{gyr}[x, c](c - a) \\
&\overset{(3)}{=\!=\!=} x + \{b + \text{gyr}[b, x]\text{gyr}[x + b, -(x + c)]\text{gyr}[x, c](c - a)\} \\
&\overset{(4)}{=\!=\!=} x + \{b + \text{gyr}[b, x]\text{gyr}[x, b]\text{gyr}[b, -c]\text{gyr}[c, x]\text{gyr}[x, c](c - a)\} \\
&\overset{(5)}{=\!=\!=} x + \{b + \text{gyr}[b, -c](c - a)\} \\
&\overset{(6)}{=\!=\!=} x + \{b + (\text{gyr}[b, -c]c - \text{gyr}[b, -c]a)\} \\
&\overset{(7)}{=\!=\!=} x + \{(b + \text{gyr}[b, -c]c) - \text{gyr}[b, \text{gyr}[b, -c]c]\text{gyr}[b, -c]a\} \\
&\overset{(8)}{=\!=\!=} x + \{(b + \text{gyr}[b, -c]c) - \text{gyr}[c, -b]\text{gyr}[b, -c]a\} \\
&\overset{(9)}{=\!=\!=} x + \{(b \boxplus c) - a\}
\end{aligned} \tag{3.65}$$

as desired.

The derivation of the equalities in (3.65) follows:

(1) Follows from (2.124).
(2) Follows from the Gyrotranslation Theorem 3.13. Since we use the second identity in Theorem 3.13 rather than the first identity, gyrocommutativity must be imposed.
(3) Derived by right gyroassociativity.
(4) Follows from Theorem 3.14 and the gyroautomorphism even property (2.92).
(5) Derived by gyroautomorphism inversion, Theorem 2.27.
(6) Derived by automorphism expansion.
(7) Derived by left gyroassociativity.
(8) Follows from the nested gyration identity (2.94).
(9) Follows from the cooperation definition, 2.18 and gyroautomorphism inversion.

$\square$

The identity in (3.62) and (3.64) uncovers an interesting symmetry called *covariance* under left gyrotranslations and automorphisms. Under left gyrotranslations and automorphisms, the elements $a, b, c \in G$ and their image under T, $T(a, b, c)$, vary together, that is, they co-vary. Accordingly, we say that the map T is covariant with respect to left gyrotranslations and automorphisms or, simply, the map T is gyrocovariant. Thus, gyrocovariance is covariance with respect to left gyrotranslations and automorphisms.

The special case of Identity (3.64) when $a = -x$ is interesting, giving rise to the identity

$$(x + b) \boxplus (x + c) = x + \{(b \boxplus c) + x\} \tag{3.66}$$

in any gyrocommutative gyrogroup. It further specializes, when $c = -b$, to the interesting cancellation law,

$$(x + b) \boxplus (x - b) = 2 \otimes x \tag{3.67}$$

where we use the notation $2 \otimes x = x + x$.

As an application of the gyrocovariance of $T(a, b, c)$, (3.61), we verify an interesting cancellation law in the following theorem.

Theorem 3.20 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$a + \{(-a + b) \boxplus (-a + c)\} = (b \boxplus c) - a \tag{3.68}$$

for all $a, b, c \in G$.

Proof. Let

$$d = (b \boxplus c) - a \tag{3.69}$$

Owing to the gyrocovariance of d, Theorem 3.19, we have

$$x + d = \{(x + b) \boxplus (x + c)\} - (x + a) \tag{3.70}$$

for all $a, b, c, x \in G$. In the special case when $x = -a$ Identity (3.70) reduces to

$$-a + d = \{(-a + b) \boxplus (-a + c)\} \tag{3.71}$$

implying, by a left cancellation,

$$d = a + \{(-a + b) \boxplus (-a + c)\} \tag{3.72}$$

Finally, (3.69) and (3.72) imply (3.68) as desired. □

Identities (3.66) – (3.67) and Theorem 3.20, which result from the gyrocovariance of the map T in Theorem 3.19, demonstrate the rich structure that gyrocovariant maps encode, suggesting the following two definitions and a theorem.

Definition 3.21 (Gyrogroup Motions). *Let $(G, +)$ be a gyrogroup, let L_x be the left gyrotranslation of G by $x \in G$,*

$$L_x : G \to G, \quad L_x : a \longmapsto x + a \tag{3.73}$$

and let $Aut_0(G, +)$ be a gyroautomorphism group of the gyrogroup $(G, +)$, Def. 2.22. Elements of the set $G \times Aut_0(G, +)$ are called motions of the gyrogroup in the sense that each element $(x, X) \in G \times Aut_0(G, +)$ gives rise to the motion (L_x, X),

$$(L_x, X)a = x + Xa \tag{3.74}$$

of G.

Theorem 3.22 (Gyrogroup Group Of Motions). *The set of motions $G \times Aut_0(G, +)$ of a gyrogroup $(G, +)$ forms a group with group operation given by motion composition.*

Proof. Let (L_x, X) and (L_y, Y) be two successive motions of a gyrogroup $(G, +)$. Then

$$\begin{aligned}(L_x, X)(L_y, Y)a &= (L_x, X)(y + Ya) \\ &= x + X(y + Ya) \\ &= x + (Xy + XYa) \\ &= (x + Xy) + \mathrm{gyr}[x, Xy]XYa \\ &= (L_{x+Xy}, \mathrm{gyr}[x, Xy]XY)a\end{aligned} \tag{3.75}$$

for all $x, y, a \in G$ and $X, Y \in Aut_0(G, +)$.

Hence, the composition of two motions is, again, a motion,

$$(L_x, X)(L_y, Y) = (L_{x+Xy}, \mathrm{gyr}[x, Xy]XY) \tag{3.76}$$

The composite motion $(L_{x+Xy}, \mathrm{gyr}[x, Xy]XY)$ is recognized as the gyrosemidirect product of its generating motions (L_x, X) and (L_y, Y). The latter, in turn, is a group operation, Theorem 2.23. Hence, the set of motions is a gyrosemidirect product group, with group operation, the gyrosemidirect product, given by motion composition. In particular, the identity motion is (L_0, I), and the inverse motion is

$$(L_x, X)^{-1} = (L_{-X^{-1}x}, X^{-1}) \tag{3.77}$$

where X^{-1} is the inverse automorphism of $X \in Aut_0(G, +)$. □

Definition 3.23 (Gyrocovariance, Gyrogroup Objects). *Let $(G, +)$ be a gyrogroup, let $G \times Aut_0(G, +)$ be its group of motions, and let T: $G^n \to G$ be a map from n copies, G^n, of G to G, $n > 1$. The map T is covariant (with respect to the motions of G) if it obeys the laws*

$$\begin{aligned}x + T(a_1, \ldots, a_n) &= T(x + a_1, \ldots, x + a_n) \\ \tau T(a_1, \ldots, a_n) &= T(\tau a_1, \ldots, \tau a_n)\end{aligned} \tag{3.78}$$

for all $a_1, \ldots, a_n, x \in G$ and all $\tau \in Aut_0(G, +)$.

Furthermore, the set of elements

$$\{a_1, \ldots, a_n,\ T(a_1, \ldots, a_n)\} \tag{3.79}$$

of G is called a gyrogroup object in G.

Example 3.24 (Gyrocommutative Gyrogroup Objects). According to Theorem 3.19, the set

$$\{a, b, c, (a \boxplus b) - c\} \tag{3.80}$$

of four elements of any gyrocommutative gyrogroup $(G, +)$ forms a gyrogroup object. The gyrogroup object will be recognized in Sec. 6.7, p. 160, as the gyroparallelogram.

Example 3.25 Slightly modifying (3.80), the set

$$\{a, b, c, (a \boxminus b) + c\} \tag{3.81}$$

of four elements of a gyrocommutative gyrogroup $(G, +)$ does not form, in general, a gyrogroup object.

We will study more about gyrocovariance in Sec. 6.6, p. 158.

3.2 Nested Gyroautomorphism Identities

We study in this section several nested gyroautomorphism identities in addition to the two nested gyroautomorphism identities already studied in (2.94). We use the notation $g_{a,b} = \mathrm{gyr}[a, b]$, etc., whenever convenient.

Lemma 3.26 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] = \mathrm{gyr}[c, (b \boxminus c) + (a \boxminus b)] \tag{3.82}$$

for all $a, b, c \in G$.

Proof. Renaming the elements $a, b, c \in G$, $(a, b, c) \to (c, b, a)$, (3.43) becomes

$$g_{c,b}g_{b,a}a = c - \{(c \boxminus b) + (b \boxminus a)\} \tag{3.83}$$

Owing to the gyroautomorphic inverse property of the gyrogroup operation $+$, Theorem 3.2, and the commutativity of the gyrogroup cooperation $\boxplus$, Theorem 3.4, Identity (3.83) can be written as

$$\begin{aligned} g_{c,b}g_{b,a}a &= c + \{(b \boxminus c) + (a \boxminus b)\} \\ &= c + x \end{aligned} \tag{3.84}$$

where

$$x = (b \boxminus c) + (a \boxminus b) \tag{3.85}$$

Hence, by (3.84) and (3.85), and by Theorem 2.30, we have

$$\begin{aligned}\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] &= \mathrm{gyr}[c + x, -c] \\ &= \mathrm{gyr}[c, x] \\ &= \mathrm{gyr}[c, (b \boxminus c) + (a \boxminus b)]\end{aligned} \tag{3.86}$$

□

Lemma 3.27 *Let $(G, +)$ be a gyrocommutative gyrogroup. Then*

$$\mathrm{gyr}[a \boxminus b, b \boxminus c] = \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] \tag{3.87}$$

for all $a, b, c \in G$.

Proof. The equalities in the following chain verify Identity (3.87). They are numbered for subsequent explanation.

$$\begin{aligned}\mathrm{gyr}[a \boxminus b, b \boxminus c] &\overset{(1)}{=\!=} \mathrm{gyr}[-b \boxplus a, b \boxminus c] \\ &\overset{(2)}{=\!=} \mathrm{gyr}[-b + g_{b,a}a, b - g_{b,c}c] \\ &\overset{(3)}{=\!=} \mathrm{gyr}[b, -g_{b,a}a]\mathrm{gyr}[g_{b,a}a, -g_{b,c}c]\mathrm{gyr}[g_{b,c}c, -b] \\ &\overset{(4)}{=\!=} \mathrm{gyr}[a, b]\mathrm{gyr}[g_{b,a}a, -g_{b,c}c]\mathrm{gyr}[g_{b,c}c, -b] \\ &\overset{(5)}{=\!=} \mathrm{gyr}[a, b]\mathrm{gyr}[g_{b,a}a, -g_{b,c}c]\mathrm{gyr}[b, c] \\ &\overset{(6)}{=\!=} \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[g_{c,b}g_{b,a}a, -c]\end{aligned} \tag{3.88}$$

The derivation of the equalities in (3.88) follows.

(1) Follows from the commutativity of the cooperation, Theorem 3.4.
(2) Follows from the cooperation definition in Def. 2.7.
(3) Follows from Theorem 3.14.
(4) Follows from the nested gyroautomorphism identity (2.94).
(5) Follows from (i) the nested gyroautomorphism identity (2.94), and (ii) the gyroautomorphism even property, (2.92).
(6) Follows from Theorem 2.19 and (2.93).

□

Lemma 3.28 *Let $(G, +)$ be a gyrocommutative gyrogroup, and let $a, b, c \in G$ be three elements satisfying the condition*

$$\mathrm{gyr}[b \boxminus c, a \boxminus b] = I \tag{3.89}$$

Then

$$\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] = \mathrm{gyr}[c, a] \tag{3.90}$$

Proof. The equalities in the following chain are numbered for subsequent explanation.

$$\begin{aligned}
\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] &\overset{(1)}{=\!=} \mathrm{gyr}[c, (b \boldsymbol{\boxminus} c) + (a \boxminus b)] \\
&\overset{(2)}{=\!=} \mathrm{gyr}[c, \mathrm{gyr}[b \boldsymbol{\boxminus} c, a \boxminus b]\{(a \boxminus b) + (b \boxminus c)\}] \\
&\overset{(3)}{=\!=} \mathrm{gyr}[c, (a \boxminus b) + (b \boxminus c)] \\
&\overset{(4)}{=\!=} \mathrm{gyr}[c, \{(a \boxminus b) + (b \boxminus c)\} + c] \\
&\overset{(5)}{=\!=} \mathrm{gyr}[c, (a \boxminus b) + \{(b \boxminus c) + c\}] \\
&\overset{(6)}{=\!=} \mathrm{gyr}[c, (a \boldsymbol{\boxminus} b) + b] \\
&\overset{(7)}{=\!=} \mathrm{gyr}[c, a]
\end{aligned} \tag{3.91}$$

The derivation of the equalities in (3.91) follows.

(1) Follows from Lemma 3.26.
(2) Follows from the gyrocommutative law.
(3) Follows from Condition (3.89).
(4) Follows from a right loop.
(5) Follows from the right gyroassociative law and Condition (3.89).
(6) Follows from a right cancellation.
(7) Follows from a right cancellation.

□

Theorem 3.29 *Let $(G, +)$ be a gyrocommutative gyrogroup, and let $a, b, c \in G$ be three elements satisfying the condition*

$$\mathrm{gyr}[b \boxminus c, a \boxminus b] = I \tag{3.92}$$

Then

$$\mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[c, a] = I \tag{3.93}$$

Proof. Inverting the gyroautomorphism in Condition (3.92), we have

$$\mathrm{gyr}[a \boxminus b, b \boxminus c] = I \tag{3.94}$$

By (3.94), Lemma 3.27, and Lemma 3.28 we have, as desired,

$$\begin{aligned} I &= \mathrm{gyr}[a \boxminus b, b \boxminus c] \\ &= \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[g_{c,b}g_{b,a}a, -c] \\ &= \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[c, a] \end{aligned} \tag{3.95}$$

□

Table 3.1 List of identities in gyrocommutative gyrogroups $(G, \oplus)$.

	Formula	Source
1	$\ominus(a \oplus b) = \ominus a \ominus b$ (Gyroautomorphic inverse ...)	Eq. (3.1), Thm 3.2
2	$a \boxplus b = b \boxplus a$	Eq. (3.11), Thm 3.4
3	$a \boxplus b = a \oplus \{(\ominus a \oplus b) \oplus a\}$	Eq. (3.29), Thm 3.11
4	$a \boxplus (a \oplus b) = a \oplus (b \oplus a)$	Eq. (3.31), Thm 3.12
5	$\mathrm{gyr}[a, b]\{b \oplus (a \oplus c)\} = (a \oplus b) \oplus c$	Eq. (3.26), Thm 3.9
6	$\mathrm{gyr}[a, b]b = (a \oplus b) \ominus a$	Eq. (3.28), Thm 3.10
7	$\ominus(a \oplus b) \oplus (a \oplus c) = \mathrm{gyr}[a, b](\ominus b \oplus c)$	Eq. (3.33), Thm 3.13
8	$(a \oplus b) \ominus (a \oplus c) = \mathrm{gyr}[a, b](b \ominus c)$	Eq. (3.33), Thm 3.13
9	$\mathrm{gyr}[a, b]\mathrm{gyr}[b + a, c] = \mathrm{gyr}[a, b + c]\mathrm{gyr}[b, c]$	Eq. (3.15), Thm 3.5
10	$\mathrm{gyr}[a, \ominus b]\mathrm{gyr}[b, \ominus c]\mathrm{gyr}[c, \ominus a] = \mathrm{gyr}[\ominus a \oplus b, \oplus a \ominus c]$	Eq. (3.34), Thm 3.14
11	$\mathrm{gyr}[a, \ominus b]\mathrm{gyr}[b, \ominus c]\mathrm{gyr}[c, \ominus d] = \mathrm{gyr}[a, \ominus d]$ The Gyroparallelogram Condition: $d = (b \boxplus c) \ominus a$	Eq. (3.18), Thm 3.6
12	$\mathrm{gyr}[a, \ominus b] = \mathrm{gyr}[\ominus a \oplus b, a \oplus b]\mathrm{gyr}[a, b]$	Eq. (3.40), Lmm 3.15
13	$a \ominus \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]c = a \boxminus \mathrm{gyr}[a \boxminus b, b \boxminus c]c$	Eq. (3.41), Thm 3.16
14	$(a \boxminus b) \oplus (b \boxminus c) = a \boxminus \mathrm{gyr}[a \boxminus b, b \boxminus c]c$	Eq. (3.42), Thm 3.16
15	$(x \oplus a) \boxplus (x \oplus b) = x \oplus \{(a \boxplus b) \oplus x\}$	Eq. (3.66)
16	$(x \oplus a) \boxplus (x \ominus a) = 2 \otimes x$	Eq. (3.67)
17	$a \oplus \{(b \boxminus a) \oplus (c \boxminus b)\} = \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]c$	Eq. (3.43), Thm 3.16
18	$\mathrm{gyr}[c, (b \boxminus c) \oplus (a \boxminus b)] = \mathrm{gyr}[g_{c,b}g_{b,a}a, \ominus c]$	Eq. (3.82), Lmm 3.26
19	$\mathrm{gyr}[a \boxminus b, b \boxminus c] = \mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[g_{c,b}g_{b,a}a, \ominus c]$	Eq. (3.87), Lmm 3.27
20	$\mathrm{gyr}[g_{c,b}g_{b,a}a, \ominus c] = \mathrm{gyr}[c, a]$ Condition: $\mathrm{gyr}[b \boxminus c, a \boxminus b] = I$	Eq. (3.89), Lmm 3.28
21	$\mathrm{gyr}[a, b]\mathrm{gyr}[b, c]\mathrm{gyr}[c, a] = I$ Condition: $\mathrm{gyr}[b \boxminus c, a \boxminus b] = I$	Eq. (3.93), Lmm 3.29
22	$x \oplus \{(a \boxplus b) \ominus c\} = \{(x \oplus a) \boxplus (x \oplus b)\} \ominus (x \oplus c)$	Eq. (3.64), Thm 3.19

Useful identities in gyrocommutative gyrogroups and pointers to their proofs are listed in Table 3.1.

3.3 Two-Divisible Two-Torsion Free Gyrocommutative Gyrogroups

Definition 3.30 (Two-Torsion Free Gyrogroups). *Let $(G, +)$ be a gyrogroup. An element $g \in G$ satisfying $g + g = 0$ is called a two-torsion element. The gyrogroup $(G, +)$ is two-torsion free if the only two-torsion element in G is $g = 0$.*

Definition 3.31 (Two-Divisible Gyrogroups). *Let $(G, +)$ be a gyrogroup. The half of $g \in G$, denoted $\frac{1}{2} \otimes g$, is an element of G satisfying*

$$\tfrac{1}{2} \otimes g + \tfrac{1}{2} \otimes g = g \tag{3.96}$$

A gyrogroup in which every element possesses a half is called a two-divisible gyrogroup.

Theorem 3.32 *Let $(G, +)$ be a two-divisible, two-torsion free, gyrocommutative gyrogroup. Then, the half $\frac{1}{2} \otimes g$ of any $g \in G$ is unique.*

Proof. Let each of a and b be half of g in G. Then

$$a + a = b + b \tag{3.97}$$

Hence,

$$-b + (a + a) = -b + (b + b) \tag{3.98}$$

Applying the left gyroassociative law to both sides of (3.98), noting that $\mathrm{gyr}[-b, b] = I$ is trivial, we have

$$(-b + a) + \mathrm{gyr}[-b, a]a = b \tag{3.99}$$

Solving (3.99) for $(-b + a)$ by a right cancellation, we have

$$\begin{aligned} -b + a &= b - \mathrm{gyr}[b, \mathrm{gyr}[-b, a]a]\mathrm{gyr}[-b, a]a \\ &= b - \mathrm{gyr}[a, -b]\mathrm{gyr}[-b, a]a \\ &= b - a \\ &= -(-b + a) \end{aligned} \tag{3.100}$$

Hence,

$$(-b+a)+(-b+a)=0 \tag{3.101}$$

so that $-b+a$ is a two-torsion element of two-torsion free gyrogroup. Hence, $-b+a=0$ so that $b=a$ as desired. □

Since both $-(\frac{1}{2}\otimes g)$ and $\frac{1}{2}\otimes(-g)$ are halves of $-g\in G$, it follows from Theorem 3.32 that

$$-(\tfrac{1}{2}\otimes g)=\tfrac{1}{2}\otimes(-g) \tag{3.102}$$

in any two-divisible, two torsion-free gyrocommutative gyrogroup $(G,+)$.

Theorem 3.33 *Let $(G,+)$ be a two-divisible, torsion-free, gyrocommutative gyrogroup. Then,*

$$\mathrm{gyr}[a,b](\tfrac{1}{2}\otimes g)=\tfrac{1}{2}\otimes\mathrm{gyr}[a,b]g \tag{3.103}$$

for all $a,b,g\in G$.

Proof. Gyroautomorphisms are automorphisms. Hence,

$$\begin{aligned}\mathrm{gyr}[a,b](\tfrac{1}{2}\otimes g)+\mathrm{gyr}[a,b](\tfrac{1}{2}\otimes g)&=\mathrm{gyr}[a,b](\tfrac{1}{2}\otimes g+\tfrac{1}{2}\otimes g)\\&=\mathrm{gyr}[a,b]g\end{aligned} \tag{3.104}$$

implying (3.103) since, by Theorem 3.32, the half is unique. □

Theorem 3.34 **(The Gyration Exclusion Theorem).** *Let $(G,+)$ be a two-divisible, torsion-free, gyrocommutative gyrogroup. Then,*

$$\mathrm{gyr}[a,b]\neq -I \tag{3.105}$$

for all $a,b\in G$.

Proof. Seeking a contradiction, we assume that $\mathrm{gyr}[a,b]=-I$ for some $a,b\in G$. We have $b=\frac{1}{2}\otimes b+\frac{1}{2}\otimes b=\frac{1}{2}\otimes b\boxplus\frac{1}{2}\otimes b$ so that by a right cancellation, $b-\frac{1}{2}\otimes b=\frac{1}{2}\otimes b$. Hence,

$$\begin{aligned}a+\tfrac{1}{2}\otimes b&=a+(b-\tfrac{1}{2}\otimes b)\\&=(a+b)-\mathrm{gyr}[a,b]\tfrac{1}{2}\otimes b\\&=(a+b)-\tfrac{1}{2}\otimes\mathrm{gyr}[a,b]b\\&=(a+b)+\tfrac{1}{2}\otimes b\end{aligned} \tag{3.106}$$

Right cancellation of $\frac{1}{2}\otimes b$ in (3.106) gives $a = a \oplus b$, implying by a left cancellation that $b = 0$. Hence, by Theorem 2.8(11), $\mathrm{gyr}[a, b] = I$, thus contradicting the assumption. $\square$

Theorem 3.35 *Let $(G, +)$ be a two-divisible, torsion-free, gyrocommutative gyrogroup. If*

$$\mathrm{gyr}[a, b]b = -b \tag{3.107}$$

in G, then $b = 0$.

Proof. As in (3.106), and by Theorem 3.33, we have

$$\begin{aligned} a + \tfrac{1}{2}\otimes b &= a + (b - \tfrac{1}{2}\otimes b) \\ &= (a + b) - \mathrm{gyr}[a, b]\tfrac{1}{2}\otimes b \\ &= (a + b) - \tfrac{1}{2}\otimes \mathrm{gyr}[a, b]b \\ &= (a + b) + \tfrac{1}{2}\otimes b \end{aligned} \tag{3.108}$$

implying $a = a + b$ so that $b = 0$. $\square$

As an application of Theorem 3.35, let us verify the following theorem:

Theorem 3.36 *Let $(G, +)$ be a two-divisible, torsion-free, gyrocommutative gyrogroup. Then, the equation*

$$x - (y - x) = y \tag{3.109}$$

$x, y \in G$, holds if and only if $x = y$.

Proof. It follows from (3.109) that

$$\begin{aligned} -y + x &= -x + y \\ &= -\mathrm{gyr}[x, -y](-y + x) \\ &= -\mathrm{gyr}[x, -y + x](-y + x) \end{aligned} \tag{3.110}$$

so that

$$t = -\mathrm{gyr}[x, t]t \tag{3.111}$$

where $t = -y + x$. Hence, by Theorem 3.35, $t = 0$, so that $x = y$. Conversely, if $x = y$ then (3.109) clearly holds. $\square$

Since the cooperation $\boxplus$ of a gyrocommutative gyrogroup is commutative, a natural candidate for the gyromidpoint m_{ab} of any two distinct points a and b in a two-divisible, torsion-free, gyrocommutative gyrogroup arises.

Definition 3.37 (Gyromidpoints, I). *The gyromidpoint m_{ab} of any two distinct points a and b in a two-divisible, torsion-free, gyrocommutative gyrogroup $(G, +)$ is given by the equation*

$$m_{ab} = \tfrac{1}{2}\otimes(a \boxplus b) \tag{3.112}$$

A different, but equivalent, definition of the gyromidpoint will be presented in Def. 6.31 p. 156. As the book unfolds, it will become clear that the ability to find gyromidpoints and their iterates lead to the construction of gyrolines, as noticed in [Ungar (1996); Lawson and Lim (2004)].

Considering the gyromidpoint concept as a primitive one, Lawson and Lim studied symmetric sets with midpoints, obtaining an important structure, called a *dyadic symmetric set* or a "dyadic symset", for short. Interestingly, the latter is remarkably general, as Lawson and Lim emphasize, and it turns out to be identical to a two-divisible, torsion-free, gyrocommutative gyrogroup [Lawson and Lim (2004), Theorem 8.8]. A classic illustration of the generality that Lawson and Lim present in [Lawson and Lim (2004)] is the well known Cartan decomposition of semisimple Lie groups; see also [Kasparian and Ungar (2004)].

3.4 The Möbius Complex Disc Gyrogroup

Gyrogroups, both gyrocommutative and non-gyrocommutative, finite and infinite, abound in the theory of groups [Foguel and Ungar (2000); Foguel and Ungar (2001); Feder (2003)], loops [Issa (1999)], quasigroup [Issa (2001); Kuznetsov (2003)], and Lie groups [Kasparian and Ungar (2004)]. Historically, the first gyrogroup structure was discovered in the study of Einstein velocity addition [Ungar (1988a); Ungar (1991); Ungar (1997); Ungar (1998)]. However, the best way to introduce the gyrogroup notion by example is provided by the Möbius transformation group of the complex open unit disc [Ungar (1994); Kinyon and Ungar (2000)].

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\} \tag{3.113}$$

in the complex plane $\mathbb{C}$ is given by the polar decomposition [Ahlfors (1973); Krantz (1990)],

$$z \mapsto e^{i\theta}\frac{a+z}{1+\bar{a}z} = e^{i\theta}(a \oplus_{\mathrm{M}} z) \tag{3.114}$$

It induces the Möbius addition $\oplus_{\text{M}}$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus_{\text{M}} z = \frac{a+z}{1+\overline{a}z} \tag{3.115}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a.

Möbius addition $a \oplus_{\text{M}} z$ and subtraction $a \ominus_{\text{M}} z = a \oplus_{\text{M}} (-z)$ are found useful in the geometric viewpoint of complex analysis; see, for instance, [Ungar (1999); Ungar (2004c)],[Krantz (1990), pp. 52–53, 56–57, 60], and the Schwarz-Pick Lemma in [Goebel and Reich (1984), Thm 1.4, p. 64]. However, prior to the appearance of [Ungar (2001)] in 2001 these were not considered "addition" and "subtraction" since it has gone unnoticed that, being *gyrocommutative* and *gyroassociative* they share analogies with common addition and subtraction, as we will see below.

Möbius addition $\oplus_{\text{M}}$ is neither commutative nor associative. The breakdown of commutativity in Möbius addition is "repaired" by the introduction of a *gyrator*

$$\text{gyr} : \mathbb{D} \times \mathbb{D} \to Aut(\mathbb{D}, \oplus_{\text{M}}) \tag{3.116}$$

that generates gyroautomorphisms according to the equation

$$\text{gyr}[a, b] = \frac{a \oplus_{\text{M}} b}{b \oplus_{\text{M}} a} = \frac{1 + a\overline{b}}{1 + \overline{a}b} \in Aut(\mathbb{D}, \oplus_{\text{M}}) \tag{3.117}$$

where $Aut(\mathbb{D}, \oplus_{\text{M}})$ is the automorphism group of the Möbius groupoid $(\mathbb{D}, \oplus_{\text{M}})$.

The inverse of the automorphism $\text{gyr}[a, b]$ is clearly $\text{gyr}[b, a]$,

$$\text{gyr}^{-1}[a, b] = \text{gyr}[b, a] \tag{3.118}$$

The gyrocommutative law of Möbius addition $\oplus_{\text{M}}$ that follows from the definition of gyr in (3.117),

$$a \oplus_{\text{M}} b = \text{gyr}[a, b](b \oplus_{\text{M}} a) \tag{3.119}$$

is not terribly surprising since it is generated by definition, but we are not finished.

Coincidentally, the gyroautomorphism $\text{gyr}[a, b]$ that repairs the breakdown of commutativity of $\oplus_{\text{M}}$ in (3.119), repairs the breakdown of associativity of $\oplus_{\text{M}}$ as well, giving rise to the respective *left and right gyroassociative* *laws*

$$\begin{aligned}a\oplus_{M}(b\oplus_{M}z) &= (a\oplus_{M}b)\oplus_{M}\mathrm{gyr}[a,b]z\\(a\oplus_{M}b)\oplus_{M}z &= a\oplus_{M}(b\oplus_{M}\mathrm{gyr}[b,a]z)\end{aligned}\tag{3.120}$$

for all $a, b, z \in \mathbb{D}$. Moreover, Möbius gyroautomorphisms possess the two elegant identities

$$\begin{aligned}\mathrm{gyr}[a\oplus_{M}b, b] &= \mathrm{gyr}[a,b]\\\mathrm{gyr}[a, b\oplus_{M}a] &= \mathrm{gyr}[a,b]\end{aligned}\tag{3.121}$$

One can now readily check that the Möbius complex disc groupoid $(\mathbb{D}, \oplus_{M})$ is a gyrocommutative gyrogroup.

As any coincidence in mathematics, the coincidences that Möbius addition exhibits in the gyrocommutative and the gyroassociative laws that it obeys are not accidental. Rather, they stem from the polar decomposition structure in (3.114), as demonstrated in most general settings by Foguel and Ungar in [Foguel and Ungar (2000); Foguel (2002)].

3.5 Möbius Gyrogroups

Identifying vectors in the Euclidean plane $\mathbb{R}^2$ with complex numbers in the complex plane $\mathbb{C}$ in the usual way we have

$$\mathbb{R}^2 \ni \mathbf{u} = (u_1, u_2) \leftrightarrow u_1 + iu_2 = u \in \mathbb{C}\tag{3.122}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &\leftrightarrow \mathrm{Re}(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2}\\\|\mathbf{u}\| &\leftrightarrow |u|\end{aligned}\tag{3.123}$$

Under the translation (3.123) of elements of the open disc

$$\mathbb{R}^2_{s=1} = \{\mathbf{u} \in \mathbb{R}^2 : \|\mathbf{u}\| < 1\}\tag{3.124}$$

of the Euclidean plane $\mathbb{R}^2$ to elements of the complex open unit disc $\mathbb{D}$, Möbius addition (3.114) in $\mathbb{V}_{s=1} = \mathbb{R}^2_{s=1}$ takes the form

$$\begin{aligned}
\mathbf{u}\oplus_{_M}\mathbf{v} &= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&\leftrightarrow \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{u+v}{1+\bar{u}v} \\
&= u\oplus_{_M} v
\end{aligned} \tag{3.125}$$

for all $\mathbf{u},\mathbf{v} \in \mathbb{R}^2_{s=1}$, and all $u, v \in \mathbb{D}$. In (3.125) we recover the Möbius addition $\oplus_{_M}$ in the open unit disc $\mathbb{D}$ of $\mathbb{C}$, (3.115), from a corresponding Möbius vector addition in the open unit disc $\mathbb{R}^2_{s=1}$ of the Euclidean plane $\mathbb{R}^2$.

Suggestively, we extend (3.125) in the following definition of Möbius vector addition in the ball.

Definition 3.38 (Möbius Addition in the Ball). *Let $\mathbb{V} = (\mathbb{V}, +, \cdot)$ be a real inner product space with a binary operation $+$ and a positive definite inner product $\cdot$ ([Marsden (1974), p. 21]; following [Kowalsky (1977)], also known as Euclidean space) and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,*

$$\mathbb{V}_s = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < s\} \tag{3.126}$$

for any fixed $s > 0$. Möbius addition $\oplus_{_M}$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\mathbf{u}\oplus_{_M}\mathbf{v} = \frac{(1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{3.127}$$

where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

In the limit of large s, $s \to \infty$, the ball $\mathbb{V}_s$ expands to the whole of its space $\mathbb{V}$, and Möbius addition reduces to vector addition in $\mathbb{V}$. Accordingly, the right hand side of (3.127) is known as a Möbius translation [Ratcliffe (1994), p. 129]. An earlier study of Möbius translation in several dimensions, using the notation $-\mathbf{u}\oplus_{_M}\mathbf{v} = T_{\mathbf{u}}\mathbf{v}$, is found in [Ahlfors (1981)] and in

[Ahlfors (1984)], where it is attributed to Poincaré. Both Ahlfors [Ahlfors (1981)] and Ratcliffe [Ratcliffe (1994)], who studied the Möbius translation in several dimensions, did not call it a Möbius addition since it has gone unnoticed at the time that Möbius translation is regulated by algebraic laws analogous to those that regulate vector addition.

Möbius addition $\oplus_{\text{M}}$ in the open unit ball $\mathbb{V}_s$ of any real inner product space $\mathbb{V}$ is thus a most natural extension of the Möbius addition in the open complex unit disc. Like the Möbius disc $(\mathbb{D}, \oplus_{\text{M}})$, the Möbius ball $(\mathbb{V}_s, \oplus_{\text{M}})$ turns out to be a gyrocommutative gyrogroup, as one can readily check by computer algebra. Interestingly, the gyrocommutative law of Möbius addition was already known to Ahlfors [Ahlfors (1981), Eq. 39]. The accompanied gyroassociative law of Möbius addition, however, had gone unnoticed.

Möbius addition satisfies the gamma identity

$$\gamma_{\mathbf{u}\oplus_{\text{M}}\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\sqrt{1 + \frac{2}{s^2}\mathbf{u}\cdot\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \tag{3.128}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$, where $\gamma_{\mathbf{u}}$ is the gamma factor

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \dfrac{\|\mathbf{v}\|^2}{s^2}}} \tag{3.129}$$

in the s-ball $\mathbb{V}_s$.

The gamma factor appears also in Einstein addition, and it is known in special relativity theory as the Lorentz factor.

The Möbius gyrogroup cooperation (2.2) is given by Möbius coaddition

$$\mathbf{u} \boxplus_{\text{M}} \mathbf{v} = \frac{\gamma_{\mathbf{u}}^2\mathbf{u} + \gamma_{\mathbf{v}}^2\mathbf{v}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 - 1} \tag{3.130}$$

satisfying the gamma identity

$$\gamma_{\mathbf{u}\boxplus_{\text{M}}\mathbf{v}} = \frac{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 - 1}{\sqrt{1 + 2\gamma_{\mathbf{u}}^2\gamma_{\mathbf{v}}^2(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) - (\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2)}} \tag{3.131}$$

Möbius coaddition is commutative, as expected from Theorem 3.4.

When the vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{V}_s$ of $\mathbb{V}$ are parallel in $\mathbb{V}$, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, Möbius addition reduces to

$$\mathbf{u}\oplus_{\text{M}}\mathbf{v} = \frac{\mathbf{u} + \mathbf{v}}{1 + \frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|}, \qquad \mathbf{u}\|\mathbf{v} \tag{3.132}$$

and, accordingly,

$$\|\mathbf{u}\| \oplus_{\mathrm{M}} \|\mathbf{v}\| = \frac{\|\mathbf{u}\| + \|\mathbf{v}\|}{1 + \frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|} \tag{3.133}$$

The restricted Möbius addition in (3.132) and (3.133) is both commutative and associative.

Möbius gyrations

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{V}_s \to \mathbb{V}_s \tag{3.134}$$

are automorphisms of the Möbius gyrogroup $(\mathbb{V}_s, \oplus_{\mathrm{M}})$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in Aut(\mathbb{V}_s, \oplus_{\mathrm{M}}) \tag{3.135}$$

given by the equation, Theorem 2.8(10),

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus_{\mathrm{M}}(\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v})\oplus_{\mathrm{M}}\{\mathbf{u}\oplus_{\mathrm{M}}(\mathbf{v}\oplus_{\mathrm{M}}\mathbf{w})\} \tag{3.136}$$

and they preserve the inner product that the ball $\mathbb{V}_s$ inherits from its real inner product space $\mathbb{V}$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{3.137}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_s$.

3.6 Einstein Gyrogroups

Attempts to measure the absolute velocity of the earth through the hypothetical ether had failed. The most famous of these experiments is one performed by Michelson and Morley in 1887 [Feynman and Sands (1964)]. It was 18 years later before the null results of these experiments were finally explained by Einstein in terms of a new velocity addition law that bears his name, which he introduced in his 1905 paper that founded the special theory of relativity [Einstein (1905); Einstein (1998)].

Contrasting Newtonian velocities, which are vectors in the Euclidean three-space $\mathbb{R}^3$, Einsteinian velocities must be relativistically admissible, that is, their magnitude must not exceed the vacuum speed of light, which is about 3×10^5 $km{\cdot}sec^{-1}$.

Let c be the vacuum speed of light, and let

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\} \tag{3.138}$$

be the c-ball of all relativistically admissible velocities of material particles. It is the open ball of radius c, centered at the origin of the Euclidean three-space $\mathbb{R}^3$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^3$ with magnitude $\|\mathbf{v}\|$ smaller than c. Einstein addition $\oplus_{\rm E}$ in the c-ball is given by the equation

$$\mathbf{u}\oplus_{\rm E}\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\times(\mathbf{u}\times\mathbf{v}))\right\} \tag{3.139}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, where $\mathbf{u}\cdot\mathbf{v}$ is the inner product that the ball $\mathbb{R}^3_c$ inherits from its space $\mathbb{R}^3$, and where $\gamma_{\mathbf{u}}$ is the gamma factor (3.129) in the c-ball.

Owing to the vector identity,

$$(\mathbf{x}\times\mathbf{y})\times\mathbf{z} = -(\mathbf{y}\cdot\mathbf{z})\mathbf{x} + (\mathbf{x}\cdot\mathbf{z})\mathbf{y} \tag{3.140}$$

$\mathbf{x},\mathbf{y},\mathbf{z}\in\mathbb{R}^3$, that holds in $\mathbb{R}^3$, Einstein addition (3.139) can also be written in the form [Sexl and Urbantke (2001); Ungar (2001)]

$$\mathbf{u}\oplus_{\rm E}\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{3.141}$$

that remains valid in higher dimensions.

Einstein addition (3.141) of relativistically admissible velocities was introduced by Einstein in his 1905 paper [Einstein (1998), p. 141] where the magnitudes of the two sides of Einstein addition (3.141) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [Einstein (1905)] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (3.141).

In the Newtonian limit, $c\rightarrow\infty$, the ball $\mathbb{R}^3_c$ of all relativistically admissible velocities expands to the whole of its space $\mathbb{R}^3$, as we see from (3.138), and Einstein addition $\oplus_{\rm E}$ in $\mathbb{R}^3_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^3$, as we see from (3.141) and (3.129).

Suggestively, we extend Einstein addition of relativistically admissible velocities by abstraction in the following definition of Einstein addition in the ball.

Definition 3.39 **(Einstein Addition in the Ball).** *Let $\mathbb{V}$ be a real inner product space and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,*

$$\mathbb{V}_s = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < s\} \tag{3.142}$$

Einstein addition $\oplus_E$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\mathbf{u}\oplus_E\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}}\left\{\mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{3.143}$$

where $\gamma_\mathbf{u}$ is the gamma factor in $\mathbb{V}_s$, and where $\cdot$ and $\|\cdot\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

Like Möbius addition in the ball, one can show by computer algebra that Einstein addition in the ball is a gyrocommutative gyrogroup operation, giving rise to the Einstein ball gyrogroup $(\mathbb{V}_s, \oplus_E)$.

Einstein addition satisfies the mutually equivalent gamma identities

$$\gamma_{\mathbf{u}\oplus_E\mathbf{v}} = \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right) \tag{3.144}$$

and

$$\gamma_{\mathbf{u}\ominus_E\mathbf{v}} = \gamma_\mathbf{u}\gamma_\mathbf{v}\left(1-\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}\right) \tag{3.145}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$. The ratio

$$\frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_\mathbf{u}\gamma_\mathbf{v}} = 1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2} \tag{3.146}$$

is a kind of an inner product that will be extended by abstraction in (4.1).

The gamma identity (3.145) signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [Sommerfeld (1909)] and Varičak [Varičak (1908); Varičak (1910a)] in terms of *rapidities*, a term coined by Robb [Robb (1914)]. The rapidity $\phi_\mathbf{v}$ of a relativistically admissible velocity $\mathbf{v}$ is defined by the equation [Levy-Leblond (1979)]

$$\phi_\mathbf{v} = \tanh^{-1}\frac{\|\mathbf{v}\|}{s} \tag{3.147}$$

so that,

$$\begin{aligned} \cosh\phi_\mathbf{v} &= \gamma_\mathbf{v} \\ \sinh\phi_\mathbf{v} &= \gamma_\mathbf{v}\frac{\|\mathbf{v}\|}{s} \end{aligned} \tag{3.148}$$

In the years 1910–1914, the period which experienced a dramatic flowering of creativity in the special theory of relativity, the Croatian physicist and mathematician Vladimir Varičak (1865–1942), professor and rector of

Zagreb University, showed in [Varičak (1910a)], that this theory has a natural interpretation in the hyperbolic geometry of Bolyai and Lobachevsky [Barrett (1998)] [Rosenfeld (1988)]. Indeed, written in terms of rapidities, identity (3.145) takes the form

$$\cosh \phi_{\mathbf{u}\ominus\mathbf{v}} = \cosh \phi_{\mathbf{u}} \cosh \phi_{\mathbf{v}} - \sinh \phi_{\mathbf{u}} \sinh \phi_{\mathbf{v}} \cos A \tag{3.149}$$

where, according to J.F. Barrett [Barrett (2001)], the angle A has been interpreted by Sommerfeld [Sommerfeld (1909)], and Varičak [Varičak (1910a)], as a hyperbolic angle in the relativistic "triangle of velocities" in the Beltrami ball model of hyperbolic geometry. The role of Constantin Carathéodory [Georgiadou (2004)] in this approach to special relativity and hyperbolic geometry has been described by J.F. Barrett [Barrett (2001)], emphasizing that (3.149) is the "cosine rule" in hyperbolic geometry.

When the vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{V}_s$ of $\mathbb{V}$ are parallel in $\mathbb{V}$, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, Einstein addition reduces to

$$\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|}, \qquad \mathbf{u}\|\mathbf{v} \tag{3.150}$$

and, accordingly,

$$\|\mathbf{u}\|\oplus_{\mathrm{E}}\|\mathbf{v}\| = \frac{\|\mathbf{u}\|+\|\mathbf{v}\|}{1+\frac{1}{s^2}\|\mathbf{u}\|\|\mathbf{v}\|} \tag{3.151}$$

The restricted Einstein addition in (3.150) and (3.151) is both commutative and associative. Accordingly, the restricted Einstein addition is a group operation, as Einstein noted in [Einstein (1905)]; see [Einstein (1998), p. 142]. In contrast, Einstein made no remark about group properties of his addition of velocities that need not be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure that was discovered only in 1988 [Ungar (1988a)].

Interestingly, Einstein addition (3.150) of parallel vectors coincides with Möbius addition (3.132) of parallel vectors.

Einstein gyrations

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}] : \mathbb{V}_s \to \mathbb{V}_s \tag{3.152}$$

are automorphisms of the Einstein gyrogroup $(\mathbb{V}_s, \oplus_{\mathrm{E}})$,

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}] \in Aut(\mathbb{V}_s, \oplus_{\mathrm{E}}) \tag{3.153}$$

given by the equation, Theorem 2.8(10),

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \ominus_{_E}(\mathbf{u}\oplus_{_E}\mathbf{v})\oplus_{_E}\{\mathbf{u}\oplus_{_E}(\mathbf{v}\oplus_{_E}\mathbf{w})\} \tag{3.154}$$

and they preserve the inner product that the ball $\mathbb{V}_s$ inherits from its real inner product space $\mathbb{V}$,

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{3.155}$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{V}_s$.

3.7 Einstein Coaddition

Einstein gyrogroup cooperation (2.2) in an Einstein gyrogroup $(\mathbb{V}_s, \oplus_{_E})$ is given by Einstein coaddition

$$\begin{aligned}\mathbf{u}\boxplus_{_E}\mathbf{v} &= \frac{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2+\gamma_{\mathbf{v}}^2+\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2})-1}(\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v})\\ &= \frac{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}}{(\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}})^2-(\gamma_{\mathbf{u}\ominus\mathbf{v}}+1)}(\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v})\\ &= 2\otimes_{_E}\frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}}\end{aligned} \tag{3.156}$$

where the scalar multiplication by the factor 2 is defined by the equation $2\otimes_{_E}\mathbf{v} = \mathbf{v}\oplus_{_E}\mathbf{v}$. A more general definition of the scalar multiplication by any real number will be studied in Chap. 6.

Einstein coaddition is commutative, as expected from Theorem 3.4, satisfying the gamma identity

$$\begin{aligned}\gamma_{\mathbf{u}\boxplus_{_E}\mathbf{v}} &= \frac{\gamma_{\mathbf{u}}^2+\gamma_{\mathbf{v}}^2+\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2})-1}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1-\frac{\mathbf{u}\cdot\mathbf{v}}{s^2})+1}\\ &= \frac{(\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}})^2-(\gamma_{\mathbf{u}\ominus\mathbf{v}}+1)}{\gamma_{\mathbf{u}\ominus\mathbf{v}}+1}\end{aligned} \tag{3.157}$$

We will see in Chap. 10 that, unlike Einstein addition, Einstein coaddition admits a gyroparallelogram addition law, Fig. 10.10, p. 381. The latter, in turn, will be found to be *covariant* with respect to left gyrotranslations. Accordingly, it will become evident that in order to capture analogies with classical results, both Einstein addition and coaddition must be considered.

Einstein coaddition of two elements, $\mathbf{u}$ and $\mathbf{v}$, of an Einstein gyrogroup is symmetric in $\mathbf{u}$ and $\mathbf{v}$, as we see from (3.156). To capture deeper analogies one may wish to find an Einstein coaddition of three elements, $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, of an Einstein gyrogroup, which is symmetric in $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and which admits a 3-dimensional gyroparallelepiped addition law. Indeed, the extension to Einstein coaddition of any finite number, k, of elements, $\mathbf{v}_i$, $i = 1, \dots k$, which is symmetric in $\mathbf{v}_i$ and which admits a k-dimensional gyroparallelepiped addition law, will be uncovered in (10.66), p. 383. The 3-dimensional case will be illustrated graphically in Fig. 10.13, p. 391.

3.8 PV Gyrogroups

Definition 3.40 **(PV Addition).** *Let $(\mathbb{V}, +, \cdot)$ be a real inner product space with addition, $+$, and inner product, $\cdot$. The PV (Proper Velocity) gyrogroup $(\mathbb{V}, \oplus_U)$ is the real inner product space $\mathbb{V}$ equipped with addition $\oplus_U$, given by*

$$\mathbf{u}\oplus_U\mathbf{v} = \mathbf{u} + \mathbf{v} + \left\{ \frac{\beta_\mathbf{u}}{1+\beta_\mathbf{u}} \frac{\mathbf{u}\cdot\mathbf{v}}{s^2} + \frac{1-\beta_\mathbf{v}}{\beta_\mathbf{v}} \right\} \mathbf{u} \tag{3.158}$$

where $\beta_\mathbf{v}$, called the beta factor, is given by the equation

$$\beta_\mathbf{v} = \frac{1}{\sqrt{1+\frac{\|\mathbf{v}\|^2}{s^2}}} \tag{3.159}$$

PV addition is the relativistic addition of *proper velocities* rather than *coordinate velocities* as in Einstein addition [Ungar (2001), p. 143]. It can be shown by computer algebra that, as anticipated in Def. 3.40, PV addition is a gyrocommutative gyrogroup operation, giving rise to the PV gyrogroup $(\mathbb{V}, \oplus_U)$.

PV addition satisfies the *beta identity*

$$\frac{1}{\beta_{\mathbf{u}\oplus_U\mathbf{v}}} = \frac{1}{\beta_\mathbf{u}}\frac{1}{\beta_\mathbf{v}} + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2} \tag{3.160}$$

or, equivalently,

$$\beta_{\mathbf{u}\oplus_U\mathbf{v}} = \frac{\beta_\mathbf{u}\beta_\mathbf{v}}{1+\beta_\mathbf{u}\beta_\mathbf{v}\frac{\mathbf{u}\cdot\mathbf{v}}{s^2}} \tag{3.161}$$

The PV gyrogroup cooperation (2.2) is given by PV coaddition,

$$\mathbf{u} \boxplus_{\mathrm{U}} \mathbf{v} = \frac{\beta_{\mathbf{u}} + \beta_{\mathbf{v}}}{1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{s^2})}(\mathbf{u} + \mathbf{v}) \tag{3.162}$$

satisfying the beta identity

$$\frac{1}{\beta_{\mathbf{u}\boxplus_{\mathrm{U}}\mathbf{v}}} = \frac{\frac{1}{\beta_{\mathbf{u}}^2} + \frac{1}{\beta_{\mathbf{v}}^2} + \frac{1}{\beta_{\mathbf{u}}}\frac{1}{\beta_{\mathbf{v}}} - (1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2})}{\frac{1}{\beta_{\mathbf{u}}}\frac{1}{\beta_{\mathbf{v}}} + 1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \tag{3.163}$$

or, equivalently,

$$\beta_{\mathbf{u}\boxplus_{\mathrm{U}}\mathbf{v}} = \frac{\beta_{\mathbf{u}}\beta_{\mathbf{v}}(1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}))}{\beta_{\mathbf{u}}\beta_{\mathbf{v}}(1 - \beta_{\mathbf{u}}\beta_{\mathbf{v}}(1 - \frac{\mathbf{u}\cdot\mathbf{v}}{s^2})) + \beta_{\mathbf{u}}^2 + \beta_{\mathbf{v}}^2} \tag{3.164}$$

PV coaddition is commutative, as expected from Theorem 3.4.

When the vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{V}$ are parallel, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, PV addition reduces to

$$\mathbf{u}\oplus_{\mathrm{U}}\mathbf{v} = \frac{1}{\beta_{\mathbf{v}}}\mathbf{u} + \frac{1}{\beta_{\mathbf{u}}}\mathbf{v}, \qquad \mathbf{u}\|\mathbf{v} \tag{3.165}$$

and, accordingly,

$$\|\mathbf{u}\|\oplus_{\mathrm{U}}\|\mathbf{v}\| = \frac{1}{\beta_{\mathbf{v}}}\|\mathbf{u}\| + \frac{1}{\beta_{\mathbf{u}}}\|\mathbf{v}\| \tag{3.166}$$

The restricted PV addition in (3.165) and (3.166) is both commutative and associative.

PV gyrations

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{V} \to \mathbb{V} \tag{3.167}$$

are automorphisms of the PV gyrogroup $(\mathbb{V}, \oplus_{\mathrm{U}})$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in Aut(\mathbb{V}, \oplus_{\mathrm{U}}) \tag{3.168}$$

given by the equation, Theorem 2.8(10),

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus_{\mathrm{U}}(\mathbf{u}\oplus_{\mathrm{U}}\mathbf{v})\oplus_{\mathrm{U}}\{\mathbf{u}\oplus_{\mathrm{U}}(\mathbf{v}\oplus_{\mathrm{U}}\mathbf{w})\} \tag{3.169}$$

and they preserve the inner product in $\mathbb{V}$,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{3.170}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$.

3.9 Points and Vectors in a Real Inner Product Space

Elements of a real inner product space $\mathbb{V} = (\mathbb{V}, +, \cdot)$, called *points* and denoted by capital italic letters, $A, B, P, Q, \ldots$, give rise to vectors, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}, \ldots$. 'Vector' means 'carrier' in Latin. The most basic meaning is a relationship between two points, namely, the displacement that would carry one into the other. Any two points $P, Q \in \mathbb{V}$ give rise to a unique rooted vector $\mathbf{v} = PQ \in \mathbb{V}$, rooted at the point P. It has a tail at the point P and a head at the point Q,

$$\mathbf{v} = PQ = -P + Q \tag{3.171}$$

The length of the rooted vector $\mathbf{v} = PQ$ is the distance between the points P and Q, given by the equation

$$\|\mathbf{v}\| = |PQ| = \| - P + Q\| \tag{3.172}$$

The *degenerate* rooted vector PP is called the zero rooted vector. Any two zero rooted vectors are equivalent. Two nonzero rooted vectors PQ and RS are equivalent if $PQSR$ is a parallelogram. A *vector* is defined to be a collection of equivalent rooted vectors. Accordingly, two vectors are equal if they have the same length and direction.

A point $P \in \mathbb{V}$ is identified with the rooted vector OP, O being the origin of the space $\mathbb{V}$. Hence, the algebra of vectors can be applied to points as well.

The vector $\mathbf{v} = -P+Q$ translates the point P into the point Q according to the equation

$$Q = \mathbf{v} + P \tag{3.173}$$

Vector addition in $\mathbb{V}$ is given by the composition of two successive translations of a point. Hence, if $\mathbf{u} = QR = -Q + R$ then $\mathbf{u} + \mathbf{v}$ translates P to R according to the equation

$$\begin{aligned} R &= \mathbf{u} + Q \\ &= \mathbf{u} + (\mathbf{v} + P) \\ &= (\mathbf{u} + \mathbf{v}) + P \end{aligned} \tag{3.174}$$

so that

$$\mathbf{u} + \mathbf{v} = (-P + Q) + (-Q + R) = -P + R = PR \tag{3.175}$$

The elements of Möbius and Einstein gyrovector spaces in Secs. 3.5 and 3.6 are either points or vectors in the ball $\mathbb{V}_s$ of a real inner product space $\mathbb{V}$. They are, however, subjected to a binary operation different from the one in their space $\mathbb{V}$ in order to keep the ball closed under its binary operation. Hence, the vector algebra of the space $\mathbb{V}$ does not suit the algebra of its ball $\mathbb{V}_s$. Indeed, the adjustment of vector algebra to gyrocommutative gyrogroups will be presented in Chap. 5 and applied to gyrovector spaces in Chap. 6.

3.10 Exercises

(1) Identify the algebraic laws that allow the chains of equations (3.57) and (3.58).

(2) Employing a computer system for technical computing, like MATHEMATICA or MAPLE, verify Identities (3.144) – (3.157).

(3) Verify directly that the expression J in Theorem 3.17 and the expression J in Theorem 3.18 are identical.

(4) Verify the Identity in Theorem 3.17 by comparing two different expansions of the expression $a + \{(b + c) + x\}$ in a gyrocommutative gyrogroup $(G, +)$. Find a similar identity that is valid in gyrogroups that need not be gyrocommutative.

(5) Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Show that each of the composite gyrations J_1, J_2, and J_3 of G,

$$J_1 = \mathrm{gyr}[a, b\oplus x]\mathrm{gyr}[b, x]\mathrm{gyr}[x, b\oplus a] \tag{3.176}$$

$$J_2 = \mathrm{gyr}[\mathrm{gyr}[a, b]x, a]\mathrm{gyr}[a\oplus\mathrm{gyr}[a, b]x, b\oplus x]\mathrm{gyr}[b, x] \tag{3.177}$$

$$J_3 = \mathrm{gyr}[a\oplus b, \ominus\mathrm{gyr}[a, b \boxplus x]x]\mathrm{gyr}[a, b \boxplus x]\mathrm{gyr}[x, \ominus b] \tag{3.178}$$

$a, b, x \in G$, is independent of x. Use (3.176) to establish the identity

$$\mathrm{gyr}[a, \ominus b] = \mathrm{gyr}[\ominus a, b\oplus a]\mathrm{gyr}[b, a]\mathrm{gyr}[a, b\ominus a] \tag{3.179}$$

Use (3.177) with $x = 0$ and $x = a$ to establish the identity

$$(\mathrm{gyr}[a, b])^2 = \mathrm{gyr}[\mathrm{gyr}[a, b]a, a]\mathrm{gyr}[a\oplus\mathrm{gyr}[a, b]a, b\oplus a] \tag{3.180}$$

Use (3.178) to establish the identity

$$(\mathrm{gyr}[a, b])^{-1} = \mathrm{gyr}[a, \ominus\mathrm{gyr}[a, b]b] \tag{3.181}$$

(6) Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Prove that the gyrogroup expression

$$J_4 = \text{gyr}[c, x]\{\ominus(x\oplus c)\oplus\tfrac{1}{2}\otimes[(x\oplus a) \boxplus (x\oplus b)]\} \tag{3.182}$$

$a, b, c, x \in G$, is independent of x.

(7) Show that the solution of the equation

$$(x\otimes a) \boxplus (x\oplus b) = p \tag{3.183}$$

in a gyrocommutative gyrogroup $(G, \oplus)$ for the unknown $x\in G$ and any given $a, b, p\in G$ is

$$x = \tfrac{1}{2}\otimes p \boxminus \tfrac{1}{2}\otimes(a \boxplus b) \tag{3.184}$$

Hint: Use (3.182) with $c = 0$.

(8) Verify the following two associative-like laws in any gyrocommutative gyrogroup $(G, \oplus)$. For all $a, b, c \in G$,

$$(a \boxplus b)\oplus c = a\oplus \text{gyr}[a, \ominus b](b\oplus c) \tag{3.185}$$

and

$$a\oplus(b \boxplus c) = (a\oplus b) \boxplus \text{gyr}[a, b \boxplus c]c \tag{3.186}$$

Note that (3.185) gives rise to the right cancellation law (2.39), while (3.186) gives rise to the cancellation-like law,

$$a\oplus(\ominus a \boxplus b) = \text{gyr}[a, b]b \tag{3.187}$$

The latter also follows from the left cancellation law (2.38) and the definition of the gyrogroup coaddition $\boxplus$.

Chapter 4

Gyrogroup Extension

We show in this chapter that gyrogroups that are equipped with the so called *gyrofactor* admit special extensions, giving rise to new gyrogroups that admit the notion of inner product and norm, and possess transformation groups that keep their inner product and norm invariant. In gyrolanguage, these inner product and norm are called *gyroinner product* and *gyronorm*.

To appreciate the usefulness of the gyrogroup extension that we study in this chapter we may note that the gyrogroup extension of the Einstein relativistic gyrogroup $(\mathbb{R}_c^3, \oplus_E)$ of all relativistically admissible velocities gives rise to the gyrocommutative gyrogroup of "Lorentz boosts". A Lorentz boost, in turn, is a "Lorentz transformation without rotation" in the jargon. The Lorentz transformation group of spacetime in special relativity theory will turn out in Chap. 10 to be a gyrosemidirect product group of the gyrogroup of Lorentz boosts and a group of space rotations.

4.1 Gyrogroup Extension

Definition 4.1 **(Gyrofactors).** *Let ρ be a positive function, $\rho : G \to \mathbb{R}^{>0}$, of a gyrogroup $(G, \oplus)$. The function $\rho(v)$, $v \in G$, is called a gyrofactor of the gyrogroup $(G, \oplus)$ if it satisfies the following conditions:*

(1) *ρ is normalized, $\rho(0) = 1$, where 0 is the identity element of G.*
(2) *ρ is even, $\rho(\ominus v) = \rho(v)$ for all $v \in G$.*
(3) *ρ is gyroinvariant, that is, $\rho(Vv) = \rho(v)$ for all $v \in G$ and all $V \in Aut_0(G, \oplus)$. Here $Aut_0(G, \oplus)$ is a gyroautomorphism group of the gyrogroup $(G, \oplus)$, Def. 2.22.*

A gyrogroup $(G, \oplus)$ equipped with a gyrofactor $\rho : G \to \mathbb{R}^{>0}$ is denoted by $(G, \oplus, \rho)$. Accordingly, an automorphism group $Aut_0(G, \oplus)$ of $(G, \oplus) = (G, \oplus, \rho)$ may also be denoted by $Aut_0(G, \oplus, \rho)$ if one wishes to emphasize the presence of the gyrofactor ρ.

Definition 4.2 **(Gyrogroup Extension by a Gyrofactor).** *Let $(G, \oplus, \rho)$ be a gyrogroup with a gyrofactor $\rho : G \to \mathbb{R}^{>0}$. The gyrogroup $\mathbb{R}^{>0} \times G$ of pairs $(s, u)^t$ (exponent t denotes transposition), $s \in \mathbb{R}^{>0}$, $u \in G$, with gyrogroup operation given by*

$$\begin{pmatrix} s \\ u \end{pmatrix} \cdot \begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \dfrac{\rho(u \oplus v)}{\rho(u)\rho(v)} st \\ u \oplus v \end{pmatrix} \tag{4.1}$$

$(s, u)^t, (t, v)^t \in \mathbb{R}^{>0} \times G$, is said to be the gyrogroup extended from the gyrogroup $(G, \oplus, \rho)$, or the extended gyrogroup of $(G, \oplus, \rho)$, and is denoted $(\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$.

The seemingly non-intuitive term $\rho(u \oplus v)/(\rho(u)\rho(v))$ in (4.1) is the abstraction of (3.146), to which it reduces in the special case when $\rho(v) = \gamma_v$ is the Lorentz factor.

Interestingly, the composition law (4.1), written additively, arises in the study of commutative groups, where $\oplus$ is a commutative group operation rather than a gyrogroup operation; see, for instance, [Jessen, Karpf and Thorup (1968)].

It is anticipated in Def. 4.2 that the groupoid $(\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ forms a gyrogroup. The following theorem states that this is, indeed, the case.

Theorem 4.3 **(Gyrogroup Extension).** *Let $(G, \oplus, \rho)$ be a (gyrocommutative) gyrogroup with a gyrofactor, and let $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ be the groupoid extended from the gyrogroup $(G, \oplus, \rho)$ according to Def. 4.2. Then, the groupoid E forms a (gyrocommutative) gyrogroup.*

Proof. *Identity Element:* The identity element of E is $(1, 0)^t$, where 0 is the identity element of $(G, \oplus)$.

Inverse: The inverse of $(s, u)^t$ in E is

$$\begin{pmatrix} s \\ u \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\rho^2(u)}{s} \\ \ominus u \end{pmatrix} \tag{4.2}$$

where we use the notation $\rho^2(u) = (\rho(u))^2$.

Gyroautomorphisms: If E is a gyrogroup then, by (4.1), (4.2), and Theorem 2.8(10), its gyroautomorphisms are recovered by the following chain of equations.

$$
\begin{aligned}
\mathrm{gyr}\left[\begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right]\begin{pmatrix} r \\ w \end{pmatrix} &= \left(\begin{pmatrix} s \\ u \end{pmatrix}\cdot\begin{pmatrix} t \\ v \end{pmatrix}\right)^{-1}\cdot\left\{\begin{pmatrix} s \\ u \end{pmatrix}\cdot\left(\begin{pmatrix} t \\ v \end{pmatrix}\cdot\begin{pmatrix} r \\ w \end{pmatrix}\right)\right\} \\
&= \begin{pmatrix} \frac{\rho(u\oplus v)}{\rho(u)\rho(v)}st \\ u\oplus v \end{pmatrix}^{-1}\cdot\left\{\begin{pmatrix} s \\ u \end{pmatrix}\cdot\begin{pmatrix} \frac{\rho(v\oplus w)}{\rho(v)\rho(w)}tr \\ v\oplus w \end{pmatrix}\right\} \\
&= \begin{pmatrix} \rho^2(u\oplus v)\frac{\rho(u)\rho(v)}{\rho(u\oplus v)st} \\ \ominus(u\oplus v) \end{pmatrix}\cdot\begin{pmatrix} \frac{\rho(u\oplus(v\oplus w))\rho(v\oplus w)}{\rho(u)\rho(v\oplus w)\rho(v)\rho(w)}str \\ u\oplus(v\oplus w) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(\ominus(u\oplus v)\oplus\{u\oplus(v\oplus w)\})}{\rho(\ominus(u\oplus v))\rho(u\oplus(v\oplus w))}\rho^2(u\oplus v)\frac{\rho(u)\rho(v)}{\rho(u\oplus v)st}\frac{\rho(u\oplus(v\oplus w))\rho(v\oplus w)}{\rho(u)\rho(v\oplus w)\rho(v)\rho(w)}str \\ \ominus(u\oplus v)\oplus\{u\oplus(v\oplus w)\} \end{pmatrix} \\
&= \begin{pmatrix} \rho(\ominus(u\oplus v)\oplus\{u\oplus(v\oplus w)\})\frac{r}{\rho(w)} \\ \ominus(u\oplus v)\oplus\{u\oplus(v\oplus w)\} \end{pmatrix} \\
&= \begin{pmatrix} \rho(\mathrm{gyr}[u,v]w)\frac{r}{\rho(w)} \\ \mathrm{gyr}[u,v]w \end{pmatrix} \\
&= \begin{pmatrix} r \\ \mathrm{gyr}[u,v]w \end{pmatrix}
\end{aligned}
\tag{4.3}
$$

Hence, if E is a gyrogroup, then its gyrations are induced by the gyrations $\mathrm{gyr}[u,v]$, $u,v\in G$, of its underlying gyrogroup $(G,\oplus)$ according to the following equation, (4.3),

$$
\mathrm{gyr}\left[\begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right]\begin{pmatrix} r \\ w \end{pmatrix} = \begin{pmatrix} r \\ \mathrm{gyr}[u,v]w \end{pmatrix} \tag{4.4}
$$

We, therefore, adopt (4.4) as the definition of the gyrator gyr of E.

It follows from (4.4) that, indeed, the gyrations $\mathrm{gyr}[(s,u)^t,(t,v)^t]$ of

E in (4.4) obey the gyrogroup axioms that the gyrations $\mathrm{gyr}[u,v]$ of the underlying gyrogroup $(G,\oplus)$ obey, so that if $(G,\oplus)$ is a (gyrocommutative) gyrogroup then also E is a (gyrocommutative) gyrogroup. As an illustrative example that demonstrates that E inherits gyrogroup properties of G, let us verify the loop property for E. By (4.1) and (4.4), and by the loop property of $(G,\oplus)$, we have

$$\begin{aligned}
\mathrm{gyr}\left[\begin{pmatrix} s \\ u \end{pmatrix}\cdot\begin{pmatrix} t \\ v \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right]\begin{pmatrix} r \\ w \end{pmatrix} &= \mathrm{gyr}\left[\begin{pmatrix} \frac{\rho(u\oplus v)}{\rho(u)\rho(v)}st \\ u\oplus v \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right]\begin{pmatrix} r \\ w \end{pmatrix} \\
&= \begin{pmatrix} r \\ \mathrm{gyr}[u\oplus v, v]w \end{pmatrix} \\
&= \begin{pmatrix} r \\ \mathrm{gyr}[u, v]w \end{pmatrix} \\
&= \mathrm{gyr}\left[\begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right]\begin{pmatrix} r \\ w \end{pmatrix}
\end{aligned} \tag{4.5}$$

□

4.2 The Gyroinner Product, the Gyronorm, and the Gyroboost

The extended gyrogroup admits a gyroinner product and a gyronorm, the definition of which follows.

Definition 4.4 **(The Gyroinner Product and the Gyronorm).** *Let $E = (\mathbb{R}^{>0}\times G, \cdot\,; \oplus, \rho)$ be the gyrogroup extended from a gyrogroup $(G,\oplus,\rho)$ with a gyrofactor. The inner product, $< (s,u)^t, (t,v)^t >$, of any two elements $(s,u)^t$ and $(t,v)^t$ of E is a nonnegative number given by the equation*

$$\left\langle \begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix} \right\rangle = \frac{\rho(u\ominus v)}{\rho(u)\rho(v)}st \tag{4.6}$$

The squared norm of any element $(t,v)^t$ of E is, accordingly, defined by the equation

$$\left\| \begin{pmatrix} t \\ v \end{pmatrix} \right\|^2 = \left\langle \begin{pmatrix} t \\ v \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix} \right\rangle \tag{4.7}$$

so that

$$\left\| \begin{pmatrix} t \\ v \end{pmatrix} \right\| = \frac{t}{\rho(v)} \tag{4.8}$$

for any element $(t, v)^t$ *of* E.

It follows from (4.8) that the elements $(\rho(v), v)^t$ of E, $v \in G$, are unimodular,

$$\left\| \begin{pmatrix} \rho(v) \\ v \end{pmatrix} \right\| = 1 \tag{4.9}$$

Definition 4.5 (Gyroboosts). *A gyroboost is a unimodular element of an extended gyrogroup* E. *Specifically, let* $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ *be the gyrogroup extended from the gyrogroup* $(G, \oplus, \rho)$. *Then the gyroboost* $B(v)$, *parametrized by* $v \in G$, *is the element*

$$B(v) = \begin{pmatrix} \rho(v) \\ v \end{pmatrix} \tag{4.10}$$

of E.

The identity gyroboost is $B(0)$, where 0 is the identity element of G, and the inverse of a gyroboost $B(v)$ is the gyroboost

$$(B(v))^{-1} = B(\ominus v) \tag{4.11}$$

for all $v \in (G, \oplus)$.

The product of two gyroboosts in an extended gyrogroup E is, again, a gyroboost in E given by gyroboost parameter gyroaddition,

$$B(u) \cdot B(v) = \begin{pmatrix} \rho(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} \rho(v) \\ v \end{pmatrix} = \begin{pmatrix} \rho(u \oplus v) \\ u \oplus v \end{pmatrix} = B(u \oplus v) \tag{4.12}$$

for all $u, v \in (G, \oplus)$. Hence, the set of all gyroboosts $B(v)$ in an extended gyrogroup $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ forms a gyrogroup which is *isomorphic* (in the gyrogroup sense: two isomorphic gyrogroups are, algebraically, the same gyrogroup) to the underlying parameter gyrogroup $(G, \oplus)$.

Gyroboosts act on their extended gyrogroup according to the following definition.

Definition 4.6 (Gyroboost Application). *Let $(G, \oplus, \rho)$ be a gyrogroup with a gyrofactor. The application*

$$B(u): E \to E \tag{4.13}$$

$u \in G$, of the gyroboost $B(u)$ to elements $(t, v)^t$ of its extended gyrogroup $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ is given by the extended gyrogroup operation,

$$B(u)\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \rho(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} t \\ v \end{pmatrix} \tag{4.14}$$

It follows from Defs. 4.6 and 4.2 that the gyroboost application to elements of its extended gyrogroup E is given by the equation

$$B(u)\begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \rho(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} t \\ v \end{pmatrix} = \begin{pmatrix} \frac{\rho(u \oplus v)}{\rho(v)} t \\ \\ u \oplus v \end{pmatrix} \tag{4.15}$$

Gyroboosts are important transformations of E since they keep the gyroinner product in E invariant. In the context of Einstein's special theory of relativity these are nothing else but the Lorentz transformations without rotation, known in the jargon as boosts, as we will see in Chap. 10.

Theorem 4.7 *Gyroboosts preserve the gyroinner product, that is,*

$$\left\langle B(a)\begin{pmatrix} s \\ u \end{pmatrix}, B(a)\begin{pmatrix} t \\ v \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix} \right\rangle = \frac{\rho(u \ominus v)}{\rho(u)\rho(v)} st \tag{4.16}$$

for all $a, u, v \in G$ and $s, t \in \mathbb{R}^{>0}$.

Hence, in particular, gyroboosts preserve the gyronorm,

$$\left\| B(u)\begin{pmatrix} t \\ v \end{pmatrix} \right\| = \left\| \begin{pmatrix} t \\ v \end{pmatrix} \right\| = \frac{t}{\rho(v)} \tag{4.17}$$

Proof. By the gyroboost and the gyroinner product definition, by Theorem 3.13, and by the gyroinvariance of the gyrofactor ρ, we have the following chain of equations,

$$\begin{aligned}
\left\langle B(a)\begin{pmatrix} s \\ u \end{pmatrix}, B(a)\begin{pmatrix} t \\ v \end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix} \rho(a) \\ a \end{pmatrix}\cdot\begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} \rho(a) \\ a \end{pmatrix}\cdot\begin{pmatrix} t \\ v \end{pmatrix}\right\rangle \\
&= \left\langle \begin{pmatrix} \frac{\rho(a\oplus u)}{\rho(u)}s \\ a\oplus u \end{pmatrix}, \begin{pmatrix} \frac{\rho(a\oplus v)}{\rho(v)}t \\ a\oplus v \end{pmatrix}\right\rangle \\
&= \frac{\rho((a\oplus u)-(a\oplus v))}{\rho(a\oplus u)\rho(a\oplus v)}\frac{\rho(a\oplus u)}{\rho(u)}\frac{\rho(a\oplus v)}{\rho(v)}st \\
&= \frac{\rho((a\oplus u)-(a\oplus v))}{\rho(u)\rho(v)}st \qquad (4.18) \\
&= \frac{\rho(\mathrm{gyr}[a,u](u\ominus v))}{\rho(u)\rho(v)}st \\
&= \frac{\rho(u\ominus v))}{\rho(u)\rho(v)}st \\
&= \left\langle \begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right\rangle
\end{aligned}$$

as desired. □

In general, gyroboosts of an extended gyrogroup do not form a group since the application of two successive gyroboosts is not equivalent to a single gyroboost but, rather, to a single gyroboost preceded, or followed, by a gyration.

Accordingly, the following theorem presents the gyroboost composition law, from which we find that it is owing to the presence of gyrations that the composition of two gyroboosts is not a gyroboost.

Theorem 4.8 (Gyroboost Composition Law). *The application of two successive gyroboosts is equivalent to the application of a single gyroboost preceded by a gyration.*

Proof. Applying successively $B(v)$ and $B(u)$ to an element $(t, w)^t$ of E we have

$$
\begin{aligned}
B(u)\left\{B(v)\begin{pmatrix} t \\ w \end{pmatrix}\right\} &= \begin{pmatrix} \rho(u) \\ u \end{pmatrix} \cdot \left\{\begin{pmatrix} \rho(v) \\ v \end{pmatrix} \cdot \begin{pmatrix} t \\ w \end{pmatrix}\right\} \\
&= \begin{pmatrix} \rho(u) \\ u \end{pmatrix} \cdot \begin{pmatrix} \frac{\rho(v\oplus w)}{\rho(w)} t \\ v\oplus w \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(u\oplus(v\oplus w))}{\rho(w)} t \\ u\oplus(v\oplus w) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho((u\oplus v)\oplus \mathrm{gyr}[u,v]w))}{\rho(\mathrm{gyr}[u,v]w)} t \\ (u\oplus v)\oplus \mathrm{gyr}[u,v]w \end{pmatrix} \\
&= \begin{pmatrix} \rho(u\oplus v) \\ u\oplus v \end{pmatrix} \cdot \begin{pmatrix} t \\ \mathrm{gyr}[u,v]w \end{pmatrix} \\
&= B(u\oplus v) \begin{pmatrix} t \\ \mathrm{gyr}[u,v]w \end{pmatrix} \\
&= \{B(u) \cdot B(v)\} \begin{pmatrix} t \\ \mathrm{gyr}[u,v]w \end{pmatrix}
\end{aligned}
\tag{4.19}
$$

Hence, the application of the two successive gyroboosts $B(v)$ and $B(u)$ is equivalent to the application of the single gyroboost $B(u\oplus v)$ preceded by a gyration.

In the chain of equations (4.19) we employ the gyroassociative law of the gyrogroup operation $\oplus$, the invariance of the gyrofactor under gyrations, and (4.12). $\square$

Suggestively, we use the notation

$$
\mathrm{Gyr}[u,v] \begin{pmatrix} t \\ w \end{pmatrix} = \begin{pmatrix} t \\ \mathrm{gyr}[u,v]w \end{pmatrix} \tag{4.20}
$$

so that the result of the chain of equations (4.19) can be written as the identity

$$B(u)B(v)\begin{pmatrix} t \\ w \end{pmatrix} = B(u\oplus v)\mathrm{Gyr}[u,v]\begin{pmatrix} t \\ w \end{pmatrix} \tag{4.21}$$

Similarly, one can also establish the identity

$$B(u)B(v)\begin{pmatrix} t \\ w \end{pmatrix} = \mathrm{Gyr}[u,v]B(v\oplus u)\begin{pmatrix} t \\ w \end{pmatrix} \tag{4.22}$$

see, for details, Lemma 4.14. Hence, two successive boosts are equivalent to a single boost preceded, (4.21), or followed, (4.22), by a gyration.

The extended gyrogroup identities (4.21) and (4.22) are valid for any element $(t, w)^t$ of their extended gyrogroup. Hence, they can be written as the gyroboost identities in the following

Theorem 4.9 *Let $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ be the extended gyrogroup of the gyrogroup $(G, \oplus, \rho)$ with the gyrofactor ρ, and let $B(v)$ and $\mathrm{Gyr}[u,v]$, $u, v \in G$, be its gyroboosts and gyrations. Then*

$$\begin{aligned} B(u)B(v) &= B(u\oplus v)\mathrm{Gyr}[u,v] \\ B(u)B(v) &= \mathrm{Gyr}[u,v]B(v\oplus u) \end{aligned} \tag{4.23}$$

Contrasting the general application of successive boosts, which involves gyrations, a "symmetric" successive boost application is gyration free, that is, it is equivalent to the application of a single boost. Three examples that illustrate the general case are presented in the following three boost identities.

$$\begin{aligned} B(u)B(u)\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} &= B(2\otimes u)\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} \\ B(v)B(u)B(v)\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} &= B(v\oplus(u\oplus v))\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} \\ B(v)B(u)B(u)B(v)\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} &= B(v\oplus(u\oplus(u\oplus v)))\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} \\ &= B(2\otimes(u\oplus v))\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} \end{aligned} \tag{4.24}$$

where $2\otimes u = u\oplus u$. The last identity in (4.24) will be derived in Theorem 6.7, p. 140.

4.3 The Extended Automorphisms

Extended automorphisms of a gyrogroup automorphisms are automorphisms of the extended gyrogroup. In particular, the extended gyroautomorphisms of a gyrogroup are the gyroautomorphism of the extended gyrogroup. The definition of extended automorphisms thus follows.

Definition 4.10 **(Extended Automorphisms).** *Let $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ be the extended gyrogroup of a gyrogroup $(G, \oplus, \rho)$, and let $Aut_0(G, \oplus, \rho)$ be any gyroautomorphism group of $(G, \oplus, \rho)$, Def. 4.1. For any $V \in Aut_0(G, \oplus, \rho)$, $E(V)$ is the transformation of the extended gyrogroup E given by the equation*

$$E(V)\begin{pmatrix} t \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} t \\ V\mathbf{v} \end{pmatrix} \tag{4.25}$$

$t \in \mathbb{R}, \mathbf{v} \in G$.

The transformation $E(V)$, $V \in Aut_0(G, \oplus)$, is called the gyroautomorphism extension of V.

Thus, for instance,

$$E(\mathrm{gyr}[u, v]) = \mathrm{Gyr}[u, v] \tag{4.26}$$

or, equivalently,

$$E(\mathrm{gyr}[u, v])\begin{pmatrix} t \\ w \end{pmatrix} = \begin{pmatrix} t \\ \mathrm{gyr}[u, v]w \end{pmatrix} = \mathrm{Gyr}[u, v]\begin{pmatrix} t \\ w \end{pmatrix} \tag{4.27}$$

for all $(t, w)^t$ in the extended gyrogroup E, as we see from (4.25) and (4.20).

The following lemma shows that the extended automorphism is an automorphism.

Lemma 4.11 *Let $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ be the gyrogroup extended from a gyrogroup $(G, \oplus, \rho)$, and let $E(V)$ be the extension of an automorphism $V \in Aut_0(G, \oplus, \rho)$. Then $E(V)$ is an automorphism of E.*

Proof. The map $E(V)$ is invertible, $(E(V))^{-1} = E(V^{-1})$, V^{-1} being the inverse of V in $Aut_0(G, \oplus, \rho)$. Moreover, $E(V)$ respects the binary operation $\cdot$ in E,

$$
\begin{aligned}
E(V)\left\{\begin{pmatrix} s \\ u \end{pmatrix}\cdot\begin{pmatrix} t \\ v \end{pmatrix}\right\} &= E(V)\begin{pmatrix} \frac{\rho(u\oplus v)}{\rho(u)\rho(v)}st \\ u\oplus v \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(u\oplus v)}{\rho(u)\rho(v)}st \\ V(u\oplus v) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(V(u\oplus v))}{\rho(Vu)\rho(Vv)}st \\ V(u\oplus v) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(Vu\oplus Vv)}{\rho(Vu)\rho(Vv)}st \\ Vu\oplus Vv \end{pmatrix} \\
&= \begin{pmatrix} s \\ Vu \end{pmatrix}\cdot\begin{pmatrix} t \\ Vv \end{pmatrix} \\
&= E(V)\begin{pmatrix} s \\ u \end{pmatrix}\cdot E(V)\begin{pmatrix} t \\ v \end{pmatrix}
\end{aligned}
\tag{4.28}
$$

Hence $E(V)$ is an automorphism of E. □

In particular, extended gyrations $E(\mathrm{gyr}[u,v])$ of gyrations $\mathrm{gyr}[u,v]$ are automorphisms of E. For these we use the special notation (4.26).

The importance of the automorphisms $E(V)$ of E rests on the result that they preserve the gyroinner product in E.

Theorem 4.12 *Let $(G,\oplus,\rho)$ be a gyrogroup with a gyrofactor. The automorphisms $E(V)$, $V\in Aut_0(G,\oplus,\rho)$, of E preserve the gyroinner product in $E=(\mathbb{R}^{>0}\times G,\cdot\,;\oplus,\rho)$.*

Proof. Since V is an automorphism of $(G,\oplus)$ and the gyrofactor ρ is gyroinvariant, we have

$$
\begin{aligned}
\left\langle E(V)\begin{pmatrix} s \\ u \end{pmatrix}, E(V)\begin{pmatrix} t \\ v \end{pmatrix}\right\rangle &= \left\langle \begin{pmatrix} s \\ Vu \end{pmatrix}, \begin{pmatrix} t \\ Vv \end{pmatrix}\right\rangle \\
&= \frac{\rho(Vu\ominus Vv)}{\rho(Vu)\rho(Vv)}st \\
&= \frac{\rho(V(u\ominus v))}{\rho(Vu)\rho(Vv)}st \\
&= \frac{\rho(u\ominus v)}{\rho(u)\rho(v)}st \\
&= \left\langle \begin{pmatrix} s \\ u \end{pmatrix}, \begin{pmatrix} t \\ v \end{pmatrix}\right\rangle
\end{aligned}
\tag{4.29}
$$

□

Lemma 4.13 *Let $E = (\mathbb{R}^{>0}\times G, \cdot\,; \oplus, \rho)$ be the gyrogroup extended from a gyrogroup $(G, \oplus)$, and let $E(V)$ be the automorphism of E extended from an automorphism V of $(G, \oplus)$. Then, any gyroboost $B(v)$, $v \in G$, of E "commutes" with the automorphism $E(V)$ according to the equation*

$$
E(V)B(v) = B(Vv)E(V) \tag{4.30}
$$

Proof. For any $v\in(G, \oplus)$, $(t, w)^t\in E$, $V\in Aut_0(G, \oplus)$, we have

$$
\begin{aligned}
E(V)B(v)\begin{pmatrix} t \\ w \end{pmatrix} &= E(V)\begin{pmatrix} \frac{\rho(v\oplus w)}{\rho(v)\rho(w)}t \\ v\oplus w \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(v\oplus w)}{\rho(v)\rho(w)}t \\ V(v\oplus w) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(V(v\oplus\oplus w))}{\rho(Vv)\rho(Vw)}t \\ V(v\oplus w) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(Vv\oplus Vw)}{\rho(Vv)\rho(Vw)}t \\ Vv\oplus Vw \end{pmatrix} \\
&= B(Vv)\begin{pmatrix} t \\ Vw \end{pmatrix} \\
&= B(Vv)E(V)\begin{pmatrix} t \\ w \end{pmatrix}
\end{aligned}
\tag{4.31}
$$

thus implying (4.30). □

As an application of Lemma 4.13 we verify the following

Lemma 4.14 *Let $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ be the gyrocommutative gyrogroup extended from a gyrocommutative gyrogroup $(G, \oplus, \rho)$, and let $B(v)$ be a boost of E. Then*

$$\mathrm{Gyr}[u,v]B(v\oplus u) = B(u\oplus v)\mathrm{Gyr}[u,v] \tag{4.32}$$

for all $u, v \in G$.

Proof. By Lemma 4.13 and by (4.26),

$$\begin{aligned}\mathrm{Gyr}[u,v]B(v\oplus u) &= E(\mathrm{gyr}[u,v])B(v\oplus u) \\ &= B(\mathrm{gyr}[u,v](v\oplus u))E(\mathrm{gyr}[u,v]) \\ &= B(u\oplus v)\mathrm{Gyr}[u,v]\end{aligned} \tag{4.33}$$

□

4.4 Gyrotransformation Groups

Motivated by the Lorentz transformation group of the special theory of relativity, that we will study in Chap. 10, in the following definition we define the gyrotransformation group of any given extended gyrogroup.

Definition 4.15 (Gyrotransformations). *The gyrotransformation $L(\mathbf{v}, V)$ is a self-transformation of an extended gyrogroup, $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$, parametrized by the gyrogroup parameter $v \in G$ and the automorphism parameter $V \in Aut_0(G, \oplus)$, given by the equation*

$$L(v, V)\begin{pmatrix} t \\ w \end{pmatrix} = B(v)E(V)\begin{pmatrix} t \\ w \end{pmatrix} \tag{4.34}$$

for all $(t, w)^t \in E$.

Accordingly, a gyrotransformation of an extended gyrogroup is a gyroboost $B(v)$ preceded by an automorphism $E(V)$ of the extended gyrogroup.

It follows from (4.34), (4.25), (4.15), and the gyroinvariance of the gyrofactor, Def. 4.1, that

$$\begin{aligned} L(v,V)\begin{pmatrix} t \\ w \end{pmatrix} &= B(v)E(V)\begin{pmatrix} t \\ w \end{pmatrix} \\ &= B(v)\begin{pmatrix} t \\ Vw \end{pmatrix} \\ &= \begin{pmatrix} \frac{\rho(v\oplus Vw)}{\rho(Vw)}t \\ v\oplus Vw \end{pmatrix} \\ &= \begin{pmatrix} \frac{\rho(v\oplus Vw)}{\rho(w)}t \\ v\oplus Vw \end{pmatrix} \end{aligned} \tag{4.35}$$

Theorem 4.16 *Gyrotransformations keep the gyroinner product and the gyronorm invariant.*

Proof. The proof follows from Def. 4.15 and Theorems 4.7 and 4.12. □

Theorem 4.17 *The set of all gyrotransformations of a given extended gyrogroup $E = (\mathbb{R}^{>0} \times G, \cdot\,; \oplus, \rho)$ form a group under gyrotransformation composition, given by the gyrosemidirect product*

$$L(u,U)L(v,V) = L(u\oplus Uv, \mathrm{gyr}[u, Uv]UV) \tag{4.36}$$

for all $u, v \in G$ and $U, V \in Aut_0(G, \oplus)$.

Proof. The identity gyrotransformation is $L(0, I)$, where 0 and I are the identity element and the identity automorphism of the underlying gyrogroup $(G, \oplus)$.

The inverse gyrotransformation is

$$L^{-1}(v,V) = L(\ominus V^{-1}v, V^{-1}) \tag{4.37}$$

so that gyrotransformations are bijective.

Calculating the gyrotransformation composition, we have by (4.35)

$$
\begin{aligned}
L(u,U)L(v,V)\begin{pmatrix} t \\ w \end{pmatrix} &= L(u,U)\begin{pmatrix} \frac{\rho(v\oplus Vw)}{\rho(w)}t \\ v\oplus Vw \end{pmatrix} \\
&= B(u)\begin{pmatrix} \frac{\rho(v\oplus Vw)}{\rho(w)}t \\ Uv\oplus UVw \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(u\oplus(Uv\oplus UVw))}{\rho(Uv\oplus UVw)}\frac{\rho(v\oplus Vw)}{\rho(w)}t \\ u\oplus(Uv\oplus UVw) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho(u\oplus(Uv\oplus UVw))}{\rho(v\oplus Vw)}\frac{\rho(v\oplus Vw)}{\rho(w)}t \\ u\oplus(Uv\oplus UVw) \end{pmatrix} \qquad (4.38) \\
&= \begin{pmatrix} \frac{\rho(u\oplus(Uv\oplus UVw))}{\rho(w)}t \\ u\oplus(Uv\oplus UVw) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\rho((u\oplus Uv)\oplus\mathrm{gyr}[u,Uv]UVw)}{\rho(w)}t \\ (u\oplus Uv)\oplus\mathrm{gyr}[u,Uv]UVw \end{pmatrix} \\
&= L(u\oplus Uv, \mathrm{gyr}[u,Uv]UV)\begin{pmatrix} t \\ w \end{pmatrix}
\end{aligned}
$$

for all $(t,w)^t \in E$, thus verifying (4.36).

The composition law (4.36) of gyrotransformations is given in terms of parameter composition where the latter, in turn, is recognized as a gyrosemidirect product, (2.59). Hence, by Theorem 2.23, the set of all gyrotransformations of a given extended gyrogroup E forms a gyrosemidirect product group. □

As explained in Sec. 4.5, we will find in Chap. 10 that the gyrotransformation group is, in fact, the *abstract Lorentz transformation group* in the sense that if one realizes the abstract gyrogroup $(G,\oplus)$ by the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_{\mathrm{E}})$, then (i) the resulting realization of the abstract Lorentz transformation group gives the familiar Lorentz transformation group of special relativity theory, and (ii) the resulting realization of the gyrofactor of the abstract Lorentz group gives the familiar *Lorentz factor.*

4.5 Einstein Gyrotransformation Groups

Let us realize (i) the abstract gyrogroup $(G, \oplus)$ by the Einstein gyrogroup $(\mathbb{V}_s, \oplus_{\mathrm{E}})$, and (ii) the gyrofactor $\rho(v)$ by the gamma factor $\gamma_{\mathbf{v}}$, (3.129). Then, Def. 4.15 and its resulting identity (4.35) give rise, by means of Identity (3.144), to the Einstein gyrotransformation

$$
\begin{aligned}
L_e(\mathbf{v}, V)\begin{pmatrix} t \\ \mathbf{w} \end{pmatrix} &= \begin{pmatrix} \frac{\gamma_{\mathbf{v}\oplus_{\mathrm{E}} V\mathbf{w}}}{\gamma_{V\mathbf{w}}} t \\ \mathbf{v}\oplus_{\mathrm{E}} V\mathbf{w} \end{pmatrix} \\
&= \begin{pmatrix} (1 + \frac{\mathbf{v}\cdot V\mathbf{w}}{c^2})t \\ \mathbf{v}\oplus_{\mathrm{E}} V\mathbf{w} \end{pmatrix}
\end{aligned}
\tag{4.39}
$$

for all $(t, \mathbf{w})^t \in \mathbb{R}^{>0} \times (\mathbb{V}_s, \oplus_{\mathrm{E}})$, where $\mathbf{v}\in\mathbb{V}_s$ and $V \in Aut_0(\mathbb{V}_s, \oplus_{\mathrm{E}})$.

The gyrotransformations $L_e(\mathbf{v}, V)$ of the gyrogroup

$$
\mathbb{R}^{>0} \times (\mathbb{V}_s, \oplus_{\mathrm{E}}) \tag{4.40}
$$

extended from the Einstein gyrogroup $(\mathbb{V}_s, \oplus_{\mathrm{E}})$ form a group with group operation given by the gyrosemidirect product (4.36),

$$
L_e(\mathbf{u}, U)L_e(\mathbf{v}, V) = L_e(\mathbf{u}\oplus_{\mathrm{E}} U\mathbf{v}, \mathrm{gyr}[\mathbf{u}, U\mathbf{v}]UV) \tag{4.41}
$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{V}_s$ and $U, V \in Aut_0(\mathbb{V}_s, \oplus_{\mathrm{E}})$.

In Chap. 10 we will see that in the special case when the ball $\mathbb{V}_s$ is realized by the ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$, the Einstein gyrotransformation group $L_e(\mathbf{v}, V)$ reduces to the familiar Lorentz group of special relativity theory, parametrized by "coordinate velocities" $\mathbf{v}\in\mathbb{R}^3_c$ and "orientations" $V \in SO(3)$.

4.6 PV (Proper Velocity) Gyrotransformation Groups

Let us realize (i) the abstract gyrogroup $(G, \oplus)$ by the PV gyrogroup $(\mathbb{V}, \oplus_{\mathrm{U}})$, and (ii) the gyrofactor $\rho(v)$ by the identity map $\rho(\mathbf{v}) = 1$, $\mathbf{v}\in\mathbb{V}$.

Then, Def. 4.15 and its resulting identity (4.35) give rise to the PV gyrotransformation

$$L_u(\mathbf{v}, V)\begin{pmatrix}\tau\\ \mathbf{w}\end{pmatrix} = \begin{pmatrix}\tau\\ \mathbf{v}\oplus_{\mathrm{U}} V\mathbf{w}\end{pmatrix} \tag{4.42}$$

for all $(\tau, \mathbf{w})^t \in \mathbb{R}^{>0} \times (\mathbb{V}, \oplus_{\mathrm{U}})$, where $\mathbf{v}\in\mathbb{V}$ and $V\in Aut_0(\mathbb{V}, \oplus_{\mathrm{U}})$.

The gyrotransformations $L_u(\mathbf{v}, V)$ of the gyrogroup

$$\mathbb{R}^{>0} \times (\mathbb{V}, \oplus_{\mathrm{U}}) \tag{4.43}$$

extended from the PV gyrogroup $(\mathbb{V}, \oplus_{\mathrm{U}})$ form a group with group operation given by the gyrosemidirect product (4.36),

$$L_u(\mathbf{u}, U)L_u(\mathbf{v}, V) = L_u(\mathbf{u}\oplus_{\mathrm{E}} U\mathbf{v}, \mathrm{gyr}[\mathbf{u}, U\mathbf{v}]UV) \tag{4.44}$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{V}$ and $U, V\in Aut_0(\mathbb{V}, \oplus_{\mathrm{U}})$.

In Chap. 10 we will see that in the special case when the space $\mathbb{V}$ is realized by the Euclidean 3-space $\mathbb{R}^3$, the PV gyrotransformation group reduces to the novel "proper Lorentz group" of special relativity theory, parametrized by "proper velocities" $\mathbf{v}\in\mathbb{R}^3$ and "orientations" $V\in SO(3)$.

4.7 Galilei Transformation Groups

Let us realize (i) the abstract gyrogroup $(G, \oplus)$ by the group $(\mathbb{V}, +)$, and (ii) the gyrofactor $\rho(v)$ by the identity map $\rho(\mathbf{v}) = 1$, $\mathbf{v}\in\mathbb{V}$. Then, definition 4.15 and its resulting identity (4.35) give rise to the gyrotransformation

$$L_g(\mathbf{v}, V)\begin{pmatrix}t\\ \mathbf{w}\end{pmatrix} = \begin{pmatrix}t\\ \mathbf{v} + V\mathbf{w}\end{pmatrix} \tag{4.45}$$

for all $(t, \mathbf{w})^t \in \mathbb{R}^{>0} \times (\mathbb{V}, +)$, where $\mathbf{v}\in\mathbb{V}$ and $V\in Aut_0(\mathbb{V}, +)$.

A group is a gyrogroup with trivial gyrations. Hence, the gyrotransformation (4.45) involves *no* gyrations. Accordingly, we call it a transformation rather than a gyrotransformation.

The transformations $L_g(\mathbf{v}, V)$ of the group

$$\mathbb{R}^{>0} \times (\mathbb{V}, +) \tag{4.46}$$

extended from the group $(\mathbb{V}, +)$ form a group with group operation given by the semidirect product (4.36),

$$L_g(\mathbf{u}, U)L_g(\mathbf{v}, V) = L_g(\mathbf{u} + U\mathbf{v}, UV) \tag{4.47}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and $U, V \in Aut_0(\mathbb{V}, \oplus_U)$. The semidirect product in (4.47) is, in fact, a gyrosemidirect product with trivial gyrations.

In the special case when the space $\mathbb{V}$ is realized by the Euclidean 3-space $\mathbb{R}^3$, the Galilei transformation group reduces to the familiar Galilei transformation group of classical mechanics, parametrized by velocities $\mathbf{v} \in \mathbb{R}^3$ and orientations $V \in SO(3)$.

4.8 From Gyroboosts to Boosts

Let $\mathbb{V}_s = (\mathbb{V}_s, \oplus, \rho)$ be a gyrogroup with a gyrofactor. There is a bijective (one-to-one) correspondence

$$\begin{pmatrix} t \\ \mathbf{v} \end{pmatrix} \longleftrightarrow \begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \tag{4.48}$$

between elements $(t, \mathbf{v})^t$ of the extended gyrogroup E of the gyrogroup $\mathbb{V}_s$,

$$E = (\mathbb{R}^{>0} \times \mathbb{V}_s, \cdot\,; \oplus, \rho) \tag{4.49}$$

and elements $(t, \mathbf{x})^t$ of the cone $\mathbb{S}_s$ of E,

$$\mathbb{S}_s = \{(t, \mathbf{x})^t : \ t \in \mathbb{R}^{>0},\ \mathbf{x} \in \mathbb{V},\ \text{and}\ \ \mathbf{v} = \mathbf{x}/t \in \mathbb{V}_s\} \tag{4.50}$$

where $\mathbf{v} = \mathbf{x}/t = \mathbf{x}(1/t)$ is a scalar multiplication in the real inner product space $\mathbb{V}$ of the ball $\mathbb{V}_s$.

Identifying $(t, \mathbf{v})^t$ and $(t, \mathbf{x} = \mathbf{v}t)^t$, and borrowing terms from Einstein's special theory of relativity, we call the former a *velocity representation* and the latter a *space representation* of the same *spacetime* point. The conversion of a spacetime point from one representation to the other is just a matter of notation. Hence, for instance, the conversion of (4.6) – (4.9) from velocity representation to space representation results in the following equations,

$$\left\langle \begin{pmatrix} s \\ \mathbf{y} \end{pmatrix}, \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\rangle = \frac{\rho(\mathbf{u} \ominus \mathbf{v})}{\rho(\mathbf{u})\rho(\mathbf{v})} st \tag{4.51}$$

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\|^2 = \left\langle \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}, \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\rangle \tag{4.52}$$

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \frac{t}{\rho(\mathbf{v})} \tag{4.53}$$

and

$$\left\| \begin{pmatrix} \rho(\mathbf{v}) \\ \mathbf{v}\rho(\mathbf{v}) \end{pmatrix} \right\| = 1 \tag{4.54}$$

where $\mathbf{u} = \mathbf{y}/t$ and $\mathbf{v} = \mathbf{x}/t$, $t > 0$.

In the special case when the abstract gyrogroup $(\mathbb{V}_s, \oplus, \rho(\mathbf{v}))$ is realized by the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus_{\rm E}, \gamma_{\mathbf{v}})$, the norm (4.53) becomes the familiar relativistic norm

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \frac{t}{\gamma_{\mathbf{v}}} = t\sqrt{1 - \|\mathbf{v}\|^2/s^2} = \sqrt{t^2 - \|\mathbf{x}\|^2/s^2} \tag{4.55}$$

Owing to the bijective correspondence, (4.48), between velocity and space representation of spacetime points, the gyroboost $B(\mathbf{v})$, (4.15), of the extended gyrogroup E gives rise to a boost, also denoted $B(\mathbf{v})$, of the cone $\mathbb{S}_s$, according to the following definition.

Definition 4.18 *Let $\mathbb{S}_s$, (4.50), be the cone of E, (4.49). The application*

$$B(\mathbf{u}) : \ \mathbb{S}_s \ \rightarrow \ \mathbb{S}_s \tag{4.56}$$

of the boost $B(\mathbf{u})$, $\mathbf{u} \in \mathbb{V}_s$, to elements of the cone $\mathbb{S}_s$ is given by the equation

$$B(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{x} = \mathbf{v}t \end{pmatrix} = \begin{pmatrix} \frac{\rho(\mathbf{u}\oplus\mathbf{v})}{\rho(\mathbf{v})} t \\ (\mathbf{u}\oplus\mathbf{v}) \frac{\rho(\mathbf{u}\oplus\mathbf{v})}{\rho(\mathbf{v})} t \end{pmatrix} \tag{4.57}$$

Clearly, the boost application in (4.57) corresponds bijectively, (4.48), to its gyroboost application,

$$B(\mathbf{u}) : \ E \ \rightarrow \ E \tag{4.58}$$

$\mathbf{u} \in \mathbb{V}_s$, given by the following equation, (4.15),

$$B(\mathbf{u}) \begin{pmatrix} t \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} \frac{\rho(\mathbf{u}\oplus\mathbf{v})}{\rho(\mathbf{v})} t \\ \mathbf{u}\oplus\mathbf{v} \end{pmatrix} \tag{4.59}$$

The usefulness of boosts, as opposed to gyroboosts, rests on the result that Einstein boosts are linear, and that this linearity can be used to introduce a linear structure into Einstein gyrogroups, as well as into its isomorphic gyrogroups like Möbius and PV gyrogroups. Gyrogroup isomorphisms are presented in Table 6.1, p. 202, following the introduction of scalar multiplication into gyrogroups of gyrovectors.

4.9 The Lorentz Boost

Definition 4.18 of the abstract boost reduces to the Lorentz boost, which Einstein employed in his 1905 special theory of relativity, when

(*i*) the abstract gyrogroup operation $\oplus$ in the ball $\mathbb{V}_s$ is realized by Einstein addition $\oplus_E$, (3.143), and
(*ii*) the abstract gyrofactor $\rho(\mathbf{v})$ is realized by the gamma factor $\gamma_{\mathbf{v}}$, (3.129).

Accordingly, the Lorentz boost takes the form that we develop in the following chain of equations, which are numbered for subsequent explanation.

$$
\begin{aligned}
B(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x}=\mathbf{v}t \end{pmatrix} &\overset{(1)}{=\!=} \begin{pmatrix} \frac{\rho(\mathbf{u}\oplus\mathbf{v})}{\rho(\mathbf{v})}t \\ (\mathbf{u}\oplus\mathbf{v})\frac{\rho(\mathbf{u}\oplus\mathbf{v})}{\rho(\mathbf{v})}t \end{pmatrix} \\
&\overset{(2)}{=\!=} \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{v}}}t \\ (\mathbf{u}\oplus_E\mathbf{v})\frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{v}}}t \end{pmatrix} \\
&\overset{(3)}{=\!=} \begin{pmatrix} \gamma_{\mathbf{u}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2})t \\ (\mathbf{u}\oplus_E\mathbf{v})\gamma_{\mathbf{u}}(1+\frac{\mathbf{u}\cdot\mathbf{v}}{s^2})t \end{pmatrix} \\
&\overset{(4)}{=\!=} \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{s^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t+\mathbf{x}+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix}
\end{aligned}
\tag{4.60}
$$

The derivation of the equalities in the chain of equations (4.60) follows:

(1) By Def. 4.18.
(2) Realizing the abstract gyrogroup operation $\oplus$ by Einstein addition $\oplus_E$, (3.143), and the abstract gyrofactor $\rho(\mathbf{v})$ by the gamma factor $\gamma_{\mathbf{v}}$, (3.129), that is, $\oplus = \oplus_E$ and $\rho(\mathbf{v}) = \gamma_{\mathbf{v}}$.
(3) Follows from (3.146).
(4) Follows by substituting $\mathbf{u}\oplus_E\mathbf{v}$ from (3.143), and noting $\mathbf{x} = \mathbf{v}t$.

Formalizing the result in (4.60) we have the following

Definition 4.19 *The Lorentz boost of spacetime points,*

$$B(\mathbf{u}) \ : \ \mathbb{R}\times\mathbb{S}_s \rightarrow \mathbb{R}\times\mathbb{S}_s \tag{4.61}$$

is given by the following equation, (4.60),

$$B(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}}(t+\frac{1}{s^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t+\mathbf{x}+\frac{1}{s^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} \tag{4.62}$$

for all $\mathbf{u}\in\mathbb{V}_s$, $t\in\mathbb{R}$ *and* $\mathbf{x}\in\mathbb{S}_s$.

It follows from (4.62) that a Lorentz boost is linear,

$$B(\mathbf{u})\left\{p_1\begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix} + p_2\begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix}\right\} = p_1 B(\mathbf{u})\begin{pmatrix} t_1 \\ \mathbf{x}_1 \end{pmatrix} + p_2 B(\mathbf{u})\begin{pmatrix} t_2 \\ \mathbf{x}_2 \end{pmatrix} \tag{4.63}$$

$p_1, p_2\in\mathbb{R}$. Identity (4.63) holds whenever its involved spacetime points are included in the cone $\mathbb{S}_s$. Ambiguously, the symbol $+$ in (4.62) and (4.63) represents addition in the real line $\mathbb{R}$, in the real inner product space $\mathbb{V}$, and in the Cartesian product space $\mathbb{R}\times\mathbb{V}$.

Furthermore, it follows from the second equality in (4.60), with $t=\gamma_{\mathbf{v}}$, that the Lorentz boost satisfies the most elegant identity

$$B(\mathbf{u})\begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}}(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}) \end{pmatrix} \tag{4.64}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{V}_s$, where Lorentz boosts are applied to unimodular spacetime points.

Unimodular spacetime points $(\gamma_{\mathbf{v}},\ \gamma_{\mathbf{v}}\mathbf{v})^t$ in $\mathbb{R}\times\mathbb{R}_c^3$, realized from (4.54), are known in special relativity theory as *"four-velocities"*.

Remark 4.20 *Identity* (4.64) *illustrates the passage in the special theory of relativity from*

(1) *Einstein's "three-velocities"* $\mathbf{v}\in\mathbb{R}_c^3$, *which are gyrovectors according to Def. 5.4, p. 119, as we will see in Chap. 10, to*
(2) *Minkowski's "four-velocities"* $(\gamma_{\mathbf{v}},\ \gamma_{\mathbf{v}}\mathbf{v})^t\in\mathbb{R}\times\mathbb{R}_c^3$, *which are known in Minkowskian special relativity as "four-vectors".*

4.10 The $(p:q)$-Gyromidpoint

The linearity of the Lorentz boost, (4.63), and its elegant property (4.64) allow the notion of the gyromidpoint to be extended. For simplicity we restrict our attention to the special case when the abstract real inner product space is realized by the Euclidean n-space, $\mathbb{V} = \mathbb{R}^n$, so that $\mathbb{V}_s = \mathbb{R}^n_s$.

To exploit the linearity of the Lorentz boost let us consider the linear combination of two unimodular spacetime points

$$p\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix} + q\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix} = \begin{pmatrix}p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}\\ p\gamma_{\mathbf{a}}\mathbf{a} + q\gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix} = t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix} \tag{4.65}$$

$p, q \geq 0$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n_s$, where $t \geq 0$ and $\mathbf{m} \in \mathbb{R}^n_s$ are to be determined in (4.66) and (4.68) below.

Comparing ratios between lower and upper entries in (4.65) we have

$$\mathbf{m} = \mathbf{m}(\mathbf{a}, \mathbf{b}; p, q) = \frac{p\gamma_{\mathbf{a}}\mathbf{a} + q\gamma_{\mathbf{b}}\mathbf{b}}{p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}} \tag{4.66}$$

so that, by convexity, $\mathbf{m} \in \mathbb{R}^n_s$. The point $\mathbf{m} = \mathbf{m}(\mathbf{a}, \mathbf{b}; p, q)$, $0 \leq p, q \leq 1$, is called the $(p:q)$-gyromidpoint of $\mathbf{a}$ and $\mathbf{b}$ in the Einstein gyrogroup $\mathbb{R}^n_s = (\mathbb{R}^n_s, \oplus_{\text{E}})$. This term will be justified by Identity (4.81) below.

We will find that the special $(p:q)$-gyromidpoint with $p = q$ coincides with the gyromidpoint of Defs. 3.37, p. 72, and 6.31, p. 156, in an Einstein gyrogroup. The (1:1)-gyromidpoint,

$$\mathbf{m}(\mathbf{a}, \mathbf{b}; 1, 1) = \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}} \tag{4.67}$$

called the Einstein gyromidpoint, will be studied in Theorem 6.87, p. 205, and will prove useful in Sec. 6.20.1, p. 204, in determining the gyrocentroid of gyrotriangles and gyrotetrahedrons in Einstein gyrovector spaces.

Comparing upper entries in (4.65) we have

$$t = \frac{p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}}{\gamma_{\mathbf{m}}} \tag{4.68}$$

Applying the Lorentz boost $B(\mathbf{w})$, $\mathbf{w} \in \mathbb{R}^n_s$, to (4.65) in two different ways, it follows from (4.64) and the linearity of the Lorentz boost that on the one hand

$$
\begin{aligned}
B(\mathbf{w})\left\{t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\} &= B(\mathbf{w})\left\{p\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix}+q\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}\right\}\\
&= pB(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{a}}\\ \gamma_{\mathbf{a}}\mathbf{a}\end{pmatrix}+qB(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{b}}\\ \gamma_{\mathbf{b}}\mathbf{b}\end{pmatrix}\\
&= p\begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{a}}\\ \gamma_{\mathbf{w}\oplus\mathbf{a}}(\mathbf{w}\oplus\mathbf{a})\end{pmatrix}+q\begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{b}}\\ \gamma_{\mathbf{w}\oplus\mathbf{b}}(\mathbf{w}\oplus\mathbf{b})\end{pmatrix}\\
&= \begin{pmatrix}p\gamma_{\mathbf{w}\oplus\mathbf{a}}+q\gamma_{\mathbf{w}\oplus\mathbf{b}}\\ p\gamma_{\mathbf{w}\oplus\mathbf{a}}(\mathbf{w}\oplus\mathbf{a})+q\gamma_{\mathbf{w}\oplus\mathbf{b}}(\mathbf{w}\oplus\mathbf{b})\end{pmatrix}
\end{aligned}
\tag{4.69}
$$

and on the other hand

$$
\begin{aligned}
B(\mathbf{w})\left\{t\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\} &= tB(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\\
&= t\begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{m}}\\ \gamma_{\mathbf{w}\oplus\mathbf{m}}\mathbf{w}\oplus\mathbf{m}\end{pmatrix}\\
&= \begin{pmatrix}t\gamma_{\mathbf{w}\oplus\mathbf{m}}\\ t\gamma_{\mathbf{w}\oplus\mathbf{m}}\mathbf{w}\oplus\mathbf{m}\end{pmatrix}
\end{aligned}
\tag{4.70}
$$

where $\oplus = \oplus_{\text{E}}$ is Einstein addition in the ball $\mathbb{R}^n_s$.

Comparing ratios between lower and upper entries of (4.69) and (4.70) we have

$$
\mathbf{w}\oplus\mathbf{m} = \frac{p\gamma_{\mathbf{w}\oplus\mathbf{a}}(\mathbf{w}\oplus\mathbf{a})+q\gamma_{\mathbf{w}\oplus\mathbf{b}}(\mathbf{w}\oplus\mathbf{b})}{p\gamma_{\mathbf{w}\oplus\mathbf{a}}+q\gamma_{\mathbf{w}\oplus\mathbf{b}}} \tag{4.71}
$$

so that by (4.66) and (4.71),

$$
\mathbf{w}\oplus\mathbf{m}(\mathbf{a},\mathbf{b};p,q) = \mathbf{m}(\mathbf{w}\oplus\mathbf{a},\mathbf{w}\oplus\mathbf{b};p,q) \tag{4.72}
$$

Identity (4.72) demonstrates that the structure of the $(p\colon q)$-gyromidpoint $\mathbf{m}$ of $\mathbf{a}$ and $\mathbf{b}$, as a function of points $\mathbf{a}$ and $\mathbf{b}$, is not distorted by left gyrotranslations. Similarly, it is not distorted by rotations as explained below.

A rotation τ of the Euclidean n-space $\mathbb{R}^n$ about its origin is a linear map of $\mathbb{R}^n$ that preserves the inner product in $\mathbb{R}^n$, and is represented by an $n\times n$ orthogonal matrix with determinant 1. The group of all rotations of $\mathbb{R}^n$ about its origin, denoted $SO(n)$, possesses the properties

$$\begin{aligned} \tau(\mathbf{a}+\mathbf{b}) &= \tau\mathbf{a}+\tau\mathbf{b} \\ \tau\mathbf{a}\cdot\tau\mathbf{b} &= \mathbf{a}\cdot\mathbf{b} \end{aligned} \tag{4.73}$$

for all $\mathbf{a},\mathbf{b}\in\mathbb{R}_s^n$ and $\tau\in SO(n)$.

Owing to properties (4.73) we have

$$\tau(\mathbf{a}\oplus_{\text{E}}\mathbf{b}) = \tau\mathbf{a}\oplus_{\text{E}}\tau\mathbf{b} \tag{4.74}$$

for all $\mathbf{a},\mathbf{b}\in(\mathbb{R}_s^n,\oplus_{\text{E}})$ and $\tau\in SO(n)$. Hence, $SO(n)$ is a gyroautomorphism group, Def. 2.22, of the Einstein gyrogroup $(\mathbb{R}_s^n,\oplus_{\text{E}})$,

$$SO(n) = Aut_0(\mathbb{R}_s^n,\oplus_{\text{E}}) \tag{4.75}$$

It follows from properties (4.73) of $\tau\in Aut_0(\mathbb{R}_s^n,\oplus_{\text{E}}) = SO(n)$ that

$$\tau\mathbf{m}(\mathbf{a},\mathbf{b};p,q) = \mathbf{m}(\tau\mathbf{a},\tau\mathbf{b};p,q) \tag{4.76}$$

for all $\mathbf{a},\mathbf{b}\in\mathbb{R}_s^n$ and $0\le p,q\le 1$. Hence, by Def. 3.21 and Theorem 3.22, the gyrosemidirect product group

$$\mathbb{R}_s^n\times Aut_0(\mathbb{R}_s^n,\oplus_{\text{E}}) = \mathbb{R}_s^n\times SO(n) \tag{4.77}$$

is a group of motions of the Einstein gyrogroup $(\mathbb{R}_s^n,\oplus_{\text{E}})$.

Having the group (4.77) of motions of the Einstein gyrogroup $(\mathbb{R}_s^n,\oplus_{\text{E}})$, Def. 3.21 and Theorem 3.22, and the variations (4.72) and (4.76) of the $(p\!:\!q)$-gyromidpoint $\mathbf{m}(\mathbf{a},\mathbf{b};p,q)\in(\mathbb{R}_s^n,\oplus_{\text{E}})$ under these motions, it follows from Def. 3.23 that the $(p\!:\!q)$-gyromidpoint $\mathbf{m}(\mathbf{a},\mathbf{b};p,q)$ is gyrocovariant. Accordingly, by Def. 3.23 the triple $\{\mathbf{a},\mathbf{b},\mathbf{m}(\mathbf{a},\mathbf{b};p,q)\}$ of two points and their $(p\!:\!q)$-gyromidpoint forms a gyrogroup object, for any $\mathbf{a},\mathbf{b}\in(\mathbb{R}_s^n,\oplus_{\text{E}})$ and $0\le p,q\le 1$.

Comparing the top entries of (4.69) and (4.70) we have

$$t = \frac{p\gamma_{\mathbf{w}\oplus\mathbf{a}}+q\gamma_{\mathbf{w}\oplus\mathbf{b}}}{\gamma_{\mathbf{w}\oplus\mathbf{m}}} \tag{4.78}$$

which generalizes the equation, (4.68),

$$t = \frac{p\gamma_{\mathbf{a}} + q\gamma_{\mathbf{b}}}{\gamma_{\mathbf{m}}} \tag{4.79}$$

The pair of two equations for t, (4.78) and (4.79), demonstrates that the positive scalar $t = t(\mathbf{a}, \mathbf{b}; p, q) \in \mathbb{R}^{>0}$ in (4.78) and (4.79) is invariant under left gyrotranslations of $\mathbf{a}$ and $\mathbf{b}$. Clearly, it is also invariant under rotations $\tau \in SO(n)$ of $\mathbf{a}$ and $\mathbf{b}$ so that, being invariant under the group of motions of its Einstein gyrogroup $(\mathbb{R}^n_s, \oplus_{\text{E}})$, it is a *gyrogroup scalar field* for any $0 \le p, q \le 1$.

Substituting $\mathbf{w} = \ominus\mathbf{m}$ in (4.71) we obtain the identity

$$p\gamma_{\ominus\mathbf{m}\oplus\mathbf{a}}(\ominus\mathbf{m}\oplus\mathbf{a}) + q\gamma_{\ominus\mathbf{m}\oplus\mathbf{b}}(\ominus\mathbf{m}\oplus\mathbf{b}) = \mathbf{0} \tag{4.80}$$

or, equivalently,

$$p\gamma_{\ominus\mathbf{m}\oplus\mathbf{a}}(\ominus\mathbf{m}\oplus\mathbf{a}) = \ominus q\gamma_{\ominus\mathbf{m}\oplus\mathbf{b}}(\ominus\mathbf{m}\oplus\mathbf{b}) \tag{4.81}$$

Identity (4.81) illustrates the sense in which the point $\mathbf{m} = \mathbf{m}(\mathbf{a}, \mathbf{b}; p, q)$ is the $(p\!:\!q)$-gyromidpoint of the points $\mathbf{a}, \mathbf{b} \in (\mathbb{R}^n_s, \oplus_{\text{E}})$. It is further illustrated graphically in Fig. 10.14, p. 408, on the relativistic law of the lever in the context of the Einstein gyrovector space.

The $(p\!:\!q)$-gyromidpoint is *homogeneous* in the sense that it depends on the ratio $p\!:\!q$ of the numbers p and q, as we see from (4.66). Since it is the ratio $p\!:\!q$ that is of interest, we call $(p\!:\!q)$ the *homogeneous gyrobarycentric coordinates* of $\mathbf{m}$ relative to the set $A = \{\mathbf{a}, \mathbf{b}\}$. Under the normalization condition $p + q = 1$, the homogeneous gyrobarycentric coordinates $(p : q)$ of $\mathbf{m}$ relative to the set A are called *gyrobarycentric coordinates*, denoted (p, q). Their classical counterpart, the notion of barycentric coordinates [Yiu (2000)] (also known as trilinear coordinates [Weisstein (2003)]), was first conceived by Möbius in 1827 [Mumford, Series and Wright (2002)].

When $p = q$ the $(p\!:\!q)$-gyromidpoint of $\mathbf{a}$ and $\mathbf{b}$ will turn out in Theorem 6.87, p. 205, to be the so called Einstein gyromidpoint $\mathbf{m_{ab}}$,

$$\mathbf{m_{ab}} = \mathbf{m}(\mathbf{a}, \mathbf{b}; p, p) = \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}}} \tag{4.82}$$

illustrated graphically in Fig. 10.1, p. 368. The Einstein gyromidpoint (4.82) will prove useful in the study of the gyrogeometric significance of Einstein's relativistic mass correction in Chap. 10. The relativistic mass will, accordingly, emerge as *gyromass*, that is, mass that bears a gyrogeometric fingerprint.

4.11 The $(p_1 : p_2 : \ldots : p_n)$-Gyromidpoint

Let $(\gamma_{\mathbf{v}_k}, \gamma_{\mathbf{v}_k}\mathbf{v}_k)^t$, where $\mathbf{v}_k \in \mathbb{R}^n_s$, $k = 1, \ldots, h$, be h unimodular spacetime points, and let

$$\sum_{k=1}^{h} p_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = p \begin{pmatrix} \gamma_{\mathbf{m}} \\ \gamma_{\mathbf{m}}\mathbf{m} \end{pmatrix} \tag{4.83}$$

$m_k \geq 0$, be a generic linear combination of these spacetime points, where $p \geq 0$ and $\mathbf{m} \in \mathbb{R}^n_s$ are to be determined in (4.84) and (4.94) below.

Comparing ratios between lower and upper entries in (4.83) we have

$$\mathbf{m} = \mathbf{m}(\mathbf{v}_1, \ldots, \mathbf{v}_h; p_1, \ldots, p_h) = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k}} \tag{4.84}$$

so that $\mathbf{m}$ lies in the convex hull of the set of the points $\mathbf{v}_k$ of $\mathbb{R}^n_s$, $k = 1, \ldots, h$. The convex hull of a set of points in an Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$ is the smallest convex set in $\mathbb{R}^n$ that includes the points. Hence, $\mathbf{m} \in \mathbb{R}^n_s$ as desired, suggesting the following

Definition 4.21 *Let $\mathbf{v}_k \in \mathbb{R}^n_s$, $k = 1, \ldots, h$, be h points of the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$. The point $\mathbf{m} \in \mathbb{R}^n_s$,*

$$\mathbf{m} = \mathbf{m}(\mathbf{v}_1, \ldots, \mathbf{v}_h; p_1, \ldots, p_h) = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k}} \tag{4.85}$$

$p_k \geq 0$, is called the $(p_1 : p_2 : \ldots : p_h)$-gyromidpoint of the h points $\mathbf{v}_1, \ldots, \mathbf{v}_h$. Furthermore, the homogeneous gyrobarycentric coordinates of the point $\mathbf{m}$ relative to the set

$$A = \{\mathbf{v}_1, \ldots, \mathbf{v}_h\} \tag{4.86}$$

of h points in $\mathbb{R}^n_s$ are $(p_1 : p_2 : \ldots : p_h)$.

Under the normalization condition,

$$\sum_{k=1}^{h} p_k = 1 \tag{4.87}$$

the homogeneous gyrobarycentric coordinates $(p_1 : p_2 : \ldots : p_h)$ of $\mathbf{m}$ relative to the set A are called gyrobarycentric coordinates, denoted $(p_1, p_2, \ldots, p_h)$.

Applying the Lorentz boost $B(\mathbf{w})$, $\mathbf{w}\in\mathbb{R}_s^n$, to (4.83) in two different ways, it follows from (4.64) and from the linearity of the Lorentz boost that, on the one hand

$$\begin{aligned} B(\mathbf{w})\left\{p\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\} &= \sum_{k=1}^{h} p_k B(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix}\\ &= \sum_{k=1}^{h} p_k \begin{pmatrix}\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix}\\ &= \begin{pmatrix}\sum_{k=1}^{h} p_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}\\ \sum_{k=1}^{h} p_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)\end{pmatrix}\end{aligned} \tag{4.88}$$

and on the other hand,

$$\begin{aligned} B(\mathbf{w})\left\{p\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\right\} &= pB(\mathbf{w})\begin{pmatrix}\gamma_{\mathbf{m}}\\ \gamma_{\mathbf{m}}\mathbf{m}\end{pmatrix}\\ &= \begin{pmatrix}p\gamma_{\mathbf{w}\oplus\mathbf{m}}\\ p\gamma_{\mathbf{w}\oplus\mathbf{m}}(\mathbf{w}\oplus\mathbf{m})\end{pmatrix}\end{aligned} \tag{4.89}$$

Comparing ratios between lower and upper entries of (4.88) and (4.89) we have

$$\mathbf{w}\oplus\mathbf{m} = \frac{\sum_{k=1}^{h} p_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{\sum_{k=1}^{h} p_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}} \tag{4.90}$$

so that, by (4.84) and (4.90),

$$\begin{aligned}&\mathbf{w}\oplus\mathbf{m}(\mathbf{v}_1,\ldots,\mathbf{v}_h;p_1,\ldots,p_h)\\ &\quad= \mathbf{m}(\mathbf{w}\oplus\mathbf{v}_1,\ldots,\mathbf{w}\oplus\mathbf{v}_h;p_1,\ldots,p_h)\end{aligned} \tag{4.91}$$

Identity (4.91) demonstrates that the structure of the $(p_1:p_2:\ \ldots\ :p_h)$-gyromidpoint $\mathbf{m}$, as a function of the points $\mathbf{v}_k\in\mathbb{R}_s^n$, $k=1,\ldots,h$, is not distorted by a left gyrotranslation of the points by any $\mathbf{w}\in\mathbb{R}_s^n$.

Similarly, the structure is not distorted by rotations $\tau\in SO(n)$ of $\mathbb{R}_s^n$ in the sense that

$$\tau\mathbf{m}(\mathbf{v}_1,\ldots,\mathbf{v}_h;p_1,\ldots,p_h) = \mathbf{m}(\tau\mathbf{v}_1,\ldots,\tau\mathbf{v}_h;p_1,\ldots,p_h) \tag{4.92}$$

for all $\tau \in SO(n)$.

Hence, by (4.91) and (4.92), the $(p_1 : p_2 : \ldots : p_h)$-gyromidpoint $\mathbf{m} = \mathbf{m}(\mathbf{v}_1, \ldots, \mathbf{v}_h; p_1, \ldots, p_h) \in \mathbb{R}^n_s$ is gyrocovariant, being covariant under the group of the gyrogroup motions, $\mathbb{R}^n_s \times SO(n)$, of the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$. Accordingly, by Def. 3.23 the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_h, \mathbf{m}\}$ of any h points, $\mathbf{v}_1, \ldots, \mathbf{v}_h$, in the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$ along with their $(p_1 : p_2 : \ldots : p_h)$-gyromidpoint $\mathbf{m}$ form a gyrogroup object.

Comparing the top entries of (4.88) and (4.89) we have

$$p = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{\gamma_{\mathbf{w}\oplus\mathbf{m}}} \tag{4.93}$$

But, we also have from (4.83)

$$p = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k}}{\gamma_{\mathbf{m}}} \tag{4.94}$$

implying that the positive scalar

$$\begin{aligned} p &= p(\mathbf{v}_1, \ldots, \mathbf{v}_h; p_1, \ldots, p_h) \\ &= p(\mathbf{w}\oplus\mathbf{v}_1, \ldots, \mathbf{w}\oplus\mathbf{v}_h; p_1, \ldots, p_h) \end{aligned} \tag{4.95}$$

in (4.93) and (4.94) is invariant under any left gyrotranslation of the points $\mathbf{v}_k \in \mathbb{R}^n_s$, $k = 1, \ldots, h$. Clearly, it is also invariant under any rotation $\tau \in SO(n)$ of its generating points $\mathbf{v}_k$. Hence, being invariant under the group of the gyrogroup motions, $\mathbb{R}^n_s \times SO(n)$, of the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$,

$$p = p(\mathbf{v}_1, \ldots, \mathbf{v}_h; p_1, \ldots, p_h) \tag{4.96}$$

is a gyrogroup scalar field for any arbitrarily fixed gyrobarycentric coordinates $(p_1, p_2, \ldots, p_h)$.

Theorem 4.22 *Let $\mathbf{v}_k \in \mathbb{R}^n_s$, $k = 1, \ldots, h$, be h points of the Einstein gyrogroup $(\mathbb{R}^n_s, \oplus)$. and let $\mathbf{m}$ be their $(p_1 : p_2 : \ldots : p_h)$-gyromidpoint, that is by Def. 4.21,*

$$\mathbf{m} = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k}} \tag{4.97}$$

$p_k \geq 0$. *Then*

$$\gamma_{\mathbf{m}} = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{v}_k}}{\sum_{k=1}^{h} p_k \gamma_{\ominus\mathbf{m}\oplus\mathbf{v}_k}} \tag{4.98}$$

and, moreover,

$$\gamma_{\mathbf{w}\oplus\mathbf{m}} = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{\sum_{k=1}^{h} p_k \gamma_{\ominus\mathbf{m}\oplus\mathbf{v}_k}} \tag{4.99}$$

Proof. Noting that $\gamma_0 = 1$, (4.93) with $\mathbf{w} = \ominus\mathbf{m}$ reduces to the identity

$$p = \sum_{k=1}^{h} p_k \gamma_{\ominus\mathbf{m}\oplus\mathbf{v}_k} \tag{4.100}$$

so that (4.98) follows from (4.94) and (4.100).

It follows from (4.90), (4.97) and (4.98) that

$$\gamma_{\mathbf{w}\oplus\mathbf{m}} = \frac{\sum_{k=1}^{h} p_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{\sum_{k=1}^{h} p_k \gamma_{\ominus(\mathbf{w}\oplus\mathbf{m})\oplus(\mathbf{w}\oplus\mathbf{v}_k)}} \tag{4.101}$$

Owing to the Gyrotranslation Theorem 3.13 and the invariance of the gamma factor under gyrations, we have

$$\begin{aligned} \gamma_{\ominus(\mathbf{w}\oplus\mathbf{m})\oplus(\mathbf{w}\oplus\mathbf{v}_k)} &= \gamma_{\mathrm{gyr}[\mathbf{w},\mathbf{m}](\ominus\mathbf{m}\oplus\mathbf{v}_k)} \\ &= \gamma_{\ominus\mathbf{m}\oplus\mathbf{v}_k} \end{aligned} \tag{4.102}$$

Hence, (4.99) follows from (4.101) and (4.102). □

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Chapter 5

Gyrovectors and Cogyrovectors

Vectors in a vector space form equivalence classes, where two vectors are equivalent if they are parallel and possess equal lengths. Gyrovectors, in contrast, do not admit parallelism. Yet, they do form analogous equivalence classes even at the primitive level of gyrocommutative gyrogroups where, in general, the concepts of length and parallelism do not exist. In the more advanced level of gyrovector spaces, gyrovectors will be found fully analogous to vectors, where they regulate algebraically the hyperbolic geometry of Bolyai and Lobachevsky just as vectors regulate algebraically Euclidean geometry.

5.1 Equivalence Classes

The definition of gyrovectors and cogyrovectors in gyrocommutative gyrogroups will be presented in Secs. 5.2 and 5.6 in terms of equivalence classes of pairs of points.

Definition 5.1 (Equivalence Relations and Classes). *A (binary) relation on a nonempty set S is a subset R of $S \times S$, written as $a \sim b$ if $(a, b) \in R$. A relation $\sim$ on a set S is*

(1) *Reflexive if $a \sim a$ for all $a \in S$.*
(2) *Symmetric if $a \sim b$ implies $b \sim a$ for all $a, b \in S$.*
(3) *Transitive if $a \sim b$ and $b \sim c$ implies $a \sim c$ for all $a, b, c \in S$.*

A relation is an equivalence relation if it is reflexive, symmetric and transitive.

An equivalence relation $\sim$ on a set S gives rise to equivalence classes. The equivalence class of $a \in S$ is the subset $\{x \in S : x \sim a\}$ of S of all the *elements $x \in S$ which are related to a.*

Two equivalence classes in a set S with an equivalence relation $\sim$ are either equal or disjoint, and the union of all the equivalence classes in S equals S. Accordingly, we say that the equivalence classes of a set with an equivalence relation form a *partition* of S.

5.2 Gyrovectors

Elements of a gyrocommutative gyrogroup are called points and are denoted by A, B, C, etc. In particular, the identity element is called the origin, denoted O.

Definition 5.2 (Rooted Gyrovectors). *A rooted gyrovector PQ in a gyrocommutative gyrogroup $(G, \oplus)$ is an ordered pair of points $P, Q \in G$. The rooted gyrovector PQ is rooted at the point P. The points P and Q of the rooted gyrovector PQ are called, respectively, the tail and the head of the rooted gyrovector. The value in G of the rooted gyrovector PQ is $\ominus P \oplus Q$. Accordingly, we write*

$$\mathbf{v} = PQ = \ominus P \oplus Q \tag{5.1}$$

and call $\mathbf{v} = \ominus P \oplus Q$ the rooted gyrovector, rooted at P, with tail P and head Q in G. The rooted gyrovector PQ is nonzero if $P \neq Q$.

Furthermore, any point $A \in G$ is identified with the rooted gyrovector OA with head A, rooted at the origin O.

Definition 5.3 (Rooted Gyrovector Equivalence). *Let*

$$PQ = \ominus P \oplus Q$$

$$P'Q' = \ominus P' \oplus Q'$$

be two rooted gyrovectors in a gyrocommutative gyrogroup $(G, \oplus)$, with respective tails P and P' and respective heads Q and Q'. The two rooted gyrovectors are equivalent,

$$\ominus P' \oplus Q' \sim \ominus P \oplus Q$$

if they have the same value in G, that is, if

$$\ominus P' \oplus Q' = \ominus P \oplus Q$$

The relation $\sim$ in Def. 5.3 is given in terms of an equality so that, being reflexive, symmetric, and transitive, it is an equivalence relation. The resulting equivalence classes are called *gyrovectors.* Formalizing, we thus have the following

Definition 5.4 (Gyrovectors). *Let $(G, \oplus)$ be a gyrocommutative gyrogroup with its rooted gyrovector equivalence relation. The resulting equivalence classes are called gyrovectors. The equivalence class of all rooted gyrovectors that are equivalent to a given rooted gyrovector $PQ = \ominus P \oplus Q$ is the gyrovector denoted by any element of its class, for instance, $PQ = \ominus P \oplus Q$. Any point $A \in G$ is identified with the gyrovector OA. In order to contrast with rooted gyrovectors, gyrovectors are also called free gyrovectors.*

5.3 Gyrovector Translation

The following theorem and definition allow rooted gyrovector equivalence to be expressed in terms of rooted gyrovector translation.

Theorem 5.5 *Let*

$$\begin{aligned} PQ &= \ominus P \oplus Q \\ P'Q' &= \ominus P' \oplus Q' \end{aligned} \tag{5.2}$$

be two rooted gyrovectors in a gyrocommutative gyrogroup $(G, \oplus)$. They are equivalent, that is,

$$\ominus P \oplus Q = \ominus P' \oplus Q' \tag{5.3}$$

if and only if there exists a gyrovector $\mathbf{t} \in G$ such that

$$\begin{aligned} P' &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t} \oplus P) \\ Q' &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t} \oplus Q) \end{aligned} \tag{5.4}$$

Furthermore, the gyrovector $\mathbf{t}$ is unique, given by the equation

$$\mathbf{t} = \ominus P \oplus P' \tag{5.5}$$

Proof. By the Gyrotranslation Theorem 3.13 we have

$$\ominus(\mathbf{t} \oplus P) \oplus (\mathbf{t} \oplus Q) = \mathrm{gyr}[\mathbf{t}, P](\ominus P \oplus Q) \tag{5.6}$$

or equivalently, by gyroautomorphism inversion,

$$\ominus P \oplus Q = \mathrm{gyr}[P, \mathbf{t}]\{\ominus(\mathbf{t}\oplus P)\oplus(\mathbf{t}\oplus Q)\} \tag{5.7}$$

for any $P, Q, \mathbf{t} \in G$.

Assuming (5.4), we have by (5.7) and (5.4)

$$\begin{aligned} \ominus P \oplus Q &= \mathrm{gyr}[P, \mathbf{t}]\{\ominus(\mathbf{t}\oplus P)\oplus(\mathbf{t}\oplus Q)\} \\ &= \ominus \mathrm{gyr}[P, \mathbf{t}](\mathbf{t}\oplus P)\oplus \mathrm{gyr}[P, \mathbf{t}](\mathbf{t}\oplus Q) \\ &= \ominus P' \oplus Q' \end{aligned} \tag{5.8}$$

thus verifying (5.3).

Conversely, assuming (5.3) we let

$$\mathbf{t} = \ominus P \oplus P' \tag{5.9}$$

so that by a left cancellation and by the gyrocommutative law we have

$$\begin{aligned} P' &= P \oplus \mathbf{t} \\ &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t}\oplus P) \end{aligned} \tag{5.10}$$

thus recovering the first equation in (5.4). Using the notation $g_{P,\mathbf{t}} = \mathrm{gyr}[P, \mathbf{t}]$ when convenient we have, by (5.3) with a left cancellation, (5.10), the right gyroassociative law, (2.50), and a left cancellation,

$$\begin{aligned} Q' &= P' \oplus (\ominus P \oplus Q) \\ &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t}\oplus P)\oplus(\ominus P \oplus Q) \\ &= (g_{P,\mathbf{t}}\mathbf{t} \oplus g_{P,\mathbf{t}}P)\oplus(\ominus P \oplus Q) \\ &= g_{P,\mathbf{t}}\mathbf{t} \oplus \{g_{P,\mathbf{t}}P \oplus \mathrm{gyr}[g_{P,\mathbf{t}}P, g_{P,\mathbf{t}}\mathbf{t}](\ominus P \oplus Q)\} \\ &= g_{P,\mathbf{t}}\mathbf{t} \oplus \{g_{P,\mathbf{t}}P \oplus g_{P,\mathbf{t}}(\ominus P \oplus Q)\} \\ &= g_{P,\mathbf{t}}\mathbf{t} \oplus \{g_{P,\mathbf{t}}P \oplus (\ominus g_{P,\mathbf{t}}P \oplus g_{P,\mathbf{t}}Q)\} \\ &= g_{P,\mathbf{t}}\mathbf{t} \oplus g_{P,\mathbf{t}}Q\} \\ &= g_{P,\mathbf{t}}(\mathbf{t}\oplus Q) \end{aligned} \tag{5.11}$$

thus verifying the second equation in (5.4), as desired.

Finally, the gyrovector $\mathbf{t} \in G$ is uniquely determined by P and P' as we see from the first equation in (5.4),

$$\begin{aligned} \ominus P \oplus P' &= \ominus P \oplus \mathrm{gyr}[P, \mathbf{t}](\mathbf{t} \oplus P) \\ &= \ominus P \oplus (P \oplus \mathbf{t}) \\ &= \mathbf{t} \end{aligned} \tag{5.12}$$

□

Theorem 5.5 suggests the following definition of gyrovector translation in gyrocommutative gyrogroups.

Definition 5.6 **(Gyrovector Translation).** *A gyrovector translation $T_{\mathbf{t}}$ by a gyrovector $\mathbf{t} \in G$ of a rooted gyrovector $PQ = \ominus P \oplus Q$, with tail P and head Q, in a gyrocommutative gyrogroup $(G, \oplus)$ is the rooted gyrovector $P'Q' = \ominus P' \oplus Q'$, with tail P' and head $P'Q' = Q'$, $T_{\mathbf{t}}PQ = P'Q'$, given by*

$$\begin{aligned} P' &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t} \oplus P) \\ Q' &= \mathrm{gyr}[P, \mathbf{t}](\mathbf{t} \oplus Q) \end{aligned} \tag{5.13}$$

The rooted gyrovector $\ominus P' \oplus Q'$ is said to be the $\mathbf{t}$ gyrovector translation, or the gyrovector translation by $\mathbf{t}$, of the rooted gyrovector $\ominus P \oplus Q$.

We may note that the two equations in (5.13) are not symmetric in P and Q since they share a Thomas gyration. Moreover, owing to the gyrocommutativity of $\oplus$, the first equation in (5.13) can be written as

$$P' = P \oplus \mathbf{t} \tag{5.14}$$

Clearly, a gyrovector translation by the zero gyrovector $\mathbf{0} \in G$ is trivial since (5.13) reduces to

$$\begin{aligned} P' &= \mathrm{gyr}[P, \mathbf{0}](\mathbf{0} \oplus P) = P \\ Q' &= \mathrm{gyr}[P, \mathbf{0}](\mathbf{0} \oplus Q) = Q \end{aligned} \tag{5.15}$$

Definition 5.6 allows Theorem 5.5 to be reformulated, obtaining the following theorem:

Theorem 5.7 *Two rooted gyrovectors*

$$\begin{aligned} PQ &= \ominus P \oplus Q \\ P'Q' &= \ominus P' \oplus Q' \end{aligned} \tag{5.16}$$

in a gyrocommutative gyrogroup $(G, \oplus)$ *are equivalent, that is,*

$$\ominus P \oplus Q = \ominus P' \oplus Q' \tag{5.17}$$

if and only if gyrovector $P'Q'$ *is a gyrovector translation of gyrovector* PQ. *Furthermore, if* $P'Q'$ *is a gyrovector translation of* PQ *then it is a gyrovector translation of* PQ *by*

$$\mathbf{t} = \ominus P \oplus P' \tag{5.18}$$

Theorem 5.8 **(Gyrovector Translation Head).** *Let* P, Q, P' *be any three points of a gyrocommutative gyrogroup* $(G, \oplus)$. *The gyrovector translation of the rooted gyrovector* $PQ = \ominus P \oplus Q$ *to the rooted gyrovector* $P'X = \ominus P' \oplus X$ *with tail* P' *determines its head* X,

$$X = P' \oplus (\ominus P \oplus Q) \tag{5.19}$$

Proof. $P'Q'$ is a gyrovector translation of PQ. Hence, by Theorem 5.7, PQ and $P'X$ are equivalent gyrovectors. Hence, by Def. 5.3, we have

$$\ominus P \oplus Q = \ominus P' \oplus X \tag{5.20}$$

from which (5.19) follows by a left cancellation. □

5.4 Gyrovector Translation Composition

Let $P''Q''$ be the gyrovector translation of a rooted gyrovector $P'Q'$ by $\mathbf{t}_2$ where $P'Q'$, in turn, is the Gyrovector translation of a rooted gyrovector PQ by $\mathbf{t}_1$ in a gyrocommutative gyrogroup $(G, \oplus)$. Then, by Def. 5.6,

$$\begin{aligned} P'' &= \mathrm{gyr}[P', \mathbf{t}_2](\mathbf{t}_2 \oplus P') = P' \oplus \mathbf{t}_2 \\ P' &= \mathrm{gyr}[P, \mathbf{t}_1](\mathbf{t}_1 \oplus P) = P \oplus \mathbf{t}_1 \end{aligned} \tag{5.21}$$

so that

$$\begin{aligned} P'' &= P' \oplus \mathbf{t}_2 \\ &= (P \oplus \mathbf{t}_1) \oplus \mathbf{t}_2 \\ &= P \oplus (\mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2) \end{aligned} \tag{5.22}$$

Moreover, by Theorem 5.7 the rooted gyrovector $P''Q''$ is equivalent to the rooted gyrovector $P'Q'$ and the latter, in turn, is equivalent to the rooted gyrovector PQ. Hence, $P''Q''$ is equivalent to PQ, so that, by Theorem 5.5, $P''Q''$ is a gyrovector translation of PQ by some unique $\mathbf{t}_{12} \in G$,

$$P'' = P\oplus\mathbf{t}_{12} \tag{5.23}$$

Comparing (5.23) and (5.22), we see that $\mathbf{t}_{12}$ is given by the equation

$$\mathbf{t}_{12} = \mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2 \tag{5.24}$$

Expressing the rooted gyrovector $P'Q'$ in terms of the rooted gyrovector PQ we have, by Theorem 5.5 and the gyrocommutative law,

$$\begin{aligned} P' &= P\oplus\mathbf{t}_1 \\ Q' &= \mathrm{gyr}[P, \mathbf{t}_1](\mathbf{t}_1\oplus Q) \end{aligned} \tag{5.25}$$

Similarly, expressing the rooted gyrovector $P''Q''$ in terms of the rooted gyrovector $P'Q'$ we have, by Theorem 5.5 and the gyrocommutative law,

$$\begin{aligned} P'' &= P'\oplus\mathbf{t}_2 \\ Q'' &= \mathrm{gyr}[P', \mathbf{t}_2](\mathbf{t}_2\oplus Q') \end{aligned} \tag{5.26}$$

Finally, expressing the rooted gyrovector $P''Q''$ in terms of the rooted gyrovector PQ we have, by Theorem 5.5 and the gyrocommutative law,

$$\begin{aligned} P'' &= P\oplus\mathbf{t}_{12} \\ Q'' &= \mathrm{gyr}[P, \mathbf{t}_{12}](\mathbf{t}_{12}\oplus Q) \end{aligned} \tag{5.27}$$

Substituting (5.24) in (5.27) we have

$$\begin{aligned} P'' &= P\oplus(\mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2) \\ Q'' &= \mathrm{gyr}[P, \mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2]\{(\mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2)\oplus Q\} \end{aligned} \tag{5.28}$$

Substituting (5.25) in the second equation in (5.26) we have

$$Q'' = \mathrm{gyr}[P\oplus\mathbf{t}_1, \mathbf{t}_2]\{\mathbf{t}_2\oplus\mathrm{gyr}[P, \mathbf{t}_1](\mathbf{t}_1\oplus Q)\} \tag{5.29}$$

From (5.29) and the second equation in (5.28) for Q'' we have the identity

$$\begin{aligned} &\mathrm{gyr}[P, \mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2]\{(\mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2)\oplus Q\} \\ =&\mathrm{gyr}[P\oplus\mathbf{t}_1, \mathbf{t}_2]\{\mathbf{t}_2\oplus\mathrm{gyr}[P, \mathbf{t}_1](\mathbf{t}_1\oplus Q)\} \end{aligned} \tag{5.30}$$

for all $P, Q, \mathbf{t}_1, \mathbf{t}_2 \in G$.

Thus, in our way to uncover the composition law (5.27) of gyrovector translation we obtained the new gyrocommutative gyrogroup identity (5.30) as an unintended and unforeseen by-product. Interestingly, the new identity (5.30) reduces to (3.26) when $P = O$.

The composite gyrovector translation (5.27) is trivial when $\mathbf{t}_{12} = \mathbf{0}$, that is, when $\mathbf{t}_2 = \ominus\mathrm{gyr}[P, \mathbf{t}_1]\mathbf{t}_1$, as we see from (5.24). Hence, the inverse gyrovector translation of gyrovector translation by $\mathbf{t}$ is a gyrovector translation by $\ominus\mathrm{gyr}[P, \mathbf{t}]\mathbf{t}$.

The equivalence relation between rooted gyrovectors in Def. 5.10 is expressed in Theorem 5.5 in terms of the gyrovector translation. Gyrovector translation, accordingly, gives rise to an equivalence relation. Indeed, the gyrovector translation relation is reflexive, symmetric, and transitive:

(1) Reflexivity: Any rooted gyrovector PQ is the $\mathbf{0}$ gyrovector translation of itself,

$$T_{\mathbf{0}}PQ = PQ \tag{5.31}$$

(2) Symmetry: If

 (*i*) a rooted gyrovector PQ is the $\mathbf{t}$ gyrovector translation of a rooted gyrovector $P'Q'$,

$$T_{\mathbf{t}}PQ = P'Q' \tag{5.32}$$

 then

 (*ii*) the rooted gyrovector $P'Q'$ is the inverse gyrovector translation, $T_{\mathbf{t}}^{-1}$, of the rooted gyrovector PQ, where

$$T_{\mathbf{t}}^{-1} = T_{\ominus\mathrm{gyr}[P,\mathbf{t}]\mathbf{t}} \tag{5.33}$$

 that is,

$$T_{\ominus\mathrm{gyr}[P,\mathbf{t}]\mathbf{t}}P'Q' = PQ \tag{5.34}$$

(3) Transitivity: If

 (*i*) a rooted gyrovector $P'Q'$ is the gyrovector translation of a rooted gyrovector PQ by $\mathbf{t}_1$,

$$T_{\mathbf{t}_1}PQ = P'Q' \tag{5.35}$$

 and

(*ii*) a rooted gyrovector $P''Q''$ is the gyrovector translation of the rooted gyrovector $P'Q'$ by $\mathbf{t}_2$,

$$T_{\mathbf{t}_2}P'Q' = P''Q'' \tag{5.36}$$

then

(*iii*) the rooted gyrovector $P''Q''$ is the gyrovector translation of the rooted gyrovector PQ by the composite gyrovector translation $\mathbf{t}_{12}$ where, (5.24),

$$\mathbf{t}_{12} = \mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2 \tag{5.37}$$

that is,

$$T_{\mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2} PQ = P''Q'' \tag{5.38}$$

5.5 Points and Gyrovectors

Let $(G, \oplus)$ be a gyrocommutative gyrogroup. The elements of G, points, give rise to gyrovectors by Def. 5.4. Points and gyrovectors in G are related to each other by the following properties.

1. To any two points A and B in G there corresponds a unique gyrovector $\mathbf{v}$ in G, given by the equation, (5.1),

$$\mathbf{v} = \ominus A \oplus B \tag{5.39}$$

Hence, any point B of G can be viewed as a gyrovector in G with head at the point and tail at the origin O,

$$B = \ominus O \oplus B \tag{5.40}$$

2. To any point A and any gyrovector $\mathbf{v}$ in G there corresponds a unique point B satisfying (5.39), that is (by left cancellation, Table 2.1),

$$B = A \oplus \mathbf{v} \tag{5.41}$$

Hence, the gyrovector $\mathbf{v}$ can be viewed as a translation (called a right gyrotranslation) of point A into point B. Let $\mathbf{u}$ be a right gyrotranslation of point B into point C. Then, the two successive right gyrotranslations of point A into point C is equivalent to a single right gyrotranslation,

$$\begin{aligned} C &= (A \oplus \mathbf{v}) \oplus \mathbf{u} \\ &= A \oplus (\mathbf{v} \oplus \mathrm{gyr}[\mathbf{v}, A]\mathbf{u}) \end{aligned} \tag{5.42}$$

The resulting single right gyrotranslation, however, is corrected by a gyration that depends on the right gyrotranslated point A. Indeed, while gyrovectors share remarkable analogies with vectors, they are not vectors.

3. To any point B and any gyrovector $\mathbf{v}$ in G there corresponds a unique point A satisfying (5.39), that is, by a right cancellation, Table 2.1, and the cogyroautomorphic inverse property (2.103),

$$A = \ominus \mathbf{v} \boxplus B \tag{5.43}$$

4. For any three points A, B, C in G, (2.21),

$$(\ominus A \oplus B) \oplus \mathrm{gyr}[\ominus A, B](\ominus B \oplus C) = \ominus A \oplus C \tag{5.44}$$

5.6 Cogyrovectors

As in Sec. 5.2, elements of a gyrocommutative gyrogroup are called points and, in particular, the identity element is called the identity point, denoted O.

Definition 5.9 **(Rooted Cogyrovectors).** *A rooted cogyrovector PQ in a gyrocommutative gyrogroup $(G, \oplus)$ is an ordered pair of points $P, Q \in G$. The rooted cogyrovector PQ is rooted at the point P. The points P and Q of the rooted cogyrovector PQ are called, respectively, the tail and the head of the rooted cogyrovector. The value in G of the rooted cogyrovector PQ is $\boxminus P \boxplus Q$. Accordingly, we write (see Theorem 3.4)*

$$PQ = \boxminus P \boxplus Q = Q \boxminus P \tag{5.45}$$

and call $PQ = \boxminus P \boxplus Q$ the rooted cogyrovector, rooted at P, with tail P and head Q in G.

Furthermore, any point $A \in G$ is identified with the rooted cogyrovector OA with head A, rooted at the origin O of G.

Definition 5.10 (Rooted Cogyrovector Equivalence). *Let*

$$\begin{aligned} PQ &= \boxminus P \boxplus Q \\ P'Q' &= \boxminus P' \boxplus Q' \end{aligned} \tag{5.46}$$

be two rooted cogyrovectors in a gyrocommutative gyrogroup $(G, \oplus)$, with respective tails P and P' and respective heads Q and Q'. The two rooted cogyrovectors are equivalent,

$$Q' \boxminus P' \;\sim\; Q \boxminus P \tag{5.47}$$

if they have the same value in G, that is, if

$$Q' \boxminus P' \;=\; Q \boxminus P \tag{5.48}$$

Since the relation $\sim$ in Def. 5.10 is given in terms of an equality, it is clearly reflexive, symmetric, and transitive. Hence, it is an equivalence relation. As such, it gives rise to equivalence classes called *cogyrovectors.* Formalizing, we thus have the following

Definition 5.11 (Cogyrovectors). *Let $(G, \oplus)$ be a gyrocommutative gyrogroup with its rooted cogyrovector equivalence relation. The resulting equivalence classes are called cogyrovectors. The equivalence class of all rooted cogyrovectors that are equivalent to a given rooted cogyrovector $PQ = \boxminus P \boxplus Q = Q \boxminus P$ is the cogyrovector denoted by any element of its class, for instance, $PQ = \boxminus P \boxplus Q$. Any point $A \in G$ is identified with the cogyrovector OA. In order to contrast with rooted cogyrovectors, cogyrovectors are also called free cogyrovectors.*

5.7 Cogyrovector Translation

The following theorem and definition allow rooted cogyrovector equivalence to be expressed in terms of cogyrovector translation.

Theorem 5.12 *Let*

$$\begin{aligned} PQ &= Q \boxminus P \\ P'Q' &= Q' \boxminus P' \end{aligned} \tag{5.49}$$

be two rooted cogyrovectors in a gyrocommutative gyrogroup $(G, \oplus)$. *They are equivalent, that is,*

$$Q \boxminus P = Q' \boxminus P' \tag{5.50}$$

if and only if there exists a gyrovector $\mathbf{t} \in G$ *such that*

$$\begin{aligned} P' &= P \oplus \mathbf{t} \\ Q' &= Q \oplus \operatorname{gyr}[Q, P]\mathbf{t} \end{aligned} \tag{5.51}$$

Furthermore, the gyrovector $\mathbf{t}$ *is unique, given by the equation*

$$\mathbf{t} = \ominus P \oplus P' \tag{5.52}$$

Proof. Equation (5.50) can be written as

$$\boxminus P \boxplus Q = \boxminus P' \boxplus Q' \tag{5.53}$$

since the gyrogroup cooperation $\boxplus$ is commutative by Theorem 3.4. This equation, in turn, can be written in terms of the gyrogroup operation $\oplus$, Def. 2.7,

$$\ominus P \oplus \operatorname{gyr}[P, Q]Q = \ominus P' \oplus \operatorname{gyr}[P', Q']Q' \tag{5.54}$$

Hence, by Theorem 5.5 there exists a unique gyrovector $\mathbf{t} \in G$ such that

$$\begin{aligned} P' &= \operatorname{gyr}[P, \mathbf{t}](\mathbf{t} \oplus P) \\ \operatorname{gyr}[P', Q']Q' &= \operatorname{gyr}[P, \mathbf{t}](\mathbf{t} \oplus \operatorname{gyr}[P, Q]Q) \end{aligned} \tag{5.55}$$

The first equation in (5.55) implies, by the gyrocommutative law,

$$P' = P \oplus \mathbf{t} \tag{5.56}$$

so that, by a left cancellation, the unique gyrovector $\mathbf{t}$ is given by (5.52) and, hence, the first equation in (5.51) is satisfied.

By (5.53) with a right cancellation, and by employing the commutativity of the gyrogroup cooperation, Theorem 2.35, and a left cancellation, we have

$$\begin{aligned} Q' &= (\boxminus P \boxplus Q) \oplus P' \\ &= (Q \boxminus P) \oplus (P \oplus \mathbf{t}) \\ &= Q \oplus \operatorname{gyr}[Q, P]\{\ominus P \oplus (P \oplus \mathbf{t})\} \\ &= Q \oplus \operatorname{gyr}[Q, P]\mathbf{t} \end{aligned} \tag{5.57}$$

so that also the second equation in (5.51) is satisfied. Thus, (5.50) implies (5.51).

Conversely, assuming that there exists a gyrovector $\mathbf{t} \in G$ satisfying (5.51) we have, by a left cancellation and Theorem 2.35,

$$\begin{aligned} Q' &= Q \oplus \text{gyr}[Q, P]\mathbf{t} \\ &= Q \oplus \text{gyr}[Q, P]\{\ominus P \oplus (P \oplus \mathbf{t})\} \\ &= (Q \boxminus P) \oplus (P \oplus \mathbf{t}) \\ &= (Q \boxminus P) \oplus P' \end{aligned} \tag{5.58}$$

so that, by a right cancellation,

$$Q' \boxminus P' = Q \boxminus P \tag{5.59}$$

thus verifying (5.50) as desired. □

Theorem 5.12 suggests the following definition of cogyrovector translation in gyrocommutative gyrogroups.

Definition 5.13 **(Cogyrovector Translation).** *A cogyrovector translation $S_{\mathbf{t}}$ by a gyrovector $\mathbf{t} \in G$ of a rooted cogyrovector $PQ = \boxminus P \boxplus Q$, with tail P and head Q, in a gyrocommutative gyrogroup $(G, \oplus)$ is the rooted cogyrovector $P'Q' = \boxminus P' \boxplus Q'$, with tail P' and head Q', $S_{\mathbf{t}}PQ = P'Q'$, given by*

$$\begin{aligned} P' &= P \oplus \mathbf{t} \\ Q' &= Q \oplus \text{gyr}[Q, P]\mathbf{t} \end{aligned} \tag{5.60}$$

The rooted cogyrovector $\boxminus P' \boxplus Q'$ is said to be the $\mathbf{t}$ cogyrovector translation, or the cogyrovector translation by the gyrovector $\mathbf{t}$, of the rooted cogyrovector $\boxminus P \boxplus Q$.

Clearly, a cogyrovector translation by the zero gyrovector $\mathbf{0} \in G$ is trivial since (5.60) reduces to

$$\begin{aligned} P' &= P \oplus \mathbf{0} = P \\ Q' &= Q \oplus \text{gyr}[Q, P]\mathbf{0} = Q \end{aligned} \tag{5.61}$$

Definition 5.13 allows Theorem 5.12 to be reformulated, obtaining the following theorem:

Theorem 5.14 *Two rooted cogyrovectors*

$$\begin{aligned} PQ &= Q \boxminus P \\ P'Q' &= Q' \boxminus P' \end{aligned} \tag{5.62}$$

in a gyrocommutative gyrogroup $(G, \oplus)$ are equivalent if and only if rooted cogyrovector $P'Q'$ is a cogyrovector translation of rooted cogyrovector PQ. Furthermore, if PQ is a cogyrovector translation of $P'Q'$, then it is a cogyrovector translation of $P'Q'$ by the gyrovector

$$\mathbf{t} = \ominus P \oplus P' \tag{5.63}$$

Theorem 5.15 (Cogyrovector Translation Head). *Let P, Q, P' be any three points of a gyrocommutative gyrogroup $(G, \oplus)$. The cogyrovector translation of the rooted cogyrovector $PQ = Q \boxminus P$ to the rooted cogyrovector $P'X = X \boxminus P'$ with tail P' determines its head X,*

$$X = (Q \boxminus P) \oplus P' \tag{5.64}$$

Proof. $P'Q'$ is a cogyrovector translation of PQ. Hence, by Theorem 5.14, PQ and $P'X$ are equivalent cogyrovectors. Hence, by Def. 5.10, we have

$$Q \boxminus P = X \boxminus P' \tag{5.65}$$

from which (5.64) follows by a right cancellation, $\square$

5.8 Cogyrovector Translation Composition

Let $P''Q''$ be the cogyrovector translation of rooted cogyrovector $P'Q'$ by a gyrovector $\mathbf{t}_2$ where $P'Q'$, in turn, is the cogyrovector translation of rooted cogyrovector PQ by a gyrovector $\mathbf{t}_1$ in a gyrocommutative gyrogroup $(G, \oplus)$. Then, by Def. 5.13,

$$\begin{aligned} P' &= P \oplus \mathbf{t}_1 \\ P'' &= P' \oplus \mathbf{t}_2 \end{aligned} \tag{5.66}$$

so that

$$\begin{aligned} P'' &= P' \oplus \mathbf{t}_2 \\ &= (P \oplus \mathbf{t}_1) \oplus \mathbf{t}_2 \\ &= P \oplus (\mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2) \end{aligned} \tag{5.67}$$

Moreover, by Theorem 5.14 the rooted cogyrovector $P''Q''$ is equivalent to the rooted cogyrovector $P'Q'$ and the latter, in turn, is equivalent to the rooted cogyrovector PQ. Hence, $P''Q''$ is equivalent to PQ, so that, by Theorem 5.14, $P''Q''$ is a cogyrovector translation of PQ by some unique gyrovector $\mathbf{t}_{12} \in G$,

$$P'' = P \oplus \mathbf{t}_{12} \tag{5.68}$$

Comparing (5.68) and (5.67), we see that $\mathbf{t}_{12}$ is given by the equation

$$\mathbf{t}_{12} = \mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2 \tag{5.69}$$

Expressing the rooted cogyrovector $P'Q'$ in terms of the rooted cogyrovector PQ we have, by Theorem 5.12,

$$\begin{aligned} P' &= P \oplus \mathbf{t}_1 \\ Q' &= Q \oplus \mathrm{gyr}[Q, P]\mathbf{t}_1 \end{aligned} \tag{5.70}$$

Similarly, expressing the rooted cogyrovector $P''Q''$ in terms of the rooted cogyrovector $P'Q'$ we have, by Theorem 5.12,

$$\begin{aligned} P'' &= P' \oplus \mathbf{t}_2 \\ Q'' &= Q' \oplus \mathrm{gyr}[Q', P']\mathbf{t}_2 \end{aligned} \tag{5.71}$$

Finally, expressing the rooted cogyrovector $P''Q''$ in terms of the rooted cogyrovector PQ we have, by Theorem 5.12,

$$\begin{aligned} P'' &= P \oplus \mathbf{t}_{12} \\ Q'' &= Q \oplus \mathrm{gyr}[Q, P]\mathbf{t}_{12} \end{aligned} \tag{5.72}$$

Substituting (5.69) in (5.72) we have

$$\begin{aligned} P'' &= P\oplus(\mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2) \\ Q'' &= Q\oplus\mathrm{gyr}[Q,P]\{\mathbf{t}_1\oplus\mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2\} \\ &= Q\oplus\{\mathrm{gyr}[Q,P]\mathbf{t}_1\oplus\mathrm{gyr}[Q,P]\mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2\} \\ &= \{Q\oplus\mathrm{gyr}[Q,P]\mathbf{t}_1\}\oplus\mathrm{gyr}[Q,\mathrm{gyr}[Q,P]\mathbf{t}_1]\mathrm{gyr}[Q,P]\mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2 \end{aligned} \tag{5.73}$$

noting that $\mathrm{gyr}[P,Q]$ respects the gyrogroup operation, and employing the left gyroassociative law.

Substituting (5.70) in the second equation in (5.71) we have

$$Q'' = \{Q\oplus\mathrm{gyr}[Q,P]\mathbf{t}_1\}\oplus\mathrm{gyr}[Q\oplus\mathrm{gyr}[Q,P]\mathbf{t}_1,P\oplus\mathbf{t}_1]\mathbf{t}_2 \tag{5.74}$$

Comparing (5.74) with the second equation in (5.73) we have, by a left cancellation,

$$\mathrm{gyr}[Q,\mathrm{gyr}[Q,P]\mathbf{t}_1]\mathrm{gyr}[Q,P]\mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2 = \mathrm{gyr}[Q\oplus\mathrm{gyr}[Q,P]\mathbf{t}_1,P\oplus\mathbf{t}_1]\mathbf{t}_2 \tag{5.75}$$

for all $P,Q,\mathbf{t}_1,\mathbf{t}_2 \in G$. Renaming $\mathbf{t}_1 = R$, and omitting $\mathbf{t}_2$ on both sides of (5.75) we uncover the gyroautomorphism identity

$$\mathrm{gyr}[Q,\mathrm{gyr}[Q,P]R]\mathrm{gyr}[Q,P]\mathrm{gyr}[R,P] = \mathrm{gyr}[Q\oplus\mathrm{gyr}[Q,P]R,P\oplus R] \tag{5.76}$$

for all $P,Q,R \in G$ in a gyrocommutative gyrogroup $(G,\oplus)$.

Thus, in our way to uncover the composition law (5.72) of cogyrovector translation we obtained the new gyroautomorphism identity (5.76) as an unintended and unforeseen by-product. Interestingly, when $P = O$ $(Q = O)$ the new identity (5.76) reduces to the left (right) loop property, and when $P = -Q$ it reduces to (3.40).

The composite cogyrovector translation (5.72) is trivial when $\mathbf{t}_{12} = O$, that is, when $\mathbf{t}_2 = \ominus\mathrm{gyr}[P,\mathbf{t}_1]\mathbf{t}_1$, as we see from (5.69). Hence, the inverse cogyrovector translation of cogyrovector translation by $\mathbf{t}$ is a cogyrovector translation by $\ominus\mathrm{gyr}[P,\mathbf{t}]\mathbf{t}$.

The equivalence relation between rooted cogyrovectors in Def. 5.10 is expressed in Theorem 5.12 in terms of the cogyrovector translation. The cogyrovector translation relation is, accordingly, an equivalence relation. Indeed, it is reflexive, symmetric, and transitive:

(1) Reflexivity: Any rooted cogyrovector PQ is the $\mathbf{0}$ cogyrovector

translation of itself,

$$S_0 PQ = PQ \tag{5.77}$$

(2) Symmetry: If

(*i*) a rooted cogyrovector PQ is the $\mathbf{t}$ cogyrovector translation of a rooted cogyrovector $P'Q'$,

$$S_{\mathbf{t}} PQ = P'Q' \tag{5.78}$$

then

(*ii*) the rooted cogyrovector $P'Q'$ is the inverse cogyrovector translation, $S_{\mathbf{t}}^{-1}$, of the rooted cogyrovector PQ, where

$$S_{\mathbf{t}}^{-1} = S_{\ominus \mathrm{gyr}[P,\mathbf{t}]\mathbf{t}} \tag{5.79}$$

that is,

$$S_{\ominus \mathrm{gyr}[P,\mathbf{t}]\mathbf{t}} P'Q' = PQ \tag{5.80}$$

(3) Transitivity: If

(*i*) a rooted cogyrovector $P'Q'$ is the cogyrovector translation of a rooted cogyrovector PQ by $\mathbf{t}_1$,

$$S_{\mathbf{t}_1} PQ = P'Q' \tag{5.81}$$

and

(*ii*) a rooted cogyrovector $P''Q''$ is the cogyrovector translation of the rooted cogyrovector $P'Q'$ by $\mathbf{t}_2$,

$$S_{\mathbf{t}_2} P'Q' = P''Q'' \tag{5.82}$$

then

(*iii*) the rooted cogyrovector $P''Q''$ is the cogyrovector translation of the rooted cogyrovector PQ by the composite cogyrovector translation $\mathbf{t}_{12}$ where, (5.69),

$$\mathbf{t}_{12} = \mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1, P]\mathbf{t}_2 \tag{5.83}$$

that is,

$$S_{\mathbf{t}_1 \oplus \mathrm{gyr}[\mathbf{t}_1,P]\mathbf{t}_2} PQ = P''Q'' \tag{5.84}$$

5.9 Points and Cogyrovectors

Let $(G, \oplus)$ be a gyrocommutative gyrogroup. Elements of G, points, give rise to cogyrovectors. Points and cogyrovectors in G are related to each other by the following properties.

1. To any two points A and B in G there corresponds a unique gyrovector $\mathbf{v}$ in G, given by the equation, (5.45),

$$\mathbf{v} = \boxminus A \boxplus B \tag{5.85}$$

   Hence, any point B of G can be viewed as a cogyrovector in G with head at the point and tail at the origin O,

$$B = \boxminus O \boxplus B \tag{5.86}$$

2. To any point A and any gyrovector $\mathbf{v}$ in G there corresponds a unique point B satisfying (5.85), that is (by right cancellation, Table 2.1, noting that $\boxminus A \boxplus B = B \boxminus A$),

$$B = \mathbf{v} \oplus A \tag{5.87}$$

3. To any point B and any gyrovector $\mathbf{v}$ in G there corresponds a unique point A satisfying (5.85), that is (by right cancellation, Table 2.1, and the gyroautomorphic inverse property (3.1)),

$$A = \ominus \mathbf{v} \oplus B \tag{5.88}$$

4. For any three points A, B, C in G, (3.42),

$$(\boxminus A \boxplus B) \oplus (\boxminus B \boxplus C) = \boxminus A \boxplus \mathrm{gyr}[\boxminus A \boxplus B, \boxminus B \boxplus C]C \tag{5.89}$$

5.10 Exercises

(1) Show that the composite gyroautomorphism J of a gyrocommutative gyrogroup $(G, \oplus)$,

$$J = \mathrm{gyr}[P, Q]\mathrm{gyr}[\mathrm{gyr}[Q, P]R, Q]\mathrm{gyr}[Q \oplus \mathrm{gyr}[Q, P]R, P \oplus R] \tag{5.90}$$

is a gyroautomorphism, and is independent of Q, for all $P, Q, R \in G$ (Hint: Note (5.76)). Use the result to uncover several gyroautomorphism identities (for instance, set $Q = O$ in J to recover (5.76); or set $Q = \ominus P$ to recover the special case (3.40) of (3.34)).

(2) Show that Identity (5.89) can be written as

$$(\boxminus A \boxplus B) \boxplus g(\boxminus B \boxplus C) = \boxminus A \boxplus gC \tag{5.91}$$

where

$$g = \mathrm{gyr}[\boxminus A \boxplus B, \boxminus B \boxplus C] \tag{5.92}$$

and conclude that the identity

$$(\boxminus A \boxplus B) \boxplus (\boxminus B \boxplus C) = \boxminus A \boxplus C \tag{5.93}$$

holds in the special case when

$$\mathrm{gyr}[\boxminus A \boxplus B, \boxminus B \boxplus C] = I \tag{5.94}$$

What does the condition (5.94) mean for the points A, B, C?

Chapter 6

Gyrovector Spaces

Some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces. The latter, in turn, are analogous to vector spaces just as gyrogroups are analogous to groups. Indeed, gyrovector spaces provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry.

The elements of a gyrovector space are called points. Any two points of a gyrovector space give rise to a gyrovector. Points give rise to geodesics and cogeodesics that share analogies with Euclidean geodesics, the straight lines.

6.1 Definition and First Gyrovector Space Theorems

Definition 6.1 (Real Inner Product Vector Spaces). *A real inner product vector space* $(\mathbb{V}, +, \cdot)$ *(vector space, in short) is a real vector space together with a map*

$$\mathbb{V}\times\mathbb{V} \to \mathbb{R}, \qquad (\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u}\cdot\mathbf{v} \tag{6.1}$$

called a real inner product, satisfying the following properties for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ *and* $r \in \mathbb{R}$*:*

(1) $\mathbf{v}\cdot\mathbf{v} \geq 0$, *with equality if, and only if,* $\mathbf{v} = \mathbf{0}$.
(2) $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$
(3) $(\mathbf{u} + \mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w}$
(4) $(r\mathbf{u})\cdot\mathbf{v} = r(\mathbf{u}\cdot\mathbf{v})$

The norm $\|\mathbf{v}\|$ *of* $\mathbf{v} \in \mathbb{V}$ *is given by the equation* $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$.

Note that the properties of vector spaces imply (i) the Cauchy-Schwarz inequality $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\|\mathbf{v}\|$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$; and (ii) the *positive definiteness* of the inner product, according to which $\mathbf{u}\cdot\mathbf{v} = 0$ for all $\mathbf{u} \in \mathbb{V}$ implies $\mathbf{v} = 0$.

Definition 6.2 **(Real Inner Product Gyrovector Spaces).** *A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:*

(1) G is a subset of a real inner product vector space $\mathbb{V}$ called the carrier of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$$

for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V1)	$1\otimes\mathbf{a} = \mathbf{a}$	
(V2)	$(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$	*Scalar Distributive Law*
(V3)	$(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$	*Scalar Associative Law*
(V4)	$\dfrac{\lvert r\rvert\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}$	*Scaling Property*
(V5)	$\mathrm{gyr}[\mathbf{u}, \mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{a}$	*Gyroautomorphism Property*
(V6)	$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$	*Identity Automorphism.*

(3) Real vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors"

$$\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$$

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V7)	$\|r\otimes\mathbf{a}\| = \lvert r\rvert\otimes\|\mathbf{a}\|$	*Homogeneity Property*
(V8)	$\|\mathbf{a}\oplus\mathbf{b}\| \le \|\mathbf{a}\|\oplus\|\mathbf{b}\|$	*Gyrotriangle Inequality.*

Remark 6.3 *One can readily verify that $(-1)\otimes\mathbf{a} = \ominus\mathbf{a}$, and $\|\ominus\mathbf{a}\| = \|\mathbf{a}\|$. We use the notation $\mathbf{a}\otimes r = r\otimes\mathbf{a}$. Our ambiguous use of $\oplus$ and $\otimes$, Def. 6.2,*

as interrelated operations in the gyrovector space $(G, \oplus, \otimes)$ *and in its associated vector space* $(\|G\|, \oplus, \otimes)$ *should raise no confusion, since the sets in which these operations operate are always clear from the text. These operations in the former (gyrovector space* $(G, \oplus, \otimes)$*) are nonassociative-nondistributive gyrovector space operations, and in the latter (vector space* $(\|G\|, \oplus, \otimes)$*) are associative-distributive vector space operations. Additionally, the gyro-addition* $\oplus$ *is gyrocommutative in the former and commutative in the latter. Note that in the vector space* $(\|G\|, \oplus, \otimes)$ *gyrations are trivial so that* $\boxplus = \oplus$ *in* $\|G\|$.

While the operations $\oplus$ *and* $\otimes$ *have distinct interpretations in the gyrovector space* G *and in the vector space* $\|G\|$*, they are related to one another by the gyrovector space axioms* $(V7)$ *and* $(V8)$*. The analogies that conventions about the ambiguous use of* $\oplus$ *and* $\otimes$ *in* G *and* $\|G\|$ *share with similar vector space conventions are obvious. In vector spaces we use the same notation,* $+$*, for the addition operation between vectors and between their magnitudes, and same notation for the scalar multiplication between two scalars and between a scalar and a vector.*

Owing to the scalar distributive law, the condition for $1\otimes\mathbf{a}$ in $(V1)$ is equivalent to the condition

$$n\otimes\mathbf{a} = \mathbf{a}\oplus \quad \dots \quad \oplus\mathbf{a} \qquad \textit{(gyroadding } \mathbf{a}\ n \textit{ times)} \tag{6.2}$$

and

$$\mathbf{a}\otimes(-t) = \ominus\mathbf{a}\otimes t \tag{6.3}$$

Clearly, in the special case when all the gyrations of a gyrovector space are trivial, the gyrovector space reduces to a vector space.

In general, gyroaddition does not distribute with scalar multiplication. However, gyrovector spaces possess a weak distributive law, called the monodistributive law, presented in the following theorem.

Theorem 6.4 (The Monodistributive Law). *A gyrovector space* $(G, \oplus, \otimes)$ *possesses the monodistributive law*

$$r\otimes(r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}) = r\otimes(r_1\otimes\mathbf{a})\oplus r\otimes(r_2\otimes\mathbf{a}) \tag{6.4}$$

Proof. The proof follows from $(V2)$ and $(V3)$,

$$
\begin{aligned}
r\otimes(r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}) &= r\otimes\{(r_1+r_2)\otimes\mathbf{a}\} \\
&= (r(r_1+r_2))\otimes\mathbf{a} \\
&= (rr_1+rr_2)\otimes\mathbf{a} \qquad (6.5)\\
&= (rr_1)\otimes\mathbf{a}\oplus(rr_1)\otimes\mathbf{a} \\
&= r\otimes(r_1\otimes\mathbf{a})\oplus r\otimes(r_1\otimes\mathbf{a})
\end{aligned}
$$

□

Definition 6.5 **(Gyrovector Space Automorphisms).** *An automorphism τ of a gyrovector space $(G,\oplus,\otimes)$, $\tau\in Aut(G,\oplus,\otimes)$, is a bijective self-map of G*

$$\tau : G \rightarrow G \tag{6.6}$$

which preserves its structure, that is, (i) binary operation, (ii) scalar multiplication, and (iii) inner product,

$$
\begin{aligned}
\tau(\mathbf{a}\oplus\mathbf{b}) &= \tau\mathbf{a}\oplus\tau\mathbf{b} \\
\tau(r\otimes\mathbf{a}) &= r\otimes\tau\mathbf{a} \qquad (6.7)\\
\tau(\mathbf{a}\cdot\mathbf{b}) &= \tau\mathbf{a}\cdot\tau\mathbf{b}
\end{aligned}
$$

The automorphisms of the gyrovector space $(G,\oplus,\otimes)$ form a group denoted $Aut(G,\oplus,\otimes)$, with group operation given by automorphism composition.

Clearly, gyroautomorphisms are special automorphisms.

Definition 6.6 **(Motions of Gyrovector Spaces).** *The motions of a gyrovector space $(G,\oplus,\otimes)$ are all its left gyrotranslations L_x, $x\in G$, Def. 2.18, and its automorphisms $\tau\in Aut(G,\oplus,\otimes)$, Def. 6.5.*

Scalar multiplication in a gyrovector space does not distribute with the gyrovector space operation. Hence, the *Two-Sum Identity* in the following theorem proves useful.

Theorem 6.7 **(The Two-Sum Identity).** *Let $(G,\oplus,\otimes)$ be a gyrovector space. Then*

$$
\begin{aligned}
2\otimes(\mathbf{a}\oplus\mathbf{b}) &= \mathbf{a}\oplus(2\otimes\mathbf{b}\oplus\mathbf{a}) \\
&= \mathbf{a}\boxplus(\mathbf{a}\oplus 2\otimes\mathbf{b})
\end{aligned}
\tag{6.8}
$$

for any $\mathbf{a},\mathbf{b}\in G$.

Proof. Employing the right gyroassociative law, the identity $\mathrm{gyr}[\mathbf{b},\mathbf{b}] = I$, the left gyroassociative law, and the gyrocommutative law we have the following chain of equation that results in the desired identity,

$$\begin{aligned}
\mathbf{a}\oplus(2\otimes\mathbf{b}\oplus\mathbf{a}) &= \mathbf{a}\oplus((\mathbf{b}\oplus\mathbf{b})\oplus\mathbf{a}) \\
&= \mathbf{a}\oplus(\mathbf{b}\oplus(\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\mathbf{b}]\mathbf{a})) \\
&= \mathbf{a}\oplus(\mathbf{b}\oplus(\mathbf{b}\oplus\mathbf{a})) \\
&= (\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\mathbf{a},\mathbf{b}](\mathbf{b}\oplus\mathbf{a}) \\
&= (\mathbf{a}\oplus\mathbf{b})\oplus(\mathbf{a}\oplus\mathbf{b}) \\
&= 2\otimes(\mathbf{a}\oplus\mathbf{b})
\end{aligned} \tag{6.9}$$

The second equality in the theorem follows from the first one and Theorem 3.12. □

A gyrovector space is a gyrometric space with a gyrodistance function that obeys the gyrotriangle inequality.

Definition 6.8 **(The Gyrodistance Function).** *Let $G = (G, \oplus, \otimes)$ be a gyrovector space. Its gyrometric is given by the gyrodistance function $d_\oplus(\mathbf{a},\mathbf{b}) : G\times G \to \mathbb{R}^{\geq 0}$,*

$$d_\oplus(\mathbf{a},\mathbf{b}) = \|\ominus\mathbf{a}\oplus\mathbf{b}\| = \|\mathbf{b}\ominus\mathbf{a}\| \tag{6.10}$$

where $d_\oplus(\mathbf{a},\mathbf{b})$ is the gyrodistance of $\mathbf{a}$ *to* $\mathbf{b}$*.*

By Def. 6.2, gyroautomorphisms preserve the inner product. Hence, they are isometries, that is, they preserve the norm as well. The identity $\|\ominus\mathbf{a}\oplus\mathbf{b}\| = \|\mathbf{b}\ominus\mathbf{a}\|$ in Def. 6.8 thus follows from the gyrocommutative law,

$$\begin{aligned}
\|\ominus\mathbf{a}\oplus\mathbf{b}\| &= \|\mathrm{gyr}[\ominus\mathbf{a},\mathbf{b}](\mathbf{b}\ominus\mathbf{a})\| \\
&= \|\mathbf{b}\ominus\mathbf{a}\|
\end{aligned} \tag{6.11}$$

Theorem 6.9 **(The Gyrotriangle Inequality).** *The gyrometric of a gyrovector space $(G, \oplus, \otimes)$ satisfies the gyrotriangle inequality*

$$\|\ominus\mathbf{a}\oplus\mathbf{c}\| \leq \|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\| \tag{6.12}$$

Proof. By Theorem 2.11 we have,

$$\ominus\mathbf{a}\oplus\mathbf{c} = (\ominus\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\ominus\mathbf{a},\mathbf{b}](\ominus\mathbf{b}\oplus\mathbf{c}) \tag{6.13}$$

Hence, by the gyrotriangle inequality ($V8$) we have

$$\begin{aligned}\|\ominus\mathbf{a}\oplus\mathbf{c}\| &= \|(\ominus\mathbf{a}\oplus\mathbf{b})\oplus\mathrm{gyr}[\ominus\mathbf{a},\mathbf{b}](\ominus\mathbf{b}\oplus\mathbf{c})\| \\ &\le \|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\mathrm{gyr}[\ominus\mathbf{a},\mathbf{b}](\ominus\mathbf{b}\oplus\mathbf{c})\| \\ &= \|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\|\end{aligned} \tag{6.14}$$

□

The basic properties of the gyrodistance function $d_\oplus$ are

(i) $d_\oplus(\mathbf{a},\mathbf{b}) \ge 0$
(ii) $d_\oplus(\mathbf{a},\mathbf{b}) = 0$ if and only if $\mathbf{a} = \mathbf{b}$.
(iii) $d_\oplus(\mathbf{a},\mathbf{b}) = d_\oplus(\mathbf{b},\mathbf{a})$
(iv) $d_\oplus(\mathbf{a},\mathbf{c}) \le d_\oplus(\mathbf{a},\mathbf{b})\oplus d_\oplus(\mathbf{b},\mathbf{c})$ (gyrotriangle inequality),

$\mathbf{a},\mathbf{b},\mathbf{c} \in G$.

Curves on which the gyrotriangle inequality reduces to an equality, called *gyrolines*, will be identified in Theorem 6.47.

In addition of being gyrometric, a gyrovector space is cogyrometric with a cogyrodistance function that obeys the cogyrotriangle inequality.

Definition 6.10 **(Cogyrodistance).** *Let $G = (G,\oplus,\otimes)$ be a gyrovector space. Its cogyrometric is given by the cogyrodistance function $d_\boxplus(\mathbf{a},\mathbf{b}) : G\times G \to \mathbb{R}^{\ge 0}$,*

$$d_\boxplus(\mathbf{a},\mathbf{b}) = \|\mathbf{b}\boxminus\mathbf{a}\| \tag{6.15}$$

Theorem 6.11 **(The Cogyrotriangle Inequality).** *The cogyrometric of a gyrovector space $(G,\oplus,\otimes)$ satisfies the cogyrotriangle inequality*

$$\|\mathbf{a}\boxminus\mathrm{gyr}[\mathbf{a}\boxminus\mathbf{b},\mathbf{b}\boxminus\mathbf{c}]\mathbf{c}\| \le \|\mathbf{a}\boxminus\mathbf{b}\| \boxplus \|\mathbf{b}\boxminus\mathbf{c}\| \tag{6.16}$$

or, equivalently,

$$\|\boxminus\mathbf{a}\boxplus\mathrm{gyr}[\mathbf{a}\boxminus\mathbf{b},\mathbf{b}\boxminus\mathbf{c}]\mathbf{c}\| \le \|\boxminus\mathbf{a}\boxplus\mathbf{b}\| \boxplus \|\boxminus\mathbf{b}\boxplus\mathbf{c}\| \tag{6.17}$$

Proof. By (3.42) and by the gyrotriangle inequality ($V8$) we have

$$\begin{aligned}\|\mathbf{a}\boxminus\mathrm{gyr}[\mathbf{a}\boxminus\mathbf{b},\mathbf{b}\boxminus\mathbf{c}]\mathbf{c}\| &= \|(\mathbf{a}\boxminus\mathbf{b})\oplus(\mathbf{b}\boxminus\mathbf{c})\| \\ &\le \|\mathbf{a}\boxminus\mathbf{b}\|\oplus\|\mathbf{b}\boxminus\mathbf{c}\| \\ &= \|\mathbf{a}\boxminus\mathbf{b}\| \boxplus \|\mathbf{b}\boxminus\mathbf{c}\|\end{aligned} \tag{6.18}$$

thus verifying (6.16). The equivalence between (6.16) and (6.17) follows from the Cogyroautomorphic Inverse Theorem 2.31, implying $\|\mathbf{a}\boxminus\mathbf{b}\| = \|\boxminus\mathbf{a}\boxplus\mathbf{b}\|$ etc. □

Note that the cogyrotriangle inequality (6.16) has the form of the gyrotriangle inequality (6.12) except that one of its terms is gyro-corrected.

The basic properties of the cogyrodistance function $d_\boxplus$ are

(*i*) $d_\boxplus(\mathbf{a}, \mathbf{b}) \geq 0$
(*ii*) $d_\boxplus(\mathbf{a}, \mathbf{b}) = 0$ if and only if $\mathbf{a} = \mathbf{b}$.
(*iii*) $d_\boxplus(\mathbf{a}, \mathbf{b}) = d_\boxplus(\mathbf{b}, \mathbf{a})$
(*iv*) $d_\boxplus(\mathbf{a}, \mathrm{gyr}[\mathbf{a} \boxminus \mathbf{b}, \mathbf{b} \boxminus \mathbf{c}]\mathbf{c}) \leq d_\boxplus(\mathbf{a}, \mathbf{b}) \boxplus d_\boxplus(\mathbf{b}, \mathbf{c})$ (cogyrotriangle inequality),

$\mathbf{a}, \mathbf{b}, \mathbf{c} \in G$.

Curves on which the cogyrotriangle inequality reduces to an equality, called *cogyrolines*, will be identified in Theorems 6.74 and 6.75.

Theorem 6.12 *The gyrodistance is invariant under automorphisms and left gyrotranslations.*

Proof. By Def. 6.5, automorphisms $\tau \in Aut(G, \oplus, \otimes)$ preserve the inner product. As such they preserve the norm and, hence, the gyrodistance,

$$\begin{aligned} \|\tau\mathbf{b}\ominus\tau\mathbf{a}\| &= \|\tau(\mathbf{b}\ominus\mathbf{a})\| \\ &= \|\mathbf{b}\ominus\mathbf{a}\| \end{aligned} \tag{6.19}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b}$ in a gyrovector space $(G, \oplus, \otimes)$. Hence, the gyrodistance is invariant under automorphisms.

Let $\mathbf{a}, \mathbf{b}, \mathbf{x} \in G$ be any three points in a gyrovector space $(G, \oplus, \otimes)$, and let the points $\mathbf{a}$ and $\mathbf{b}$ be left gyrotranslated by $\mathbf{x}$ into $\mathbf{a}'$ and $\mathbf{b}'$ respectively,

$$\begin{aligned} \mathbf{a}' &= \mathbf{x}\oplus\mathbf{a} \\ \mathbf{b}' &= \mathbf{x}\oplus\mathbf{b} \end{aligned} \tag{6.20}$$

Then, by the Gyrotranslation Theorem 3.13 we have

$$\begin{aligned} \mathbf{b}'\ominus\mathbf{a}' &= (\mathbf{x}\oplus\mathbf{b})\ominus(\mathbf{x}\oplus\mathbf{a}) \\ &= \mathrm{gyr}[\mathbf{x}, \mathbf{b}](\mathbf{b}\ominus\mathbf{a}) \end{aligned} \tag{6.21}$$

so that

$$\begin{aligned} \|\mathbf{b}'\ominus\mathbf{a}'\| &= \|\mathrm{gyr}[\mathbf{x}, \mathbf{b}](\mathbf{b}\ominus\mathbf{a})\| \\ &= \|\mathbf{b}\ominus\mathbf{a}\| \end{aligned} \tag{6.22}$$

Hence, the gyrodistance is invariant under left gyrotranslations. □

Like gyrodistance, cogyrodistance is invariant under automorphisms.

Theorem 6.13 *The cogyrodistance is invariant under automorphisms.*

Proof. By (2.52) with $a = \mathbf{a}$, $b = \ominus\mathbf{b}$, and $\tau \in Aut(G, \oplus, \otimes)$, we have

$$\tau(\mathbf{a} \boxminus \mathbf{b}) = \tau\mathbf{a} \boxminus \tau\mathbf{b} \tag{6.23}$$

Hence,

$$\begin{aligned} \|\tau\mathbf{a} \boxminus \tau\mathbf{b}\| &= \|\tau(\mathbf{a} \boxminus \mathbf{b})\| \\ &= \|\mathbf{a} \boxminus \mathbf{b}\| \end{aligned} \tag{6.24}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{a}, \mathbf{b} \in (G, \oplus, \otimes)$, and all $\tau \in Aut(G, \oplus, \otimes)$. □

Unlike gyrodistance, cogyrodistance is not invariant under left gyrotranslations. It is also not invariant under right gyrotranslation. However, it is invariant under appropriately gyrated right gyrotranslations, as we will see in the following Theorem 6.14 and in Theorem 6.73.

Theorem 6.14 *The cogyrodistance in a gyrovector space $(G, \oplus, \otimes)$ is invariant under appropriately gyrated right gyrotranslations,*

$$\mathbf{a} \boxminus \mathbf{b} = (\mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{x}) \boxminus (\mathbf{b} \oplus \mathbf{x}) \tag{6.25}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{x} \in G$.

Proof. This theorem is just a reformulation of Identity (2.42) in Theorem 2.16 in terms of a right gyrotranslation. □

Theorem 6.14 seemingly attributes asymmetry to $\mathbf{a}$ and $\mathbf{b}$ in (6.25). Following the discovery of the Cogyroline Gyration Transitive Law in Theorem 6.62, Theorem 6.14 will be generalized in Theorem 6.73, where the seemingly lost symmetry in $\mathbf{a}$ and $\mathbf{b}$ will appear.

6.2 Solving a System of Two Equations in a Gyrovector Space

Theorem 6.15 *Let $(G, \oplus, \otimes)$ be a gyrovector space, and let $a, b \in G$ be any two elements of G. The unique solution of the system of two equations*

$$\begin{aligned} x \oplus y &= a \\ \ominus x \oplus y &= b \end{aligned} \tag{6.26}$$

for the unknowns x and y is

$$\begin{aligned} x &= \tfrac{1}{2}\otimes(a \boxminus b) \\ y &= \tfrac{1}{2}\otimes(a \boxminus b)\oplus b \end{aligned} \tag{6.27}$$

Proof. Solving the first equation in (6.26) for y, we have by Theorem 2.15

$$y = x\oplus b \tag{6.28}$$

Eliminating y between (6.28) and the first equation in (6.26), we have

$$a = x\oplus(x\oplus b) = (2\otimes x)\oplus b$$

so that, by Theorem 2.15, $2\otimes x = a \boxminus b$, implying

$$x = \tfrac{1}{2}\otimes(a \boxminus b) \tag{6.29}$$

It then follows from (6.28) and (6.29) that

$$y = \tfrac{1}{2}\otimes(a \boxminus b)\oplus b \tag{6.30}$$

Hence if (6.26) possesses a solution it must be the unique one given by (6.27). The latter is, indeed, a solution of the former since (i) by the left gyroassociative law

$$\begin{aligned} x\oplus y &= \tfrac{1}{2}\otimes(a \boxminus b)\oplus\{\tfrac{1}{2}\otimes(a \boxminus b)\oplus b\} \\ &= \{\tfrac{1}{2}\otimes(a \boxminus b)\oplus\tfrac{1}{2}\otimes(a \boxminus b)\}\oplus b \\ &= 2\otimes\tfrac{1}{2}\otimes(a \boxminus b)\oplus b \\ &= (a \boxminus b)\oplus b \\ &= a \end{aligned} \tag{6.31}$$

and since (ii) by a left cancellation

$$\ominus x\oplus y = \ominus\tfrac{1}{2}\otimes(a \boxminus b)\oplus\{\tfrac{1}{2}\otimes(a \boxminus b)\oplus b\} = b \tag{6.32}$$

□

Interchanging a and b in the system (6.26) keeps x invariant and reverses the sign of y. Hence, it follows from Theorem 6.15 that x in (6.27) is antisymmetric in a and b,

$$\tfrac{1}{2}\otimes(a \boxminus b) = -\tfrac{1}{2}\otimes(b \boxminus a) \tag{6.33}$$

and that y in (6.27) is symmetric in a and b,

$$\tfrac{1}{2}\otimes(a\boxminus b)\oplus b = \tfrac{1}{2}\otimes(b\boxminus a)\oplus a \tag{6.34}$$

Indeed, (6.33) also follows from Theorems 3.4 and 2.31.

As an application of Theorem 6.15 we substitute $x\oplus y$ and $\ominus x\oplus y$ from (6.26) and y from the second equation of (6.27) in (3.53), with a and b replaced by x and y, obtaining the identity

$$\begin{aligned}\mathrm{gyr}[b,a] &= \mathrm{gyr}[b, \tfrac{1}{2}\otimes(a\boxminus b)\oplus b]\mathrm{gyr}[\tfrac{1}{2}\otimes(a\boxminus b)\oplus b, a]\\ &= \mathrm{gyr}[b, m_{ab}^{s}]\mathrm{gyr}[m_{ab}^{s}, a]\end{aligned} \tag{6.35}$$

where

$$m_{ab}^{s} = \tfrac{1}{2}\otimes(a\boxminus b)\oplus b \tag{6.36}$$

is the so called cogyromidpoint of a and b, Def. 6.70, satisfying the symmetry condition $m_{ab}^{s} = m_{ba}^{s}$. It shares duality symmetries with the gyromidpoint that will be defined in Def. 6.31. Interestingly, Identity (6.35) is a special case of the cogyroline gyration transitive law in Theorem 6.62.

We may note that y in (6.27) can be written as

$$\begin{aligned}y &= \tfrac{1}{2}\otimes(a\boxminus b)\oplus b\\ &= \tfrac{1}{2}\otimes\mathrm{gyr}[\tfrac{1}{2}\otimes(a\boxminus b), \tfrac{1}{2}\otimes(b\oplus a)](b\oplus a)\end{aligned} \tag{6.37}$$

We may also note the nice related identity

$$a\oplus\tfrac{1}{2}\otimes(\ominus a\boxplus b) = \tfrac{1}{2}\otimes(a\oplus b) \tag{6.38}$$

This identity, in turn, may be compared with the semidual identity

$$a\oplus\tfrac{1}{2}\otimes(\ominus a\oplus b) = \tfrac{1}{2}\otimes(a\boxplus b) \tag{6.39}$$

As an application of the Identity Automorphism ($V8$) in Def. 6.2 of gyrovector spaces, we prove the following

Theorem 6.16 *Let $(G,\oplus,\otimes)$ be a gyrovector space. Then*

$$\mathrm{gyr}[(r+s)\otimes a, b] = \mathrm{gyr}[r\otimes a, s\otimes a\oplus b]\mathrm{gyr}[s\otimes a, b] \tag{6.40}$$

for all $r,s\in\mathbb{R}$ and $a,b\in G$.

Proof. Expanding $(r+s)\otimes a\oplus(b\oplus x)$ in G in two different ways we have

$$\begin{aligned}(r+s)\otimes a\oplus(b\oplus x) &= (r\otimes a\oplus s\otimes a)\oplus(b\oplus x)\\ &= r\otimes a\oplus\{s\otimes a\oplus(b\oplus x)\}\\ &= r\otimes a\oplus\{(s\otimes a\oplus b)\oplus\mathrm{gyr}[s\otimes a,b]x\}\end{aligned}\tag{6.41}$$

and

$$\begin{aligned}(r+s)\otimes a\oplus(b\oplus x) &= \{(r+s)\otimes a\oplus b\}\oplus\mathrm{gyr}[(r+s)\otimes a,b]x\\ &= \{(r\otimes a\oplus s\otimes a)\oplus b\}\oplus\mathrm{gyr}[(r+s)\otimes a,b]x\\ &= \{r\otimes a\oplus(s\otimes a\oplus\mathrm{gyr}[s\otimes a,r\otimes a]b)\}\oplus\mathrm{gyr}[(r+s)\otimes a,b]x\\ &= \{r\otimes a\oplus(s\otimes a\oplus b)\}\oplus\mathrm{gyr}[(r+s)\otimes a,b]x\\ &= r\otimes a\oplus\{(s\otimes a\oplus b)\oplus\mathrm{gyr}[s\otimes a\oplus b,r\otimes a]\mathrm{gyr}[(r+s)\otimes a,b]x\}\end{aligned}\tag{6.42}$$

for all $r,s\in\mathbb{R}$ and $a,b,x\in G$.

Comparing the extreme right hand sides of (6.41) and (6.42) we have by two successive left cancellations,

$$\mathrm{gyr}[s\otimes a\oplus b,r\otimes a]\mathrm{gyr}[(r+s)\otimes a,b]=\mathrm{gyr}[s\otimes a,b]\tag{6.43}$$

which is equivalent to the gyration identity in the theorem. □

6.3 Gyrolines and Cogyrolines

In full analogy with (i) the two identical line expressions

$$\begin{array}{ll}\mathbf{a}+\mathbf{b}t & \text{The Euclidean Line}\\ \mathbf{b}t+\mathbf{a} & \text{The Euclidean Line}\end{array}\tag{6.44}$$

$\mathbf{a},\mathbf{b}\in G$, $t\in\mathbb{R}$, in analytic Euclidean geometry, which is regulated by the (associative) algebra of vector spaces $(G,+,\cdot)$, (ii) the two distinct hyperbolic line expressions

$$\begin{array}{ll}\mathbf{a}\oplus\mathbf{b}\otimes t & \text{Gyroline, The Hyperbolic Line}\\ \mathbf{b}\otimes t\oplus\mathbf{a} & \text{Cogyroline, The Hyperbolic Dual Line}\end{array}\tag{6.45}$$

$t\in\mathbb{R}$, of hyperbolic analytic geometry are regulated by the (nonassociative) algebra of gyrovector spaces $(G,\oplus,\otimes)$.

In order to emphasize that the Euclidean line is uniquely determined by any two distinct points that it contains, one replaces the expressions in (6.44) by

$$\begin{aligned} &\mathbf{a}+(-\mathbf{a}+\mathbf{b})t && \text{The Euclidean Line} \\ &(\mathbf{b}-\mathbf{a})t+\mathbf{a} && \text{The Euclidean Line} \end{aligned} \tag{6.46}$$

calling it the line representation by the two points, **a** and **b**, that it contains.

The first line in (6.46) is the unique Euclidean line that passes through the points **a** and **b**. Considering the line parameter t as "time", the line passes through the point **a** at time $t = 0$, and owing to a *left cancellation*, it passes through the point **b** at time $t = 1$.

Similarly, the second line in (6.46) is the unique Euclidean line that passes through the points **a** and **b**. It passes through the point **a** at time $t = 0$, and owing to a *right cancellation*, it passes through the point **b** at time $t = 1$. In vector spaces, of course, left cancellations and right cancellations coincide.

In full analogy with (6.46) , in order to emphasize that the hyperbolic lines, gyrolines, are uniquely determined by any two distinct points that they contain, one replaces the expressions in (6.45) by

$$\begin{aligned} &\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t && \text{Gyroline, The Hyperbolic Line} \\ &(\mathbf{b}\boxminus\mathbf{a})\otimes t\oplus\mathbf{a} && \text{Cogyroline, The Hyperbolic Dual Line} \end{aligned} \tag{6.47}$$

calling them, respectively, the gyroline and the cogyroline representation by the two points, **a** and **b**, that each of them contains.

The gyroline in (6.47) is the unique gyroline that passes through the points **a** and **b**. It passes through the point **a** at time $t = 0$, and owing to a left cancellation, it passes through the point **b** at time $t = 1$.

Similarly, the cogyroline in (6.47) is the unique cogyroline that passes through the points **a** and **b**. It passes through the point **a** at time $t = 0$, and owing to a right cancellation, it passes through the point **b** at time $t = 1$. Unlike left cancellations and right cancellations in vector spaces, where they coincide, left cancellations and right cancellations in gyrovector spaces are distinct, forcing us to employ the cooperation $\boxplus$, rather than the operation $\oplus$, in the second expression of (6.47). It is the presence of the cooperation in the second expression in (6.47) that allows a right cancellation, (2.39), when $t = 1$. Hence, the replacement of $\ominus\mathbf{a}\oplus\mathbf{b}$ in the first equation in (6.47) by $\mathbf{b}\boxminus\mathbf{a} = \boxminus\mathbf{a}\boxplus\mathbf{b}$ in the second equation in (6.47) is a matter of necessity rather than choice.

Definition 6.17 (Origin-Intercept Gyrolines and Cogyrolines). *A gyroline (cogyroline) that passes through the origin of its gyrovector space is called an origin-intercept gyroline (cogyroline).*

Theorem 6.18 *An origin-intercept gyroline (cogyroline) is a cogyroline (gyroline).*

Proof. Let

$$L = \mathbf{a}\oplus\mathbf{b}\otimes t \tag{6.48}$$

$\mathbf{a},\mathbf{b} \in G$, $t \in \mathbb{R}$, be an origin-intercept gyroline in a gyrovector space $(G,\oplus,\otimes)$. Then, there exists $t_0\in\mathbb{R}$ such that

$$\mathbf{a}\oplus\mathbf{b}\otimes t_0 = \mathbf{0} \tag{6.49}$$

so that

$$\mathbf{a} = \ominus\mathbf{b}\otimes t_0 \tag{6.50}$$

and hence,

$$\begin{aligned} L &= \mathbf{a}\oplus\mathbf{b}\otimes t \\ &= \ominus\mathbf{b}\otimes t_0\oplus\mathbf{b}\otimes t \\ &= \mathbf{b}\otimes(-t_0 + t) \\ &= \mathbf{b}\otimes s \\ &= \mathbf{b}\otimes s\oplus\mathbf{0} \end{aligned} \tag{6.51}$$

$s\in\mathbb{R}$. The gyroline L is recognized in (6.51) as a cogyroline.

Similarly, let

$$L^c = \mathbf{b}\otimes t\oplus\mathbf{a} \tag{6.52}$$

$\mathbf{a},\mathbf{b}\in G$, $t\in\mathbb{R}$, be an origin-intercept cogyroline in the gyrovector space $(G,\oplus,\otimes)$. Then, there exists $t_0\in\mathbb{R}$ such that

$$\mathbf{b}\otimes t_0\oplus\mathbf{a} = \mathbf{0} \tag{6.53}$$

so that

$$\mathbf{a} = \ominus\mathbf{b}\otimes t_0 \tag{6.54}$$

and hence,

$$\begin{aligned} L^c &= \mathbf{b}\otimes t\oplus\mathbf{a} \\ &= \mathbf{b}\otimes t\ominus\mathbf{b}\otimes t_0 \\ &= \mathbf{b}\otimes(t-t_0) \\ &= \mathbf{b}\otimes s \\ &= \mathbf{0}\oplus\mathbf{b}\otimes s \end{aligned} \tag{6.55}$$

$s\in\mathbb{R}$. The cogyroline L^c is recognized in (6.55) as a gyroline. □

The definition of the gyroline and its associated cogyroline in (6.46) will be presented formally in Secs. 6.4 and 6.9.

6.4 Gyrolines

Definition 6.19 (Gyrolines, Gyrosegments). *Let* $\mathbf{a}, \mathbf{b}$ *be any two distinct points in a gyrovector space* $(G, \oplus, \otimes)$. *The gyroline in* G *that passes through the points* $\mathbf{a}$ *and* $\mathbf{b}$ *is the set of all points*

$$L^g = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.56}$$

in G, $t\in\mathbb{R}$. *The gyrovector space expression in* (6.56) *is called the representation of the gyroline* L^g *in terms of the two points* $\mathbf{a}$ *and* $\mathbf{b}$ *that it contains.*

A gyroline segment (or, a gyrosegment) $\mathbf{ab}$ *with endpoints* $\mathbf{a}$ *and* $\mathbf{b}$ *is the set of all points in* (6.56) *with* $0 \le t \le 1$. *The gyrolength* $|\mathbf{ab}|$ *of the gyrosegment* $\mathbf{ab}$ *is the gyrodistance between* $\mathbf{a}$ *and* $\mathbf{b}$,

$$|\mathbf{ab}| = d_\oplus(\mathbf{a}, \mathbf{b}) = \|\ominus\mathbf{a}\oplus\mathbf{b}\| \tag{6.57}$$

Two gyrosegments are congruent if they have the same gyrolength.

Considering the real parameter t as "time", the gyroline (6.56) passes through the point $\mathbf{a}$ at time $t = 0$ and, owing to the left cancellation law, it passes thought the point $\mathbf{b}$ at time $t = 1$.

It is anticipated in Def. 6.19 that the gyroline is uniquely represented by any two given points that it contains. The following theorem shows that this is indeed the case.

Theorem 6.20 *Two gyrolines that share two distinct points are coincident.*

Proof. Let

$$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.58}$$

be a gyroline that contains two given distinct points $\mathbf{p}_1$ and $\mathbf{p}_2$ in a gyrovector space $(G, \oplus, \otimes)$. Then, there exist real numbers $t_1, t_2 \in \mathbb{R}$, $t_1 \neq t_2$, such that

$$\begin{aligned}\mathbf{p}_1 &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\\ \mathbf{p}_2 &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_2\end{aligned} \tag{6.59}$$

A gyroline containing the points $\mathbf{p}_1$ and $\mathbf{p}_2$ has the form

$$\mathbf{p}_1\oplus(\ominus\mathbf{p}_1\oplus\mathbf{p}_2)\otimes t \tag{6.60}$$

which, by means of (6.59) is reducible to (6.58) with a reparametrization. Indeed, by (6.59), the Gyrotranslation Theorem 3.13, scalar distributivity and associativity, and left gyroassociativity, we have

$$\begin{aligned}&\mathbf{p}_1\oplus(\ominus\mathbf{p}_1\oplus\mathbf{p}_2)\otimes t\\ &= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\{\ominus[\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus[\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_2]\}\otimes t\\ &= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\mathrm{gyr}[\mathbf{a}, (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\{\ominus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_2\}\otimes t\\ &= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\mathrm{gyr}[\mathbf{a}, (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\{(\ominus\mathbf{a}\oplus\mathbf{b})\otimes(-t_1+t_2)\}\otimes t\\ &= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\mathrm{gyr}[\mathbf{a}, (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1](\ominus\mathbf{a}\oplus\mathbf{b})\otimes((-t_1+t_2)t)\\ &= \mathbf{a}\oplus\{(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes((-t_1+t_2)t)\}\\ &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes(t_1+(-t_1+t_2)t)\end{aligned} \tag{6.61}$$

thus obtaining the gyroline (6.58) with a reparametrization. It is a reparametrization in which the original gyroline parameter t is replaced by the new gyroline parameter $t_1 + (-t_1 + t_2)t$, $t_2 - t_1 \neq 0$.

Hence, any gyroline (6.58) that contains the two distinct points $\mathbf{p}_1$ and $\mathbf{p}_2$ coincides with the gyroline (6.60). □

Theorem 6.21 *A left gyrotranslation of a gyroline is, again, a gyroline.*

Proof. Let

$$L = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.62}$$

be a gyroline L represented by its two points $\mathbf{a}$ and $\mathbf{b}$ in a gyrovector space $(G, \oplus, \otimes)$. The left gyrotranslation, $\mathbf{x}\oplus L$ of the gyroline L is given by the equation

$$\mathbf{x}\oplus L = \mathbf{x}\oplus\{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\} \tag{6.63}$$

which can be recast in the form of a gyroline by employing the left gyroassociative law, Axiom $(V5)$ of gyrovector spaces, and Theorem 3.13,

$$\begin{aligned} \mathbf{x}\oplus L &= \mathbf{x}\oplus\{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\} \\ &= (\mathbf{x}\oplus\mathbf{a})\oplus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\{(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\} \\ &= (\mathbf{x}\oplus\mathbf{a})\oplus\{\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})\}\otimes t \\ &= (\mathbf{x}\oplus\mathbf{a})\oplus\{(\ominus\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{b})\}\otimes t \end{aligned} \tag{6.64}$$

thus obtaining a gyroline representation, (6.45), (6.47), for the left gyrotranslated gyroline, $\mathbf{x}\oplus L$. □

Definition 6.22 **(Gyrocollinearity).** *Three points, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, in a gyrovector space $(G, \oplus, \otimes)$ are gyrocollinear if they lie on the same gyroline, that is, there exist $\mathbf{a}, \mathbf{b} \in G$ such that*

$$\mathbf{a}_k = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_k \tag{6.65}$$

for some $t_k \in \mathbb{R}$, $k = 1, 2, 3$. Similarly, n points in G, $n > 3$, are gyrocollinear if any three of these points are gyrocollinear.

We should note here that we will use the similar term "cogyroline" for a "dual gyroline", Hence, to avoid a conflict with the term "cogyroline" we use here the term "gyrocollinear" rather than the seemingly more appropriate term "cogyrolinear".

Definition 6.23 **(Betweenness).** *A point $\mathbf{a}_2$ lies between the points $\mathbf{a}_1$ and $\mathbf{a}_3$ in a gyrovector space $(G, \oplus, \otimes)$ (i) if the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are gyrocollinear, that is, they are related by the equations*

$$\mathbf{a}_k = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_k \tag{6.66}$$

$k = 1, 2, 3$, for some $\mathbf{a}, \mathbf{b} \in G$, $\mathbf{a} \neq \mathbf{b}$, and some $t_k \in \mathbb{R}$, and (ii) if, in addition, either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$.

Lemma 6.24 *Three distinct points, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ in a gyrovector space $(G, \oplus, \otimes)$ are gyrocollinear if and only if any one of these points, say $\mathbf{a}_2$, can be expressed in terms of the two other points by the equation*

$$\mathbf{a}_2 = \mathbf{a}_1\oplus(\ominus\mathbf{a}_1\oplus\mathbf{a}_3)\otimes t \tag{6.67}$$

for some $t \in \mathbb{R}$.

Proof. If the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are gyrocollinear, then there exist points $\mathbf{a}, \mathbf{b} \in G$ and distinct real number t_k such that

$$\mathbf{a}_k = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_k \tag{6.68}$$

$k = 1, 2, 3$.

Let

$$t = \frac{t_2 - t_1}{t_3 - t_1} \tag{6.69}$$

Then, by the Gyrotranslation Theorem 3.13, the scalar distributive and associative law, and gyroassociativity, we have the chain of equations

$$\begin{aligned}
&\mathbf{a}_1\oplus(\ominus\mathbf{a}_1\oplus\mathbf{a}_3)\otimes t \\
&= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\{\ominus[\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus[\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_3]\}\otimes t \\
&= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\mathrm{gyr}[\mathbf{a}, (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\{\ominus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_3\}\otimes t \\
&= [\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1]\oplus\mathrm{gyr}[\mathbf{a}, (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1](\ominus\mathbf{a}\oplus\mathbf{b})\otimes((-t_1 + t_3)t) \\
&= \mathbf{a}\oplus\{(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes((-t_1 + t_3)t)\} \\
&= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes(t_1 + (-t_1 + t_3)t) \\
&= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_2 \\
&= \mathbf{a}_2
\end{aligned} \tag{6.70}$$

thus verifying (6.67).

Conversely, if (6.67) holds then the three points $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are gyrocollinear, the point $\mathbf{a}_2$ lying on the gyroline passing through the other two points, $\mathbf{a}_1$ and $\mathbf{a}_3$. □

Lemma 6.25 *A point $\mathbf{a}_2$ lies between the points $\mathbf{a}_1$ and $\mathbf{a}_3$ in a gyrovector space $(G, \oplus, \otimes)$ if and only if*

$$\mathbf{a}_2 = \mathbf{a}_1\oplus(\ominus\mathbf{a}_1\oplus\mathbf{a}_3)\otimes t \tag{6.71}$$

for some $0 < t < 1$.

Proof. If $\mathbf{a}_2$ lies between $\mathbf{a}_1$ and $\mathbf{a}_3$, the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are gyrocollinear by Def. 6.23, and there exist distinct points $\mathbf{a}, \mathbf{b} \in G$ and real numbers t_k such that

$$\mathbf{a}_k = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_k \tag{6.72}$$

$k = 1, 2, 3$ and either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$.

Let

$$t = \frac{t_2 - t_1}{t_3 - t_1} \tag{6.73}$$

Then $0 < t < 1$ and following the chain of equations (6.70), we derive the desired identity

$$\mathbf{a}_1 \oplus (\ominus \mathbf{a}_1 \oplus \mathbf{a}_3) \otimes t = \mathbf{a}_2 \tag{6.74}$$

thus verifying (6.71) for $0 < t < 1$.

Conversely, if (6.71) holds then, by Def. 6.23 with $t_1 = 0$, $t_2 = t$ and $t_3 = 1$, $\mathbf{a}_2$ lies between $\mathbf{a}_1$ and $\mathbf{a}_3$. □

Lemma 6.26 *The two equations*

$$\mathbf{b} = \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{c}) \otimes t \tag{6.75}$$

and

$$\mathbf{b} = \mathbf{c} \oplus (\ominus \mathbf{c} \oplus \mathbf{a}) \otimes (1 - t) \tag{6.76}$$

are equivalent for the parameter $t \in \mathbb{R}$ *and all points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$.

Proof. Let us assume the validity of (6.75). Then, by the scalar distributive law, gyroassociativity, left cancellation and (2.101), gyrocommutativity, and the gyroautomorphic inverse property we have the chain of equations

$$\begin{aligned}
\mathbf{c} \oplus (\ominus \mathbf{c} \oplus \mathbf{a}) \otimes (1 - t) &= \mathbf{c} \oplus \{(\ominus \mathbf{c} \oplus \mathbf{a}) \ominus (\ominus \mathbf{c} \oplus \mathbf{a}) \otimes t\} \\
&= \{\mathbf{c} \oplus (\ominus \mathbf{c} \oplus \mathbf{a})\} \ominus \mathrm{gyr}[\mathbf{c}, \ominus \mathbf{c} \oplus \mathbf{a}](\ominus \mathbf{c} \oplus \mathbf{a}) \otimes t \\
&= \mathbf{a} \ominus \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{c}](\ominus \mathbf{c} \oplus \mathbf{a}) \otimes t \\
&= \mathbf{a} \ominus (\mathbf{a} \ominus \mathbf{c}) \otimes t \\
&= \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{c}) \otimes t \\
&= \mathbf{b}
\end{aligned} \tag{6.77}$$

thus implying (6.76). Similarly, (6.76) implies (6.75). □

Lemma 6.26 suggests the following

Definition 6.27 **(Directed Gyrolines).** *Let*

$$L = \mathbf{a} \oplus \mathbf{b} \otimes t \tag{6.78}$$

be a gyroline with a parameter $t \in \mathbb{R}$ in a gyrovector space $(G, \oplus, \otimes)$, and let p_1 and p_2 be two distinct points on L,

$$\begin{aligned} p_1 &= \mathbf{a}\oplus\mathbf{b}\otimes t_1 \\ p_2 &= \mathbf{a}\oplus\mathbf{b}\otimes t_2 \end{aligned} \tag{6.79}$$

$\mathbf{a}, \mathbf{b} \in G$, $t_1, t_2 \in \mathbb{R}$. *The gyroline L is directed from p_1 to p_2 if $t_1 < t_2$.*

As an example, the gyroline in (6.75) has the gyroline parameter t and it is directed from $\mathbf{a}$ (where $t = 0$) to $\mathbf{c}$ (where $t = 1$). Similarly, the gyroline in (6.76) has the gyroline parameter $s = 1 - t$, and it is directed from $\mathbf{c}$ (where $s = 0$) to $\mathbf{a}$ (where $s = 1$).

The next lemma relates gyrocollinearity to gyrations. A similar result for cogyrocollinearity will be presented in Lemma 6.61.

Lemma 6.28 *If the three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in a gyrovector space $(G, \oplus, \otimes)$ are gyrocollinear then*

$$\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b}, \ominus\mathbf{c}] = \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{c}] \tag{6.80}$$

Proof. By Lemma 6.24 and a left cancellation

$$\ominus\mathbf{a}\oplus\mathbf{b} = (\ominus\mathbf{a}\oplus\mathbf{c})\otimes t \tag{6.81}$$

for some $t \in \mathbb{R}$. By Identity (3.34), the gyroautomorphic inverse property, Eq. (6.81), and Axiom $(V6)$ of gyrovector spaces, we have the chain of equations

$$\begin{aligned} \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b}, \ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{a}] &= \mathrm{gyr}[\ominus\mathbf{a}\oplus\mathbf{b}, \ominus(\ominus\mathbf{a}\oplus\mathbf{c})] \\ &= \mathrm{gyr}[(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t, \ominus(\ominus\mathbf{a}\oplus\mathbf{c})] \\ &= I \end{aligned} \tag{6.82}$$

from which we obtain (6.80) by gyroautomorphism inversion. □

The converse of Lemma 6.28 is not valid, a counterexample being vector spaces. Any vector space is a gyrovector space in which all the gyrations are trivial. Hence, Identity (6.80) holds in vector spaces for any three points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ while not every three points of a vector space are collinear.

The obvious extension of Lemma 6.28 to any number of gyrocollinear points results in the following

Theorem 6.29 **(The Gyroline Gyration Transitive Law).** *Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be a set of n gyrocollinear points in a gyrovector space $(G, \oplus, \otimes)$. Then*

$$\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_3] \cdots \mathrm{gyr}[\mathbf{a}_{n-1}, \ominus\mathbf{a}_n] = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_n] \tag{6.83}$$

Proof. By Lemma 6.28, Identity (6.83) of the theorem holds for $n = 3$. Let us assume, by induction, that Identity (6.83) is valid for some $n = k \geq 3$. Then, Identity (6.83) is valid for $n = k + 1$ as well,

$$\begin{aligned} &\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2] \ldots \mathrm{gyr}[\mathbf{a}_{k-1}, \ominus\mathbf{a}_k]\mathrm{gyr}[\mathbf{a}_k, \ominus\mathbf{a}_{k+1}] \\ &\qquad = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_k]\mathrm{gyr}[\mathbf{a}_k, \ominus\mathbf{a}_{k+1}] \\ &\qquad = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_{k+1}] \end{aligned} \tag{6.84}$$

Hence, Identity (6.83) is valid for all $n \geq 3$. □

Remark 6.30 *Gyrocollinearity is sufficient but not necessary for the validity of* (6.83), *a counterexample being Theorem 3.6.*

6.5 Gyromidpoints

The value $t = 1/2$ in Lemma 6.26 gives rise to a special point where the two parameters of the gyroline $\mathbf{b}$, t and $(1 - t)$ coincide. It suggests the following

Definition 6.31 **(Gyromidpoints, II).** *The gyromidpoint $\mathbf{p}_{\mathbf{ac}}^m$ of any two distinct points $\mathbf{a}$ and $\mathbf{c}$ in a gyrovector space $(G, \oplus, \otimes)$ is given by the equation*

$$\mathbf{p}_{\mathbf{ac}}^m = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes\tfrac{1}{2} \tag{6.85}$$

Theorem 6.32 *Let $\mathbf{a}$ and $\mathbf{c}$ be any two points of a gyrovector space $(G, \oplus, \otimes)$. Then,*

$$\mathbf{p}_{\mathbf{ac}}^m = \mathbf{p}_{\mathbf{ca}}^m \tag{6.86}$$

and

$$\|\mathbf{a}\ominus\mathbf{p}_{\mathbf{ac}}^m\| = \|\mathbf{c}\ominus\mathbf{p}_{\mathbf{ac}}^m\| \tag{6.87}$$

Proof. By Lemma 6.26, with $t = 1/2$, the two equations

$$\begin{aligned} \mathbf{b} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes\tfrac{1}{2} = \mathbf{p}_{\mathbf{ac}}^m \\ \mathbf{b} &= \mathbf{c}\oplus(\ominus\mathbf{c}\oplus\mathbf{a})\otimes\tfrac{1}{2} = \mathbf{p}_{\mathbf{ca}}^m \end{aligned} \tag{6.88}$$

are equivalent, thus verifying (6.86).

It follows from (6.88) by left cancellations and the gyrocommutative law that

$$\begin{aligned}\ominus\mathbf{a}\oplus\mathbf{p}^m_{\mathbf{ac}} &= (\ominus\mathbf{a}\oplus\mathbf{c})\otimes\tfrac{1}{2}\\ \ominus\mathbf{c}\oplus\mathbf{p}^m_{\mathbf{ca}} &= (\ominus\mathbf{c}\oplus\mathbf{a})\otimes\tfrac{1}{2} = \ominus\mathrm{gyr}[\ominus\mathbf{c},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})\otimes\tfrac{1}{2}\end{aligned}\tag{6.89}$$

implying

$$\begin{aligned}\|\ominus\mathbf{a}\oplus\mathbf{p}^m_{\mathbf{ac}}\| &= \|\ominus\mathbf{a}\oplus\mathbf{c}\|\otimes\tfrac{1}{2}\\ \|\ominus\mathbf{c}\oplus\mathbf{p}^m_{\mathbf{ca}}\| &= \|\ominus\mathbf{a}\oplus\mathbf{c}\|\otimes\tfrac{1}{2}\end{aligned}\tag{6.90}$$

thus verifying (6.87). □

Clearly, Identities (6.86) and (6.87) justify calling $\mathbf{p}^m_{\mathbf{ac}}$ the gyromidpoint of the points $\mathbf{a}$ and $\mathbf{c}$ in Def. 6.31.

Theorem 6.33 *The gyromidpoint of points* $\mathbf{a}$ *and* $\mathbf{b}$ *can be written as*

$$\mathbf{p}^m_{\mathbf{ab}} = \tfrac{1}{2}\otimes(\mathbf{a}\boxplus\mathbf{b})\tag{6.91}$$

so that

$$\|\mathbf{p}^m_{\mathbf{ab}}\| = \tfrac{1}{2}\otimes\|\mathbf{a}\boxplus\mathbf{b}\|\tag{6.92}$$

Proof. By Def. 6.31, the Two-Sum Identity in Theorem 6.7, the scalar associative law, left gyroassociativity, a left cancellation and Theorem 2.30, we have

$$\begin{aligned}2\otimes\mathbf{p}^m_{\mathbf{ab}} &= 2\otimes\{\mathbf{a}\oplus\tfrac{1}{2}\otimes(\ominus\mathbf{a}\oplus\mathbf{b})\}\\ &= \mathbf{a}\oplus\{(\ominus\mathbf{a}\oplus\mathbf{b})\oplus\mathbf{a}\}\\ &= \{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\}\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{a}\oplus\mathbf{b}]\mathbf{a}\\ &= \mathbf{b}\oplus\mathrm{gyr}[\mathbf{b},\ominus\mathbf{a}]\mathbf{a}\\ &= \mathbf{b}\boxplus\mathbf{a}\\ &= \mathbf{a}\boxplus\mathbf{b}\end{aligned}\tag{6.93}$$

implying

$$\mathbf{p}^m_{\mathbf{ab}} = \tfrac{1}{2}\otimes(\mathbf{a}\boxplus\mathbf{b})\tag{6.94}$$

□

Since, by Theorem 3.4, the gyrogroup cooperation $\boxplus$ in a gyrocommutative gyrogroup is commutative, we have

$$\mathbf{p}^m_{\mathbf{ab}} = \mathbf{p}^m_{\mathbf{ba}}\tag{6.95}$$

as expected from Theorem 6.32.

The gyromidpoint in (6.94) shares an obvious analogy with its classical counterpart. We thus see again that in order to capture analogies with classical results, both gyrogroup operations and cooperations must be employed.

Theorem 6.33 shows that, in the context of gyrogroups, Defs. 6.31 and 3.37 are equivalent.

An interesting related duality symmetry is uncovered in the following

Theorem 6.34 *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be any two points of a gyrovector space* $(G,\oplus,\otimes)$. *Then,*

$$\begin{aligned} \mathbf{a}\oplus\tfrac{1}{2}(\ominus\mathbf{a}\oplus\mathbf{b}) &= \tfrac{1}{2}(\mathbf{a}\boxplus\mathbf{b}) \\ \mathbf{a}\oplus\tfrac{1}{2}(\boxminus\mathbf{a}\boxplus\mathbf{b}) &= \tfrac{1}{2}(\mathbf{a}\oplus\mathbf{b}) \end{aligned} \tag{6.96}$$

Proof. The first identity in (6.96) is the result of Theorem 6.33. The proof of the second identity in (6.96) follows.

By the Two-Sum Identity in Theorem 6.7, the scalar associative law, the commutativity of the cooperation and a right cancellation we have

$$\begin{aligned} 2\otimes(\mathbf{a}\oplus\tfrac{1}{2}\otimes(\boxminus\mathbf{a}\boxplus\mathbf{b})) &= \mathbf{a}\oplus\{(\boxminus\mathbf{a}\boxplus\mathbf{b})\oplus\mathbf{a}\} \\ &= \mathbf{a}\oplus\{(\mathbf{b}\boxminus\mathbf{a})\oplus\mathbf{a}\} \\ &= \mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{6.97}$$

□

6.6 Gyrocovariance

Definition 6.35 **(Gyrocovariance, Gyrovector Space Objects).** *A map*

$$T : G^n \;\rightarrow\; G \tag{6.98}$$

from n *copies,* G^n, *of a gyrovector space* $G=(G,\oplus,\otimes)$ *into the gyrovector space* G *is a rule which assigns to each* n *points* $\mathbf{a}_1,\ldots,\mathbf{a}_n\in G$ *a new point* $T(\mathbf{a}_1,\ldots,\mathbf{a}_n)\in G$, *called the image of the points* $\mathbf{a}_1,\ldots,\mathbf{a}_n$. *The map* T *is gyrocovariant (with respect to the motions of the gyrovector space) if its image co-varies (that is, varies together) with its preimage points* $\mathbf{a}_1,\ldots,\mathbf{a}_n$ *under the gyrovector space motions, that is, if*

$$\begin{aligned} \tau T(\mathbf{a}_1,\ldots,\mathbf{a}_n) &= T(\tau\mathbf{a}_1,\ldots,\tau\mathbf{a}_n) \\ \mathbf{x}\oplus T(\mathbf{a}_1,\ldots,\mathbf{a}_n) &= T(\mathbf{x}\oplus\mathbf{a}_1,\ldots,\mathbf{x}\oplus\mathbf{a}_n) \end{aligned} \tag{6.99}$$

for all $\tau \in Aut(G, \oplus, \otimes)$ *and all* $\mathbf{x} \in G$.

Furthermore, let T_k: $G^n \to G$, $k = 1, \dots, m$, *be* m *gyrocovariant maps. The set of* $n + m$ *elements*

$$S = \{a_1, \dots, a_n,\ T_1(a_1, \dots, a_n), \dots,\ T_m(a_1, \dots, a_n)\} \tag{6.100}$$

in G *is called a gyrovector space object in* G.

Theorem 6.36 *The gyromidpoint map* T: $G^2 \to G$ *that takes two points of a gyrovector space* $(G, \oplus, \otimes)$ *into their gyromidpoint,*

$$T(\mathbf{a}, \mathbf{b}) = \tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b}) \tag{6.101}$$

is gyrocovariant.

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be any two points of a gyrovector space $(G, \oplus, \otimes)$. We have to establish the identities

$$\begin{aligned} \tau\{\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b})\} &= \tfrac{1}{2}\otimes(\tau\mathbf{a} \boxplus \tau\mathbf{b}) \\ \mathbf{x}\oplus\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b}) &= \tfrac{1}{2}\otimes\{(\mathbf{x}\oplus\mathbf{a}) \boxplus (\mathbf{x}\oplus\mathbf{b})\} \end{aligned} \tag{6.102}$$

for all $\tau \in Aut(G, \oplus, \otimes)$ and all $\mathbf{x} \in G$.

The first identity in (6.102) follows immediately from Def. 6.5 of a gyrovector space automorphism and from (2.52). To verify the second identity in (6.102) we note that

$$\begin{aligned} 2\otimes\{\mathbf{x}\oplus\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b})\} &= \mathbf{x}\oplus\{(\mathbf{a} \boxplus \mathbf{b})\}\oplus\mathbf{x}\} \\ &= (\mathbf{x}\oplus\mathbf{a}) \boxplus (\mathbf{x}\oplus\mathbf{b}) \end{aligned} \tag{6.103}$$

by the Two-Sum Identity in Theorem 6.7 and Identity (3.66). Gyromultiplying the extreme sides of (6.103) by $\frac{1}{2}$ gives the second identity in (6.102),

$$\begin{aligned} \tfrac{1}{2}\otimes\{(\mathbf{x}\oplus\mathbf{a}) \boxplus (\mathbf{x}\oplus\mathbf{b})\} &= \tfrac{1}{2}\otimes[2\otimes\{\mathbf{x}\oplus\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b})\}] \\ &= (\tfrac{1}{2}2)\otimes\{\mathbf{x}\oplus\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b})\} \\ &= \mathbf{x}\oplus\tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{b}) \end{aligned} \tag{6.104}$$

□

It follows from Theorem 6.36 that the set $\{\mathbf{a}, \frac{1}{2}(\mathbf{a} \boxplus \mathbf{b}), \mathbf{b}\}$ of any two points and their gyromidpoint in a gyrovector space $(G, \oplus, \otimes)$ is a gyrovector space object in G. As such, it can be moved in G by the motions of G without destroying its internal structure as a set of two points with their gyromidpoint.

Theorem 6.37 *The points of any gyroline*

$$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.105}$$

$\mathbf{a}, \mathbf{b} \in G$, $t \in \mathbb{R}$, *in a gyrovector space* $(G, \oplus, \otimes)$ *form a gyrovector space object.*

Proof. We have to show that

$$\tau\{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\} = \tau\mathbf{a}\oplus(\ominus\tau\mathbf{a}\oplus\tau\mathbf{b})\otimes t \tag{6.106}$$

and

$$\mathbf{x}\oplus\{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\} = (\mathbf{x}\oplus\mathbf{a})\oplus\{(\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{b})\}\otimes t \tag{6.107}$$

for all automorphisms $\tau \in Aut(G, \oplus, \otimes)$, and all $\mathbf{a}, \mathbf{b}, \mathbf{x} \in G$, and $t \in \mathbb{R}$.

Identity (6.106) follows straightforwardly from the definition of gyrovector space automorphisms in Def. 6.5. Identity (6.107) follows from the chain of equations

$$\begin{aligned}
(\mathbf{x}\oplus\mathbf{a})\oplus\{\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{b})\}\otimes t &= (\mathbf{x}\oplus\mathbf{a})\oplus\{\mathrm{gyr}[\mathbf{x}, \mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})\}\otimes t \\
&= (\mathbf{x}\oplus\mathbf{a})\oplus\mathrm{gyr}[\mathbf{x}, \mathbf{a}][(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t] \\
&= \mathbf{x}\oplus\{\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \mathbf{x}]\mathrm{gyr}[\mathbf{x}, \mathbf{a}][(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t]\} \\
&= \mathbf{x}\oplus\{\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t\}
\end{aligned} \tag{6.108}$$

in which we employ Theorem 3.13, Axiom $(V5)$ of gyrovector spaces, the right gyroassociative law, and gyration inversion (see Theorem 6.21). $\square$

6.7 Gyroparallelograms

Theorem 6.38 *Let* $(G, \oplus, \otimes)$ *be a gyrovector space and let* $T: G^3 \to G$ *be a map given by the equation*

$$T(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a} \tag{6.109}$$

Then the map T *is gyrocovariant.*

Proof. We have to verify the identities

$$\begin{aligned}
\tau\{(\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a}\} &= (\tau\mathbf{b} \boxplus \tau\mathbf{c})\ominus\tau\mathbf{a} \\
\mathbf{x}\oplus\{(\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a}\} &= \{(\mathbf{x}\oplus\mathbf{b}) \boxplus (\mathbf{x}\oplus\mathbf{c})\}\ominus(\mathbf{x}\oplus\mathbf{a})
\end{aligned} \tag{6.110}$$

for all $\tau \in Aut(G, \oplus, \otimes)$ and all $\mathbf{x} \in G$.

Any gyrovector space automorphism τ of a gyrovector space $(G, \oplus, \otimes)$, $\tau \in Aut(G, \oplus, \otimes)$ is, in particular, an automorphism of the corresponding gyrogroup, $\tau \in Aut(G, \oplus)$. Hence, by Theorem 2.21, τ is an automorphism of the groupoid $(G, \boxplus)$ as well, $\tau \in Aut(G, \boxplus)$. Hence, τ satisfies the first identity in (6.110). The validity of the second identity in (6.110) follows from Theorem 3.19. $\square$

It follows from Theorem 6.38 that the ordered set of four points

$$S = \{\mathbf{a}, \mathbf{b}, \mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a}, \mathbf{c}\} \tag{6.111}$$

in a gyrovector space $(G, \oplus, \otimes)$ is a gyrovector space object, so that it can be moved by the motions of its gyrovector space while keeping its internal structure intact.

The next theorem will enable us to recognize the gyrovector space object S in (6.111) as the *gyroparallelogram*, the analogue of the Euclidean parallelogram in vector spaces.

Theorem 6.39 *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be any three points of a gyrovector space* $(G, \oplus, \otimes)$ *and let* $\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a}$. *Then, the gyromidpoint* $\mathbf{p}^m_{\mathbf{a},\mathbf{d}}$ *of the points* $\mathbf{a}$ *and* $\mathbf{d}$ *coincides with the gyromidpoint* $\mathbf{p}^m_{\mathbf{b},\mathbf{c}}$ *of the points* $\mathbf{b}$ *and* $\mathbf{c}$.

Proof. By Theorem 6.33 and a right cancellation we have

$$\begin{aligned} 2\otimes\mathbf{p}^m_{\mathbf{b},\mathbf{c}} &= \mathbf{b} \boxplus \mathbf{c} \\ 2\otimes\mathbf{p}^m_{\mathbf{a},\mathbf{d}} &= \mathbf{a} \boxplus \mathbf{d} = \mathbf{d} \boxplus \mathbf{a} = \{(\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a}\} \boxplus \mathbf{a} = \mathbf{b} \boxplus \mathbf{c} \end{aligned} \tag{6.112}$$

so that $\mathbf{p}^m_{\mathbf{b},\mathbf{c}} = \mathbf{p}^m_{\mathbf{a},\mathbf{d}}$. $\square$

By Theorems 6.38 and 6.39, the ordered set of four points

$$S = \{\mathbf{a}, \mathbf{b}, \mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a}, \mathbf{c}\} \tag{6.113}$$

in a gyrovector space $(G, \oplus, \otimes)$ forms an object of four points that, in turn, form two pairs, $(\mathbf{a}, \mathbf{d})$ and $(\mathbf{b}, \mathbf{c})$, that share their gyromidpoints. By analogy with vector spaces, we recognize the ordered set S in (6.113) as the four vertices of a gyroparallelogram, ordered clockwise or counterclockwise, the two diagonals $\mathbf{ad}$ and $\mathbf{bc}$ of which intersect at their gyromidpoints. Hence, Theorems 6.38 and 6.39 suggest the following

Definition 6.40 **(Gyroparallelograms).** *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be any three points in a gyrovector space* $(G, \oplus, \otimes)$. *Then, the four points* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *in* G *are the vertices of the gyroparallelogram* $\mathbf{abdc}$, *ordered either clockwise of counterclockwise, Fig. 8.19, p. 290, if they satisfy the gyroparallelogram condition*

$$\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a} \tag{6.114}$$

The gyroparallelogram is degenerate if the three points $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are gyrocollinear.*

If the gyroparallelogram $\mathbf{abdc}$ *is non-degenerate, then the two vertices in each of the pairs* $(\mathbf{a}, \mathbf{d})$ *and* $(\mathbf{b}, \mathbf{c})$ *are said to be opposite to one another. The gyrosegments of adjacent vertices,* $\mathbf{ab}$, $\mathbf{bd}$, $\mathbf{dc}$ *and* $\mathbf{ca}$ *are the sides of the gyroparallelogram. The gyrosegments* $\mathbf{ad}$ *and* $\mathbf{bc}$ *of opposite vertices in the non-degenerate gyroparallelogram* $\mathbf{abdc}$ *are the diagonals of the gyroparallelogram.*

A gyroparallelogram in the Möbius (Einstein) gyrovector plane, that is, in the Poincaré (Beltrami) disc model of hyperbolic geometry, is presented in Fig. 8.19 (in Fig. 10.7).

Theorem 6.41 (Gyroparallelogram Symmetries). *Every vertex of the gyroparallelogram* $\mathbf{abdc}$ *satisfies the gyroparallelogram condition,* (6.114), *that is,*

$$\begin{aligned} \mathbf{a} &= (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{d} \\ \mathbf{b} &= (\mathbf{a} \boxplus \mathbf{d}) \ominus \mathbf{c} \\ \mathbf{c} &= (\mathbf{a} \boxplus \mathbf{d}) \ominus \mathbf{b} \\ \mathbf{d} &= (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a} \end{aligned} \tag{6.115}$$

Furthermore, the two diagonals of the gyroparallelogram are concurrent, the concurrency point being the gyromidpoint of each of the two diagonals.

Proof. The last equation in (6.115) is valid by Def. 6.40 of the gyroparallelogram. By the right cancellation law this equation is equivalent to the equation

$$\mathbf{a} \boxplus \mathbf{d} = \mathbf{b} \boxplus \mathbf{c} \tag{6.116}$$

Since the coaddition $\boxplus$ is commutative in gyrovector spaces, Eq. (6.116) is equivalent to each of the equations in (6.115) by the right cancellation law, thus verifying the first part of the theorem.

Equation (6.116) implies

$$\tfrac{1}{2} \otimes (\mathbf{a} \boxplus \mathbf{d}) = \tfrac{1}{2} \otimes (\mathbf{b} \boxplus \mathbf{c}) \tag{6.117}$$

By Theorem 6.33, the left (right) hand side of (6.117) is the gyromidpoint of the diagonal $\mathbf{ad}$ ($\mathbf{bc}$). Hence, the gyromidpoints of the two diagonals of the gyroparallelogram coincide, thus verifying the second part of the theorem. $\square$

Theorem 6.42 (The Gyroparallelogram (Addition) Law). *Let* **abdc** *be a gyroparallelogram in a gyrovector space* $(G, \oplus, \otimes)$. *Then*

$$(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{c}) = \ominus\mathbf{a}\oplus\mathbf{d} \tag{6.118}$$

Proof. By (3.66) and (6.114) we have

$$\begin{aligned}(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{c}) &= \ominus\mathbf{a}\oplus\{(\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a}\} \\ &= \ominus\mathbf{a}\oplus\mathbf{d}\end{aligned} \tag{6.119}$$

□

The gyroparallelogram law in the Möbius (Einstein) gyrovector plane, that is, in the Poincaré (Beltrami) disc model of hyperbolic geometry, is presented graphically in Figs. 8.21 and 8.22 (in Figs. 10.7 and 10.10).

The gyroparallelogram law (6.118) of gyrovector addition is analogous to the parallelogram law of vector addition in Euclidean geometry, and is given by the coaddition law of gyrovectors. Remarkably, in order to capture this analogy we must employ both the gyrocommutative operation $\oplus$ and the commutative cooperation $\boxplus$ of gyrovector spaces.

Theorem 6.43 *If the gyroparallelogram* **abdc** *in a gyrovector space* $(G, \oplus, \otimes)$ *is degenerate, then its four vertices are gyrocollinear.*

Proof. By Def. 6.40 the vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of the gyroparallelogram **abdc** are gyrocollinear. Hence, the vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$ lie on the gyroline

$$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.120}$$

In order to show that also the vertex $\mathbf{d}$,

$$\mathbf{d} = (\mathbf{a} \boxplus \mathbf{b})\ominus\mathbf{c} \tag{6.121}$$

lies on the gyroline (6.120), we represent the point $\mathbf{c}$ by the value $t_c \in \mathbb{R}$ of its gyroline parameter, t, on the gyroline (6.120),

$$\mathbf{c} = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_c \tag{6.122}$$

Then, employing various gyrogroup identities, we have the following chain of equations, where we use the notation $g_{a,-b} = \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]$ and $g_{-b,a} = \mathrm{gyr}[\ominus\mathbf{b}, \mathbf{a}]$. Equalities in the chain of equations are numbered for subsequent explanation.

$$
\begin{aligned}
\mathbf{d} =\!=\; & (\mathbf{a} \boxplus \mathbf{b}) \ominus \mathbf{c} \\
\overset{(1)}{=\!=\!=}\; & (\mathbf{a} \oplus g_{a,-b}\mathbf{b}) \oplus \{\ominus\mathbf{a} \ominus (\ominus\mathbf{a} \oplus \mathbf{b}) \otimes t_c\} \\
\overset{(2)}{=\!=\!=}\; & \{(\mathbf{a} \oplus g_{a,-b}\mathbf{b}) \ominus \mathbf{a}\} \ominus \mathrm{gyr}[\mathbf{a} \oplus g_{a,-b}\mathbf{b}, \ominus\mathbf{a}](\ominus\mathbf{a} \oplus \mathbf{b}) \otimes t_c \\
\overset{(3)}{=\!=\!=}\; & \mathrm{gyr}[\mathbf{a}, g_{a,-b}\mathbf{b}] g_{a,-b}\mathbf{b} \ominus \mathrm{gyr}[\mathbf{a}, g_{a,-b}\mathbf{b}] g_{a,-b}(\mathbf{b} \ominus \mathbf{a}) \otimes t_c \\
\overset{(4)}{=\!=\!=}\; & \mathrm{gyr}[\mathbf{a}, g_{a,-b}\mathbf{b}] g_{a,-b}\{\mathbf{b} \ominus (\mathbf{b} \ominus \mathbf{a}) \otimes t_c\} \\
\overset{(5)}{=\!=\!=}\; & g_{-b,a} g_{a,-b}\{\mathbf{b} \ominus (\mathbf{b} \ominus \mathbf{a}) \otimes t_c\} \\
\overset{(6)}{=\!=\!=}\; & \mathbf{b} \oplus (\ominus\mathbf{b} \oplus \mathbf{a}) \otimes t_c \\
\overset{(7)}{=\!=\!=}\; & \mathbf{a} \oplus (\ominus\mathbf{a} \oplus \mathbf{b}) \otimes (1 - t_c)
\end{aligned}
\tag{6.123}
$$

so that, as expected, the point $\mathbf{d}$ lies on the gyroline that contains the points $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, that is, the gyroline (6.120). The proof of the theorem is thus complete.

The derivation of the equalities in the chain of equations (6.123) follows.

(1) Follows from the Gyrogroup Cooperation Def. 2.7, and from (6.122).
(2) Follows from the left gyroassociative law.
(3) Follows by employing (3.28), (2.101) and the gyrocommutative law.
(4) Follows from the automorphism property of gyroautomorphisms.
(5) Follows from the nested gyroautomorphism identity (2.94).
(6) Follows by gyroautomorphism inversion, (2.93).
(7) Follows from Lemma 6.26.

It follows from (6.123) that $\mathbf{d}$ lies on the gyroline (6.120), as desired. □

Theorem 6.44 *If the gyroparallelogram* **abdc** *in a gyrovector space* $(G, \oplus, \otimes)$ *is degenerate, then its gyroparallelogram addition, (6.118), reduces to gyroaddition, that is, the coaddition* $\boxplus$ *in the gyroparallelogram addition law reduces to the gyrovector space addition* $\oplus$*.*

Proof. By the coaddition definition in Def. 2.7, we have

$$
(\ominus\mathbf{a} \oplus \mathbf{b}) \boxplus (\ominus\mathbf{a} \oplus \mathbf{c}) = (\ominus\mathbf{a} \oplus \mathbf{b}) \oplus \mathrm{gyr}[\ominus\mathbf{a} \oplus \mathbf{b}, \ominus(\ominus\mathbf{a} \oplus \mathbf{c})](\ominus\mathbf{a} \oplus \mathbf{c}) \tag{6.124}
$$

But, it follows from (6.122) that

$$\ominus\mathbf{a}\oplus\mathbf{c} = (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_c \tag{6.125}$$

Hence, by (6.125) and by Property ($V6$) of gyrovector spaces we have

$$\begin{aligned} \mathrm{gyr}[\ominus\mathbf{a}\oplus\mathbf{b}, \ominus(\ominus\mathbf{a}\oplus\mathbf{c})] &= \mathrm{gyr}[\ominus\mathbf{a}\oplus\mathbf{b}, \ominus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_c] \\ &= I \end{aligned} \tag{6.126}$$

so that the gyration in (6.124) is trivial, obtaining

$$(\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{c}) = (\ominus\mathbf{a}\oplus\mathbf{b})\oplus(\ominus\mathbf{a}\oplus\mathbf{c}) \tag{6.127}$$

as desired. □

In the next theorem we will uncover the relationship between opposite sides of the gyroparallelogram.

Theorem 6.45 *Opposite sides of a gyroparallelogram* **abdc** *in a gyrovector space* $(G, \oplus, \otimes)$ *form gyrovectors which are equal modulo gyrations (see* *Figs. 8.23, p. 294, and 10.6, p. 376,* *for illustration), that is,*

$$\begin{aligned} \ominus\mathbf{c}\oplus\mathbf{d} &= \mathrm{gyr}[\mathbf{c}, \ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b}, \ominus\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\mathbf{c}, \ominus\mathbf{b}](\mathbf{b}\ominus\mathbf{a}) \\ \ominus\mathbf{b}\oplus\mathbf{d} &= \mathrm{gyr}[\mathbf{b}, \ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c}) = \mathrm{gyr}[\mathbf{b}, \ominus\mathbf{c}](\mathbf{c}\ominus\mathbf{a}) \end{aligned} \tag{6.128}$$

and, equivalently,

$$\begin{aligned} \ominus\mathbf{c}\oplus\mathbf{d} &= \ominus\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{b}](\ominus\mathbf{b}\oplus\mathbf{a}) \\ \ominus\mathbf{c}\oplus\mathbf{a} &= \ominus\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{b}](\ominus\mathbf{b}\oplus\mathbf{d}) \end{aligned} \tag{6.129}$$

Accordingly, two opposite sides of a gyroparallelogram are congruent, having equal gyrolengths,

$$\begin{aligned} \|\ominus\mathbf{a}\oplus\mathbf{b}\| &= \|\ominus\mathbf{c}\oplus\mathbf{d}\| \\ \|\ominus\mathbf{a}\oplus\mathbf{c}\| &= \|\ominus\mathbf{b}\oplus\mathbf{d}\| \end{aligned} \tag{6.130}$$

Proof. By Theorem 2.11 we have

$$\ominus\mathbf{a}\oplus\mathbf{d} = (\ominus\mathbf{a}\oplus\mathbf{c})\oplus\mathrm{gyr}[\ominus\mathbf{a}, \mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{d}) \tag{6.131}$$

and by Theorem 6.42, noting the definition of the gyrogroup cooperation, we have

$$\begin{aligned} \ominus\mathbf{a}\oplus\mathbf{d} &= (\ominus\mathbf{a}\oplus\mathbf{c}) \boxplus (\ominus\mathbf{a}\oplus\mathbf{b}) \\ &= (\ominus\mathbf{a}\oplus\mathbf{c})\oplus\mathrm{gyr}[\ominus\mathbf{a}\oplus\mathbf{c}, \mathbf{a}\ominus\mathbf{b}](\ominus\mathbf{a}\oplus\mathbf{b}) \end{aligned} \tag{6.132}$$

Comparing (6.131) and (6.132), and employing a right cancellation we have

$$\mathrm{gyr}[\ominus\mathbf{a}\oplus\mathbf{c}, \mathbf{a}\ominus\mathbf{b}](\ominus\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\ominus\mathbf{a}, \mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{d}) \tag{6.133}$$

Identity (6.133) can be written, in terms of Identity (3.34), as

$$\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b}, \ominus\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b}) = \mathrm{gyr}[\ominus\mathbf{a}, \mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{d}) \tag{6.134}$$

which is reducible to the first identity in (6.128) by eliminating $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{c}]$ on both sides of (6.134). Similarly, interchanging $\mathbf{b}$ and $\mathbf{c}$, one can verify the second identity in (6.128).

The equivalence between (6.128) and (6.129) follows from the gyroautomorphic inverse property and a gyration inversion.

Finally, (6.130) follows from (6.128) since gyrations preserve the gyrolength. □

Theorem 6.46 (The Gyroparallelogram Gyration Transitive Law). *Let* **abdc** *be a gyroparallelogram in a gyrovector space* $(G, \oplus, \otimes)$. *Then*

$$\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b}, \ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c}, \ominus\mathbf{d}] = \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{d}] \tag{6.135}$$

Proof. The proof follows immediately from the gyroparallelogram condition (6.114) and Theorem 3.6. □

Following the introduction of the gyroangle in Chap. 8 we will uncover other analogies that the gyroparallelogram shares with its vector space counterpart, the parallelogram.

6.8 Gyrogeodesics

The following theorem gives a condition that reduces the gyrotriangle inequality (6.12) to an equality, enabling us to interpret gyrolines as gyrogeodesics.

Theorem 6.47 (The Gyrotriangle Equality). *If a point* **b** *lies between two points* **a** *and* **c** *in a gyrovector space* $(G, \oplus, \otimes)$ *then*

$$\|\ominus\mathbf{a}\oplus\mathbf{c}\| = \|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\| \tag{6.136}$$

Proof. If **b** lies between **a** and **c** then, by Lemma 6.25,

$$\mathbf{b} = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t \tag{6.137}$$

for some $0 < t < 1$, and hence, by Lemma 6.26

$$\mathbf{b} = \mathbf{c}\oplus(\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t) \tag{6.138}$$

Hence, by left cancellations, we have

$$\begin{aligned} \ominus\mathbf{a}\oplus\mathbf{b} &= (\ominus\mathbf{a}\oplus\mathbf{c})\otimes t \\ \ominus\mathbf{c}\oplus\mathbf{b} &= (\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t) \end{aligned} \tag{6.139}$$

Taking magnitudes, noting the homogeneity property $(V7)$, (6.139) gives

$$\begin{aligned} \|\ominus\mathbf{a}\oplus\mathbf{b}\| &= \|\ominus\mathbf{a}\oplus\mathbf{c}\|\otimes t \\ \|\ominus\mathbf{b}\oplus\mathbf{c}\| &= \|\ominus\mathbf{a}\oplus\mathbf{c}\|\otimes(1-t) \end{aligned} \tag{6.140}$$

so that, by the scalar distributive law $(V2)$,

$$\begin{aligned} \|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\| &= \|\ominus\mathbf{a}\oplus\mathbf{c}\|\otimes\{t+(1-t)\} \\ &= \|\ominus\mathbf{a}\oplus\mathbf{c}\| \end{aligned} \tag{6.141}$$

□

Remark 6.48 *Comparing Theorem 6.47 with Theorem 6.9 we see that point* $\mathbf{b}$ *between two given points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *(i) turn the gyrotriangle inequality into an equality, and hence (ii) minimize the gyrodistance gyrosum* $\|\ominus\mathbf{a}\oplus\mathbf{b}\|\oplus\|\ominus\mathbf{b}\oplus\mathbf{c}\|$.

Definition 6.49 (Gyrodistance Along Gyropolygonal Paths). *Let* $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ *be a gyropolygonal path from a point* $\mathbf{a}_0$ *to a point* $\mathbf{a}_n$ *in a gyrovector space* $(G, \oplus, \otimes)$*, Def. 2.13. The gyrodistance* $d_{P(\mathbf{a}_0,\ldots,\mathbf{a}_n)}$ *between the points* $\mathbf{a}_0$ *and* $\mathbf{a}_n$ *along the gyropolygonal path* $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ *is given by the equation*

$$d_{P(\mathbf{a}_0,\ldots,\mathbf{a}_n)} = \sum_{\oplus,\ k=1}^{n} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\| \tag{6.142}$$

In Def. 6.49 we use the notation

$$\sum_{\oplus,\ k=1}^{n} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\| = \|\ominus\mathbf{a}_0\oplus\mathbf{a}_1\|\oplus \quad \ldots \quad \oplus\|\ominus\mathbf{a}_{n-1}\oplus\mathbf{a}_n\| \tag{6.143}$$

noting that unlike the gyrooperation $\oplus$ between elements of G, which is gyrocommutative and gyroassociative, the gyrooperation $\oplus$ between norms of elements of G is commutative and associative, Remark 6.3.

By the gyrotriangle inequality (6.12) we have the inequality

$$\|\ominus\mathbf{a}_0\oplus\mathbf{a}_n\| \le \sum_{\oplus,\ k=1}^{n} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\| \tag{6.144}$$

for the vertices of any gyropolygonal path $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ that joins the points $\mathbf{a}_0$ and $\mathbf{a}_n$ in a gyrovector space $(G, \oplus, \otimes)$.

The gyropolygonal path inequality (6.144) reduces to an equality when (i) the vertices of the gyropolygonal path $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ lie on the gyroline that passes through the points $\mathbf{a}_0$ and $\mathbf{a}_n$ and when (ii) the vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$ are ordered on the gyroline by the increasing, or decreasing, order of their gyroline parameter $t \in \mathbb{R}$. Formally, we thus have the following

Theorem 6.50 *Let $(G, \oplus, \otimes)$ be a gyrovector space, and let $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ be a gyropolygonal path joining the points $\mathbf{a}_0$ and $\mathbf{a}_n$ in G. If (i) the vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$ of the gyropolygonal path lie on the gyroline passing through the points $\mathbf{a}_0$ and $\mathbf{a}_n$ and if (ii) they are ordered on the gyroline by the increasing, or decreasing, order of their gyroline parameter, then the gyropolygonal path inequality* (6.144) *reduces to the equality*

$$\|\ominus\mathbf{a}_0\oplus\mathbf{a}_n\| = \sum_{\oplus,\ k=1}^{n} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\| \tag{6.145}$$

Proof. Let

$$\mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \tag{6.146}$$

be the gyroline passing through the points $\mathbf{a}_0, \ldots, \mathbf{a}_n$ which, in turn, correspond to the gyroline parameter t in increasing order, that is,

$$\mathbf{a}_k = \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_k \tag{6.147}$$

$k = 0, \ldots, n$, and

$$t_0 \le t_1 \le \ldots t_{n-1} \le t_n \tag{6.148}$$

For $n = 1$ the equality in (6.145) clearly holds. Let us assume, by induction, that (6.145) is valid for $n = i$. Then,

$$\begin{aligned}
\sum_{\oplus,\ k=1}^{i+1} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\| &= \sum_{\oplus,\ k=1}^{i} \|\ominus\mathbf{a}_{k-1}\oplus\mathbf{a}_k\|\oplus\|\ominus\mathbf{a}_i\oplus\mathbf{a}_{i+1}\| \\
&= \|\ominus\mathbf{a}_0\oplus\mathbf{a}_i\|\oplus\|\ominus\mathbf{a}_i\oplus\mathbf{a}_{i+1}\| \\
&= \|\ominus\mathbf{a}_0\oplus\mathbf{a}_{i+1}\|
\end{aligned} \tag{6.149}$$

so that (6.145) holds for $n = i + 1$ as well.

The second equality in (6.149) follows from the induction assumption. To verify the third equality in (6.149) we note that the points $\mathbf{a}_0$, $\mathbf{a}_i$, and $\mathbf{a}_{i+1}$ are given by

$$\begin{aligned} \mathbf{a}_0 &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_0 \\ \mathbf{a}_i &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_i \\ \mathbf{a}_{i+1} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_{i+1} \end{aligned} \tag{6.150}$$

with gyroline parameters satisfying

$$t_0 \le t_i \le t_{i+1} \tag{6.151}$$

so that the point $\mathbf{a}_i$ lies between the points $\mathbf{a}_0$, and $\mathbf{a}_{i+1}$. Hence, the third equality in (6.149) follows from the gyrotriangle equality in Theorem 6.47.

Hence, by induction, (6.145) is valid for all $n \ge 1$.

The proof of the theorem for the case when the gyroline parameters in (6.148) are in decreasing order is similar. □

Remark 6.51 **(Gyrogeodesics).** *It follows from Theorem 6.50 that gyrolines minimize gyropolygonal path distances, turning an inequality,* (6.144), *into an equality,* (6.145). *Accordingly, we say that gyrolines are gyrogeodesics. The concept of gyrogeodesics coincides with that of geodesics. Accordingly, in our concrete examples of gyrovector spaces, gyrogeodesics will turn out to be identical with standard geodesics. This, however, is not the case with cogyrogeodesics for a reason that is of interest on its own right, as we will see in the sequel.*

6.9 Cogyrolines

Following the discussion leading to Def. 6.19 of the gyroline, we now present the definition of the cogyroline.

Definition 6.52 **(Cogyrolines, Cogyrosegments).** *Let* $\mathbf{a}, \mathbf{b}$ *be any two distinct points in a gyrovector space* $(G, \oplus, \otimes)$. *The cogyroline in* G *that passes through the points* $\mathbf{a}$ *and* $\mathbf{b}$ *is the set of all points*

$$L^{c} = (\mathbf{b} \boxminus \mathbf{a})\otimes t\oplus\mathbf{a} \tag{6.152}$$

$t \in \mathbb{R}$. *The gyrovector space expression in* (6.152) *is called the representation of the cogyroline* L^{c} *in terms of the two points* $\mathbf{a}$ *and* $\mathbf{b}$ *that it contains.*

A cogyroline segment (or, a cogyrosegment) **ab** *with endpoints* **a** *and* **b** *is the set of all points in* (6.152) *with* $0 \le t \le 1$. *The cogyrolength* $|\mathbf{ab}|^c$ *of the cogyrosegment* **ab** *is the cogyrodistance* $d_{\boxplus}(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} \boxminus \mathbf{a}\|$ *between* **a** *and* **b**,

$$|\mathbf{ab}|^c = d_{\boxplus}(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} \boxminus \mathbf{a}\| \tag{6.153}$$

Two cogyrosegments are congruent if they have the same cogyrolength.

Considering the real parameter t as "time", the cogyroline (6.152) passes through the point **a** at time $t = 0$ and, owing to the right cancellation law, it passes thought the point **b** at time $t = 1$.

It is anticipated in Def. 6.52 that the cogyroline is uniquely represented by any two given points that it contains. The following theorem shows that this is indeed the case.

Theorem 6.53 *Two cogyrolines that share two distinct points are coincident.*

Proof. Let

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \tag{6.154}$$

be a cogyroline that contains the two distinct points $\mathbf{p}_1$ and $\mathbf{p}_2$. Then, there exist real numbers $t_1, t_2 \in \mathbb{R}$, $t_1 \ne t_2$, such that

$$\begin{aligned} \mathbf{p}_1 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a} \\ \mathbf{p}_2 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_2 \oplus \mathbf{a} \end{aligned} \tag{6.155}$$

A cogyroline containing the points $\mathbf{p}_1$ and $\mathbf{p}_2$ has the form

$$(\mathbf{p}_2 \boxminus \mathbf{p}_1) \otimes t \oplus \mathbf{p}_1 \tag{6.156}$$

which, by means of (6.155) is reducible to (6.154) with a reparametrization. Indeed, by (6.155), Identity (2.44) of the Cogyrotranslation Theorem 2.16, scalar distributivity and associativity, and left gyroassociativity with Axiom

$(V6)$ of gyrovector spaces, we have

$$\begin{aligned}
&(\mathbf{p}_2 \boxminus \mathbf{p}_1)\otimes t\oplus\mathbf{p}_1 \\
&= \{[(\mathbf{b} \boxminus \mathbf{a})\otimes t_2\oplus\mathbf{a}] \boxminus [(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\oplus\mathbf{a}]\}\otimes t\oplus[(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\oplus\mathbf{a}] \\
&= \{(\mathbf{b} \boxminus \mathbf{a})\otimes t_2\ominus(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\}\otimes t\oplus[(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\oplus\mathbf{a}] \\
&= \{(\mathbf{b} \boxminus \mathbf{a})\otimes(t_2 - t_1)\}\otimes t\oplus[(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\oplus\mathbf{a}] \\
&= (\mathbf{b} \boxminus \mathbf{a})\otimes((t_2 - t_1)t)\oplus[(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\oplus\mathbf{a}] \\
&= \{(\mathbf{b} \boxminus \mathbf{a})\otimes((t_2 - t_1)t)\oplus(\mathbf{b} \boxminus \mathbf{a})\otimes t_1\}\oplus\mathbf{a} \\
&= (\mathbf{b} \boxminus \mathbf{a})\otimes((t_2 - t_1)t + t_1)\oplus\mathbf{a}
\end{aligned} \tag{6.157}$$

We obtain in (6.157) a reparametrization of the cogyroline (6.154) in which the original cogyroline parameter t is replaced by the new cogyroline parameter $(t_2 - t_1)t + t_1$, where $t_2 - t_1 \neq 0$.

Hence, any cogyroline (6.154) that contains the two points $\mathbf{p}_1$ and $\mathbf{p}_2$ is identical with the cogyroline (6.156). □

Definition 6.54 **(Cogyrocollinearity).** *Three points in a gyrovector space $(G, \oplus, \otimes)$ are cogyrocollinear if they lie on the same cogyroline, that is, there exist $\mathbf{a}, \mathbf{b} \in G$ such that*

$$\mathbf{a}_k = (\mathbf{b} \boxminus \mathbf{a})\otimes t_k\oplus\mathbf{a} \tag{6.158}$$

for some $t_k \in \mathbb{R}$, $k = 1, 2, 3$. Similarly, n points in G, $n > 3$, are cogyrocollinear if any three of these points are cogyrocollinear.

Definition 6.55 **(Cobetweenness).** *A point $\mathbf{a}_2$ lies cobetween the points $\mathbf{a}_1$ and $\mathbf{a}_3$ in G if the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are cogyrocollinear, that is, they are related by the equations*

$$\mathbf{a}_k = (\mathbf{b} \boxminus \mathbf{a})\otimes t_k\oplus\mathbf{a} \tag{6.159}$$

$k = 1, 2, 3$, for some $\mathbf{a}, \mathbf{b} \in G$, $\mathbf{a} \neq \mathbf{b}$, and some $t_k \in \mathbb{R}$, such that either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$.

Lemma 6.56 *Three points, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ in a gyrovector space $(G, \oplus, \otimes)$ are cogyrocollinear if and only if one of these points, say $\mathbf{a}_2$, can be expressed in terms of the two other points by the equation*

$$\mathbf{a}_2 = (\mathbf{a}_3 \boxminus \mathbf{a}_1)\otimes t\oplus\mathbf{a}_1 \tag{6.160}$$

for some $t \in \mathbb{R}$.

Proof. Since the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are cogyrocollinear there exist two distinct points $\mathbf{a}, \mathbf{b} \in G$ and real number t_k such that

$$\mathbf{a}_k = (\mathbf{b} \boxminus \mathbf{a}) \otimes t_k \oplus \mathbf{a} \tag{6.161}$$

$k = 1, 2, 3$.

Let

$$t = \frac{t_2 - t_1}{t_3 - t_1} \tag{6.162}$$

Then, by Identity (2.44) of the Cogyrotranslation Theorem 2.16, and the scalar distributive law we have

$$\begin{aligned}
(\mathbf{a}_3 \boxminus \mathbf{a}_1) \otimes t \oplus \mathbf{a}_1
&= \{[(\mathbf{b} \boxminus \mathbf{a}) \otimes t_3 \oplus \mathbf{a}] \boxminus [(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}]\} \otimes t \oplus [(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}] \\
&= \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_3 \boxminus (\mathbf{b} \boxminus \mathbf{a}) \otimes t_1\} \otimes t \oplus [(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}] \\
&= \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_3 \ominus (\mathbf{b} \boxminus \mathbf{a}) \otimes t_1\} \otimes t \oplus [(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}] \\
&= (\mathbf{b} \boxminus \mathbf{a}) \otimes ((t_3 - t_1)t + t_1) \oplus \mathbf{a} \\
&= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_2 \oplus \mathbf{a} \\
&= \mathbf{a}_2
\end{aligned} \tag{6.163}$$

thus verifying (6.160).

Conversely, if (6.160) holds then the three points are cogyrocollinear, the point $\mathbf{a}_2$ lying on the cogyroline passing through the two other points, $\mathbf{a}_1$ and $\mathbf{a}_3$. □

Lemma 6.57 *A point $\mathbf{a}_2$ lies cobetween the points $\mathbf{a}_1$ and $\mathbf{a}_3$ in a gyrovector space $(G, \oplus, \otimes)$ if and only if*

$$\mathbf{a}_2 = (\mathbf{a}_3 \boxminus \mathbf{a}_1) \otimes t \oplus \mathbf{a}_1 \tag{6.164}$$

for some $0 < t < 1$.

Proof. If $\mathbf{a}_2$ lies cobetween $\mathbf{a}_1$ and $\mathbf{a}_3$, the points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are cogyrocollinear. Hence, there exist distinct points $\mathbf{a}, \mathbf{b} \in G$ and real number t_k such that

$$\mathbf{a}_k = (\mathbf{b} \boxminus \mathbf{a}) \otimes t_k \oplus \mathbf{a} \tag{6.165}$$

$k = 1, 2, 3$, and either $t_1 < t_2 < t_3$ or $t_3 < t_2 < t_1$. Let

$$t = \frac{t_2 - t_1}{t_3 - t_1} \tag{6.166}$$

Then $0 < t < 1$ and, as in the chain of equations (6.163), we derive the desired identity

$$(\mathbf{a}_3 \boxminus \mathbf{a}_1)\otimes t \oplus \mathbf{a}_1 = \mathbf{a}_2 \tag{6.167}$$

thus verifying (6.164)

Conversely, if (6.164) holds then, by Def. 6.55 with $t_1 = 0$, $t_2 = t$ and $t_3 = 1$, $\mathbf{a}_2$ lies cobetween $\mathbf{a}_1$ and $\mathbf{a}_3$. □

Lemma 6.58 *The two equations*

$$\mathbf{b} = (\mathbf{c} \boxminus \mathbf{a})\otimes t \oplus \mathbf{a} \tag{6.168}$$

and

$$\mathbf{b} = (\mathbf{a} \boxminus \mathbf{c})\otimes (1-t) \oplus \mathbf{c} \tag{6.169}$$

are equivalent for the parameter $t \in \mathbb{R}$ and all points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in a gyrovector space $(G, \oplus, \otimes)$.

Proof. Let us assume the validity of (6.168). Then, by the scalar distributive law, (3.27), a right cancellation, and Axiom $(V6)$ of gyrovector spaces we have the chain of equations

$$\begin{aligned}
(\mathbf{a} \boxminus \mathbf{c})\otimes(1-t)\oplus\mathbf{c} &= \{(\mathbf{a} \boxminus \mathbf{c})\ominus(\mathbf{a} \boxminus \mathbf{c})\otimes t\}\oplus\mathbf{c} \\
&= \mathrm{gyr}[\ominus(\mathbf{a} \boxminus \mathbf{c})\otimes t, \mathbf{a} \boxminus \mathbf{c}]\{\ominus(\mathbf{a} \boxminus \mathbf{c})\otimes t\oplus((\mathbf{a} \boxminus \mathbf{c})\oplus\mathbf{c})\} \\
&= (\ominus(\mathbf{a} \boxminus \mathbf{c})\otimes t\oplus\mathbf{a} \\
&= (\mathbf{c} \boxminus \mathbf{a})\otimes t\oplus\mathbf{a} \\
&= \mathbf{b}
\end{aligned} \tag{6.170}$$

thus implying (6.169). Note that the gyration in (6.170) is trivial by Axiom $(V6)$ of gyrovector spaces. Similarly, (6.169) implies (6.168). □

Lemma 6.58 suggests the following

Definition 6.59 **(Directed Cogyrolines).** *Let*

$$L = \mathbf{b}\otimes t\oplus\mathbf{a} \tag{6.171}$$

be a cogyroline with a parameter $t \in \mathbb{R}$ in a gyrovector space $(G, \oplus, \otimes)$, and let p_1 and p_2 be two distinct points on L,

$$\begin{aligned}
p_1 &= \mathbf{b}\otimes t_1\oplus\mathbf{a} \\
p_2 &= \mathbf{b}\otimes t_2\oplus\mathbf{a}
\end{aligned} \tag{6.172}$$

$\mathbf{a}, \mathbf{b} \in G$, $t_1, t_2 \in \mathbb{R}$. *The cogyroline* L *is directed from* p_1 *to* p_2 *if* $t_1 < t_2$.

As an example, the cogyroline (6.168) has the cogyroline parameter t, and it is directed from $\mathbf{a}$ (where $t = 0$) to $\mathbf{c}$ (where $t = 1$). Similarly, the cogyroline (6.169) has the cogyroline parameter $s = 1 - t$, and it is directed from $\mathbf{c}$ (where $s = 0$) to $\mathbf{a}$ (where $s = 1$).

Lemma 6.60 *If the three points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *are cogyrocollinear, then*

$$\mathrm{gyr}[\mathbf{b} \boxminus \mathbf{c}, \mathbf{a} \boxminus \mathbf{b}] = I \tag{6.173}$$

Proof. By Lemma 6.56

$$\mathbf{a} = (\mathbf{c} \boxminus \mathbf{b}) \otimes t \oplus \mathbf{b} \tag{6.174}$$

for some $t \in \mathbb{R}$. Hence, by a right cancellation and the commutativity of the gyrogroup cooperation, Theorem 3.4, we have

$$\mathbf{a} \boxminus \mathbf{b} = (\mathbf{c} \boxminus \mathbf{b}) \otimes t = (\mathbf{b} \boxminus \mathbf{c}) \otimes (-t) \tag{6.175}$$

for some $t \in \mathbb{R}$. The latter, in turn, implies

$$\mathrm{gyr}[\mathbf{b} \boxminus \mathbf{c}, \mathbf{a} \boxminus \mathbf{b}] = \mathrm{gyr}[\mathbf{b} \boxminus \mathbf{c}, (\mathbf{b} \boxminus \mathbf{c}) \otimes (-t)] \tag{6.176}$$

But, the right hand side of (6.176) is the identity automorphism by Axiom $(V6)$ of gyrovector spaces, thus verifying (6.173). □

The next Lemma relates cogyrocollinearity to gyrations. A similar result for gyrocollinearity is found in Lemma 6.28.

Lemma 6.61 *If the three points* $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *are cogyrocollinear, then*

$$\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathrm{gyr}[\mathbf{b}, \mathbf{c}] = \mathrm{gyr}[\mathbf{a}, \mathbf{c}] \tag{6.177}$$

Proof. By Lemma 6.60 we have

$$\mathrm{gyr}[\mathbf{b} \boxminus \mathbf{c}, \mathbf{a} \boxminus \mathbf{b}] = I \tag{6.178}$$

so that the condition of Theorem 3.29 is satisfied. Hence, by Theorem 3.29 we have

$$\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathrm{gyr}[\mathbf{b}, \mathbf{c}]\mathrm{gyr}[\mathbf{c}, \mathbf{a}] = I \tag{6.179}$$

Identity (6.177) follows from (6.179) by gyroautomorphism inversion. □

The obvious extension of Lemma 6.61 to any number of cogyrocollinear points results in the following

Theorem 6.62 **(The Cogyroline Gyration Transitive Law).** *Let* $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *be a set of* n *cogyrocollinear points in a gyrovector space* $(G, \oplus, \otimes)$. *Then*

$$\mathrm{gyr}[\mathbf{a}_1, \mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \mathbf{a}_3] \cdots \mathrm{gyr}[\mathbf{a}_{n-1}, \mathbf{a}_n] = \mathrm{gyr}[\mathbf{a}_1, \mathbf{a}_n] \tag{6.180}$$

Proof. By Lemma 6.61, Identity (6.180) of the theorem holds for $n = 3$. Let us assume, by induction, that Identity (6.180) is valid for some $k \geq 3$. Then, Identity (6.180) is valid for $k + 1$ as well,

$$\begin{aligned}\mathrm{gyr}[\mathbf{a}_1, \mathbf{a}_2] \ldots \mathrm{gyr}[\mathbf{a}_{k-1}, \mathbf{a}_k]\mathrm{gyr}[\mathbf{a}_k, \mathbf{a}_{k+1}] \\ &= \mathrm{gyr}[\mathbf{a}_1, \mathbf{a}_k]\mathrm{gyr}[\mathbf{a}_k, \mathbf{a}_{k+1}] \\ &= \mathrm{gyr}[\mathbf{a}_1, \mathbf{a}_{k+1}]\end{aligned} \tag{6.181}$$

Hence, Identity (6.180) is valid for all $n \geq 3$. □

The duality symmetry that the gyroline and Cogyroline Gyration Transitive Law share in Theorems 6.29 and 6.62 is just a new manifestation of the gyration duality symmetry already observed in Theorem 2.10.

Definition 6.63 **(Parallelism between Cogyrolines).** *The two cogyrolines*

$$\begin{aligned}&(\mathbf{b} \boxminus \mathbf{a})\otimes t \oplus \mathbf{a} \\ &(\mathbf{b}' \boxminus \mathbf{a}')\otimes t \oplus \mathbf{a}'\end{aligned} \tag{6.182}$$

in a gyrovector space $(G, \oplus, \otimes)$ *are parallel if the two points*

$$\begin{aligned}&\mathbf{b} \boxminus \mathbf{a} \\ &\mathbf{b}' \boxminus \mathbf{a}'\end{aligned} \tag{6.183}$$

in G *are related by the equation*

$$\mathbf{b}' \boxminus \mathbf{a}' = \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \tag{6.184}$$

for some real number $\lambda \in \mathbb{R}$.

In the following theorem we will show that parallelism is a property of two cogyrolines rather than a property of pairs of points on the cogyrolines.

Theorem 6.64 *Parallelism between cogyrolines is cogyroline representation independent.*

Proof. Let

$$\begin{aligned} L = \overset{c}{L}_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \\ L = \overset{c}{L}_{\mathbf{cd}} &= (\mathbf{d} \boxminus \mathbf{c}) \otimes t \oplus \mathbf{c} \end{aligned} \tag{6.185}$$

$t \in \mathbb{R}$, be two representations of the same cogyroline L in terms of two different pairs of points that L contains in a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. In order to show that parallelism is cogyroline representation independent, we have to show that the cogyrolines $\overset{c}{L}_{\mathbf{ab}}$ and $\overset{c}{L}_{\mathbf{cd}}$ are parallel.

Since the points $\mathbf{c}$ and $\mathbf{d}$ are two distinct points lying on the cogyroline $L = \overset{c}{L}_{\mathbf{ab}}$, there are real numbers $t_1, t_2 \in \mathbb{R}$, $t_1 \neq t_2$, such that

$$\begin{aligned} \mathbf{c} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a} \\ \mathbf{d} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_2 \oplus \mathbf{a} \end{aligned} \tag{6.186}$$

The condition

$$\operatorname{gyr}[(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1, (\mathbf{b} \boxminus \mathbf{a}) \otimes t_2] = I \tag{6.187}$$

of Theorem 2.16 is satisfied by Axiom $(V6)$ of gyrovector spaces. Hence, by Theorem 2.16 we have from (6.186),

$$\begin{aligned} \mathbf{d} \boxminus \mathbf{c} &= \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_2 \oplus \mathbf{a}\} \boxminus \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1 \oplus \mathbf{a}\} \\ &= \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_2\} \boxminus \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1\} \\ &= \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_2\} \ominus \{(\mathbf{b} \boxminus \mathbf{a}) \otimes t_1\} \\ &= (\mathbf{b} \boxminus \mathbf{a}) \otimes (t_2 - t_1) \end{aligned} \tag{6.188}$$

The third equality in (6.188) follows from condition (6.187) and (2.4), and the fourth equality in (6.188) follows from the scalar distributive law.

Hence, by Def. 6.63, the cogyrolines $\overset{c}{L}_{\mathbf{ab}}$ and $\overset{c}{L}_{\mathbf{cd}}$ are parallel. □

Theorem 6.65 *The family of all cogyrolines that are parallel to a given cogyroline*

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \tag{6.189}$$

in a gyrovector space $(G, \oplus, \otimes)$ is given by

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{c} \tag{6.190}$$

with the parameter $\mathbf{c} \in G$ running over the points of G.

Proof. The cogyrolines (6.189) and (6.190) are parallel by Def. 6.63 with $\lambda = 1$.

Conversely, if the cogyroline (6.189) is parallel to the cogyroline

$$(\mathbf{f} \boxminus \mathbf{e}) \otimes t \oplus \mathbf{e} \tag{6.191}$$

then the latter can be recast in the form of (6.190). To see this we note that by the parallelism in Def. 6.63,

$$\mathbf{f} \boxminus \mathbf{e} = \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \tag{6.192}$$

implying the following equivalent equations

$$\begin{aligned} \ominus\mathbf{e} \boxplus \mathbf{f} &= \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \\ \ominus\mathbf{e} &= \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \ominus \mathbf{f} \\ \mathbf{e} &= \ominus\lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{f} \end{aligned} \tag{6.193}$$

so that (6.191) takes the form

$$\begin{aligned} (\mathbf{f} \ominus \mathbf{e}) \otimes t \oplus \mathbf{e} &= \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \{\ominus\lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{f}\} \\ &= (\mathbf{b} \boxminus \mathbf{a}) \otimes (\lambda t) \oplus \{\ominus\lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{f}\} \end{aligned} \tag{6.194}$$

The extreme right hand side of (6.194) has the desired form of (6.190) (i) with

$$\mathbf{c} = \ominus\lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{f} \tag{6.195}$$

and (ii) with a reparametrization from the cogyroline parameter t to the new cogyroline parameter λt. □

The left and right gyrotranslations of $\mathbf{a}$ by $\mathbf{x}$ in a gyrovector space $(G, \oplus, \otimes)$ are, respectively, $\mathbf{x} \oplus \mathbf{a}$ and $\mathbf{a} \oplus \mathbf{x}$. Similarly, the left and right cogyrotranslations of $\mathbf{a}$ by $\mathbf{x}$ in in G are, respectively, $\mathbf{x} \boxplus \mathbf{a}$ and $\mathbf{a} \boxplus \mathbf{x}$. Left and right gyrotranslations have, in general, different effects. In contrast, left and right cogyrotranslations coincide owing to the commutativity of the gyrovector space cooperation $\boxplus$. Hence, we call them collectively cogyrotranslations.

Theorem 6.66 *Let* $\mathbf{d}$ *be any point on the cogyroline*

$$\mathbf{b} \otimes t \oplus \mathbf{a} \tag{6.196}$$

in a gyrovector space $(G, \oplus, \otimes)$. *The cogyrotranslation of the cogyroline* (6.196) *by* $\ominus\mathbf{d}$,

$$(\mathbf{b}\otimes t\oplus\mathbf{a}) \boxminus \mathbf{d} \tag{6.197}$$

is, again, a cogyroline. The cogyroline (6.196) *and the cogyrotranslated cogyroline* (6.197) *are parallel and, furthermore, the cogyrotranslated cogyroline* (6.197) *passes through the origin of* G.

Proof. Let $\mathbf{d}$ be any point on the cogyroline (6.196). The cogyrotranslation (6.197) of the cogyroline (6.196) is manipulated in the following chain of numbered equalities.

$$\begin{aligned}
(\mathbf{b}\otimes t\oplus\mathbf{a}) \boxminus \mathbf{d} &\overset{(1)}{=\!=\!=} (\mathbf{b}\otimes t\oplus\mathbf{a})\ominus\mathrm{gyr}[\mathbf{b}\otimes t\oplus\mathbf{a}, \mathbf{d}]\mathbf{d} \\
&\overset{(2)}{=\!=\!=} \mathbf{b}\otimes t\oplus(\mathbf{a}\ominus\mathrm{gyr}[\mathbf{a}, \mathbf{b}\otimes t]\mathrm{gyr}[\mathbf{b}\otimes t\oplus\mathbf{a}, \mathbf{d}]\mathbf{d}) \\
&\overset{(3)}{=\!=\!=} \mathbf{b}\otimes t\oplus(\mathbf{a}\ominus\mathrm{gyr}[\mathbf{a}, \mathbf{b}\otimes t\oplus\mathbf{a}]\mathrm{gyr}[\mathbf{b}\otimes t\oplus\mathbf{a}, \mathbf{d}]\mathbf{d}) \\
&\overset{(4)}{=\!=\!=} \mathbf{b}\otimes t\oplus(\mathbf{a}\ominus\mathrm{gyr}[\mathbf{a}, \mathbf{d}]\mathbf{d}) \\
&\overset{(5)}{=\!=\!=} \mathbf{b}\otimes t\oplus(\mathbf{a} \boxminus \mathbf{d})
\end{aligned} \tag{6.198}$$

Hence, by (6.198) and Theorem 6.65, the cogyroline (6.196) and the cogyrotranslated cogyroline (6.197) are parallel.

The derivation of the equalities in (6.198) follows.

(1) Follows from (2.4).
(2) Follows from the right gyroassociative law
(3) Follows from the right loop property.
(4) Follows from Theorem 6.62 since the points $\mathbf{a}$, $\mathbf{b}\otimes t\oplus\mathbf{a}$, $\mathbf{d}$ are cogyrocolinear.
(5) Follows from (2.4).

Since the point $\mathbf{d}$ lies on the cogyroline (6.196), it is given by the equation

$$\mathbf{d} = \mathbf{b}\otimes t_0\oplus\mathbf{a} \tag{6.199}$$

for some $t_0 \in \mathbb{R}$. Hence,

$$\begin{aligned}\mathbf{a} \boxminus \mathbf{d} &= \mathbf{a} \boxminus (\mathbf{b} \otimes t_0 \oplus \mathbf{a}) \\ &= \mathbf{a} \ominus \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t_0 \oplus \mathbf{a}](\mathbf{b} \otimes t_0 \oplus \mathbf{a}) \\ &= \mathbf{a} \ominus \mathrm{gyr}[\mathbf{a}, \mathbf{b} \otimes t_0](\mathbf{b} \otimes t_0 \oplus \mathbf{a}) \\ &= \mathbf{a} \ominus (\mathbf{a} \oplus \mathbf{b} \otimes t_0) \\ &= \ominus \mathbf{b} \otimes t_0\end{aligned} \tag{6.200}$$

It follows from (6.198) and (6.200) that

$$\begin{aligned}(\mathbf{b} \otimes t \oplus \mathbf{a}) \boxminus \mathbf{d} &= \mathbf{b} \otimes t \oplus (\mathbf{a} \boxminus \mathbf{d}) \\ &= \mathbf{b} \otimes t \ominus \mathbf{b} \otimes t_0 \\ &= \mathbf{b} \otimes (t - t_0)\end{aligned} \tag{6.201}$$

demonstrating that the cogyrotranslated cogyroline (6.197) is a special cogyroline. It is both a gyroline and a cogyroline that passes through the origin of its gyrovector space when $t = t_0$. □

Clearly, the cogyroline in (6.197) is origin-intercept, Def. 6.17, suggesting the following

Definition 6.67 **(Origin-Intercept Cogyroline).** *Let* $\mathbf{d}$ *be any point on the cogyroline*

$$\mathbf{b} \otimes t \oplus \mathbf{a} \tag{6.202}$$

in a gyrovector space $(G, \oplus, \otimes)$. *The resulting origin-intercept cogyroline*

$$(\mathbf{b} \otimes t \oplus \mathbf{a}) \boxminus \mathbf{d} \tag{6.203}$$

is said to be the origin-intercept cogyroline (or equivalently, gyroline, by Theorem 6.68 below) that corresponds to the cogyroline (6.202).

Theorem 6.68 *The cogyrodifference* $\mathbf{p}_1 \boxminus \mathbf{p}_2$ *of any two points* $\mathbf{p}_1$ *and* $\mathbf{p}_2$ *lying on a cogyroline lies on a corresponding origin-intercept gyroline.*

Proof. The proof follows from (6.201) or, equivalently, from (6.186) and (6.188). □

Remark 6.69 **(Supporting Gyrodiameters).** *We will find that in Möbius and in Einstein gyrovector spaces the origin-intercept gyroline that corresponds to a given cogyroline turns out to be the "supporting gyrodiameter" of the cogyroline. The supporting gyrodiameter, in turn, defines the* *orientation of its cogyroline thus allowing the introduction of parallelism between cogyrolines.*

6.10 Cogyromidpoints

The value $t = 1/2$ in Lemma 6.58 gives rise to a special point where the two parameters of $\mathbf{b}$, t and $(1-t)$, coincide. It suggests the following

Definition 6.70 (Cogyromidpoints). *The cogyromidpoint* $\mathbf{p}^c_{\mathbf{ac}}$ *of points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *is given by the equation*

$$\mathbf{p}^c_{\mathbf{ac}} = (\mathbf{c} \boxminus \mathbf{a}) \otimes \tfrac{1}{2} \oplus \mathbf{a} \tag{6.204}$$

Theorem 6.71 *Let* $\mathbf{a}$ *and* $\mathbf{c}$ *be any two points of a gyrovector space* $(G, \oplus, \otimes)$. *Then*

$$\mathbf{p}^c_{\mathbf{ac}} = \mathbf{p}^c_{\mathbf{ca}} \tag{6.205}$$

and

$$\|\mathbf{a} \boxminus \mathbf{p}^c_{\mathbf{ac}}\| = \|\mathbf{c} \boxminus \mathbf{p}^c_{\mathbf{ac}}\| \tag{6.206}$$

Proof. By Lemma 6.58, with $t = 1/2$, the two equations

$$\begin{aligned} \mathbf{b} &= (\mathbf{c} \boxminus \mathbf{a}) \otimes \tfrac{1}{2} \oplus \mathbf{a} = \mathbf{p}^c_{\mathbf{ac}} \\ \mathbf{b} &= (\mathbf{a} \boxminus \mathbf{c}) \otimes \tfrac{1}{2} \oplus \mathbf{c} = \mathbf{p}^c_{\mathbf{ca}} \end{aligned} \tag{6.207}$$

are equivalent, thus verifying (6.205).

It follows from (6.207) by right cancellations and the gyroautomorphic inverse property, Theorem 2.31, of the gyrogroup cooperation that

$$\begin{aligned} \mathbf{p}^c_{\mathbf{ac}} \boxminus \mathbf{a} &= (\mathbf{c} \boxminus \mathbf{a}) \otimes \tfrac{1}{2} \\ \mathbf{p}^c_{\mathbf{ca}} \boxminus \mathbf{c} &= (\mathbf{a} \boxminus \mathbf{c}) \otimes \tfrac{1}{2} = \ominus(\mathbf{c} \boxminus \mathbf{a}) \otimes \tfrac{1}{2} \end{aligned} \tag{6.208}$$

implying

$$\begin{aligned} \|\mathbf{p}^c_{\mathbf{ac}} \boxminus \mathbf{a}\| &= \|\mathbf{c} \boxminus \mathbf{a}\| \otimes \tfrac{1}{2} \\ \|\mathbf{p}^c_{\mathbf{ca}} \boxminus \mathbf{c}\| &= \|\mathbf{c} \boxminus \mathbf{a}\| \otimes \tfrac{1}{2} \end{aligned} \tag{6.209}$$

thus verifying (6.206). $\square$

Theorem 6.72 *The cogyromidpoint* $\mathbf{p}^c_{\mathbf{ab}}$ *of points* $\mathbf{a}$ *and* $\mathbf{b}$ *satisfies the identity*

$$\mathbf{p}^c_{\mathbf{ab}} = \tfrac{1}{2}\otimes\mathrm{gyr}[\mathbf{p}^c_{\mathbf{ab}}, \mathbf{a}](\mathbf{a}\oplus\mathbf{b}) \tag{6.210}$$

so that

$$\|\mathbf{p}^c_{\mathbf{ab}}\| = \tfrac{1}{2}\otimes\|\mathbf{a}\oplus\mathbf{b}\| \tag{6.211}$$

Proof. By the left loop property and (6.204) we have

$$\mathrm{gyr}[(\mathbf{b}\boxminus\mathbf{a})\otimes\tfrac{1}{2}, \mathbf{a}] = \mathrm{gyr}[\mathbf{p}^c_{\mathbf{ab}}, \mathbf{a}] \tag{6.212}$$

Hence, by the gyrocommutative law, (6.212), the Two-Sum Identity in Theorem 6.7, and a right cancellation we have,

$$\begin{aligned} 2\otimes\{(\mathbf{b}\boxminus\mathbf{a})\otimes\tfrac{1}{2}\oplus\mathbf{a}\} &= 2\otimes\mathrm{gyr}[(\mathbf{b}\boxminus\mathbf{a})\otimes\tfrac{1}{2}, \mathbf{a}]\{\mathbf{a}\oplus\tfrac{1}{2}\otimes(\mathbf{b}\boxminus\mathbf{a})\} \\ &= \mathrm{gyr}[\mathbf{p}^c_{\mathbf{ab}}, \mathbf{a}]\{2\otimes[\mathbf{a}\oplus\tfrac{1}{2}\otimes(\mathbf{b}\boxminus\mathbf{a})]\} \\ &= \mathrm{gyr}[\mathbf{p}^c_{\mathbf{ab}}, \mathbf{a}]\{\mathbf{a}\oplus[(\mathbf{b}\boxminus\mathbf{a})\oplus\mathbf{a}]\} \\ &= \mathrm{gyr}[\mathbf{p}^c_{\mathbf{ab}}, \mathbf{a}](\mathbf{a}\oplus\mathbf{b}) \end{aligned} \tag{6.213}$$

thus implying (6.210). Finally, (6.211) follows from (6.210) by Axiom $V7$ of gyrovector spaces, noting that gyrations preserve the norm. □

The gyromidpoint and the cogyromidpoint share in Identities (6.211) and (6.92) a remarkable duality symmetry.

6.11 Cogyrogeodesics

Theorem 6.73 *The cogyrodistance in a gyrovector space* $(G, \oplus, \otimes)$ *is invariant under appropriately gyrated right gyrotranslations,*

$$\mathbf{a}\boxminus\mathbf{b} = (\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \mathbf{k_{ab}}]\mathbf{x}) \boxminus (\mathbf{b}\oplus\mathrm{gyr}[\mathbf{b}, \mathbf{k_{ab}}]\mathbf{x}) \tag{6.214}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{x} \in G$ *where* $\mathbf{k_{ab}}$ *is any point lying on the cogyroline passing through the distinct points* $\mathbf{a}$ *and* $\mathbf{b}$.

Proof. By Theorem 6.14 we have the identity

$$\mathbf{a}\boxminus\mathbf{b} = (\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{y}) \boxminus (\mathbf{b}\oplus\mathbf{y}) \tag{6.215}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{y} \in G$.

By the Cogyroline Gyration Transitive Law, Theorem 6.62, we have the identity

$$\mathrm{gyr}[\mathbf{a},\mathbf{b}] = \mathrm{gyr}[\mathbf{a},\mathbf{k_{ab}}]\mathrm{gyr}[\mathbf{k_{ab}},\mathbf{b}] \tag{6.216}$$

for any point $\mathbf{k_{ab}}$ lying on the cogyroline that passes through the points $\mathbf{a}$ and $\mathbf{b}$.

Selecting $\mathbf{y} = \mathrm{gyr}[\mathbf{b},\mathbf{k_{ab}}]\mathbf{x}$, and noting (6.216), Identity (6.215) reduces to (6.214). □

Theorem 6.73 reduces to Theorem 6.14 when the point $\mathbf{k_{ab}}$ on the cogyroline that passes through the points $\mathbf{a}$ and $\mathbf{b}$ is selected to be $\mathbf{k_{ab}} = \mathbf{b}$.

The following theorem will enable us to recognize that cogyrolines are *cogeodesics.*

Theorem 6.74 **(The Cogyrotriangle Equality, I).** *If a point* $\mathbf{b}$ *lies cobetween two points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G,\oplus,\otimes)$ *then*

$$\|\boxminus\mathbf{a}\boxplus\mathbf{c}\| = \|\boxminus\mathbf{a}\boxplus\mathbf{b}\| \boxplus \|\boxminus\mathbf{b}\boxplus\mathbf{c}\| \tag{6.217}$$

Proof. If $\mathbf{b}$ lies cobetween $\mathbf{a}$ and $\mathbf{c}$ then, by Lemma 6.57,

$$\mathbf{b} = (\mathbf{c}\boxminus\mathbf{a})\otimes t\oplus\mathbf{a} \tag{6.218}$$

for some $0 < t < 1$, and hence, by Lemma 6.58

$$\mathbf{b} = (\mathbf{a}\boxminus\mathbf{c})\otimes(1-t)\oplus\mathbf{c} \tag{6.219}$$

Hence, by right cancellations and by the commutativity of the cooperation $\boxplus$ we have from (6.218) and (6.219),

$$\begin{aligned}\boxminus\mathbf{a}\boxplus\mathbf{b} &= (\boxminus\mathbf{a}\boxplus\mathbf{c})\otimes t\\ \boxminus\mathbf{c}\boxplus\mathbf{b} &= (\boxminus\mathbf{c}\boxplus\mathbf{a})\otimes(1-t)\end{aligned} \tag{6.220}$$

Taking magnitudes, noting the homogeneity property $(V7)$, (6.220) gives

$$\begin{aligned}\|\boxminus\mathbf{a}\boxplus\mathbf{b}\| &= \|\boxminus\mathbf{a}\boxplus\mathbf{c}\|\otimes t\\ \|\boxminus\mathbf{b}\boxplus\mathbf{c}\| &= \|\boxminus\mathbf{a}\boxplus\mathbf{c}\|\otimes(1-t)\end{aligned} \tag{6.221}$$

so that, by the scalar distributive law $(V2)$,

$$\begin{aligned}\|\boxminus\mathbf{a}\boxplus\mathbf{b}\|\oplus\|\boxminus\mathbf{b}\boxplus\mathbf{c}\| &= \|\boxminus\mathbf{a}\boxplus\mathbf{c}\|\otimes\{t+(1-t)\}\\ &= \|\boxminus\mathbf{a}\boxplus\mathbf{c}\|\end{aligned} \tag{6.222}$$

But, the operation $\oplus$ between magnitudes equals the cooperation $\boxplus$ between magnitudes, Remark 6.3. Hence, (6.222) is equivalent to (6.217). □

Theorem 6.75 (The Cogyrotriangle Equality, II). *If a point* $\mathbf{b}$ *lies cobetween two points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *then*

$$\| \boxminus \mathbf{a} \boxplus \mathrm{gyr}[\mathbf{a} \boxminus \mathbf{b}, \mathbf{b} \boxminus \mathbf{c}]\mathbf{c} \| = \| \boxminus \mathbf{a} \boxplus \mathbf{b} \| \boxplus \| \boxminus \mathbf{b} \boxplus \mathbf{c} \| \tag{6.223}$$

Proof. It follows from Lemma 6.60 and gyroautomorphism inversion that

$$\mathrm{gyr}[\mathbf{a} \boxminus \mathbf{b}, \mathbf{b} \boxminus \mathbf{c}] = I \tag{6.224}$$

Hence, the gyration in (6.223) for cogyrocollinear points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is trivial, making (6.223) equivalent to (6.217) in Theorem 6.74. □

The cogyrotriangle equality in Theorem 6.74 has a form analogous to that of the gyrotriangle equality in Theorem 6.47. In contrast, the equivalent cogyrotriangle equality in Theorem 6.75 has a form that emphasizes the result that the cogyrotriangle equality is a special case of the cogyrotriangle inequality (6.17) corresponding to cogyrocollinear points.

Remark 6.76 *Comparing Theorem 6.75 with Theorem 6.11 we see that point* $\mathbf{b}$ *cobetween two given points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *turn the cogyrotriangle inequality into an equality in analogy with the first part of Remark 6.48.*

Definition 6.77 (Cogyrodistance Along Cogyropolygonal Paths). *A cogyropolygonal path* $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ *from a point* $\mathbf{a}_0$ *to a point* $\mathbf{a}_n$ *in a gyrovector space* $(G, \oplus, \otimes)$ *is the same as the gyropolygonal path* $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ *in Def. 2.13 except that the value of a pair* $(\mathbf{a}, \mathbf{b})$ *is now* $\boxminus \mathbf{a} \boxplus \mathbf{b}$*. The cogyrodistance* $d_{P(\mathbf{a}_0, \ldots, \mathbf{a}_n)}$ *between the points* $\mathbf{a}_0$ *and* $\mathbf{a}_n$ *along the cogyropolygonal path* $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ *is given by the equation*

$$d_{P(\mathbf{a}_0, \ldots, \mathbf{a}_n)} = \sum_{\boxplus,\ k=1}^{n} \| \boxminus \mathbf{a}_{k-1} \boxplus \mathbf{a}_k \| \tag{6.225}$$

In Def. 6.77 we use the notation

$$\sum_{\boxplus,\ k=1}^{n} \| \boxminus \mathbf{a}_{k-1} \boxplus \mathbf{a}_k \| = \| \boxminus \mathbf{a}_0 \boxplus \mathbf{a}_1 \| \boxplus \quad \ldots \quad \boxplus \| \boxminus \mathbf{a}_{n-1} \boxplus \mathbf{a}_n \| \tag{6.226}$$

noting that unlike the cogyrooperation $\boxplus$ between elements of G, which is commutative and nonassociative, the cogyrooperation $\boxplus$ between norms of elements of G is equal to the gyrooperation $\oplus$, and is both commutative and associative, Remark 6.48.

Theorem 6.78 *Let $(G, \oplus, \otimes)$ be a gyrovector space, and let $P(\mathbf{a}_0, \ldots, \mathbf{a}_n)$ be a cogyropolygonal path joining the points $\mathbf{a}_0$ and $\mathbf{a}_n$ in G. If (i) the vertices $\mathbf{a}_0, \ldots, \mathbf{a}_n$ of the cogyropolygonal path lie on the cogyroline passing through the points $\mathbf{a}_0$ and $\mathbf{a}_n$ and if (ii) they are ordered on the cogyroline by the increasing, or decreasing order of their cogyroline parameter, then we have the equality*

$$\|\boxminus \mathbf{a}_0 \boxplus \mathbf{a}_n\| = \sum_{\boxplus,\ k=1}^{n} \|\boxminus \mathbf{a}_{k-1} \boxplus \mathbf{a}_k\| \tag{6.227}$$

Proof. Let

$$(\mathbf{b} \boxminus \mathbf{a}) \otimes t \oplus \mathbf{a} \tag{6.228}$$

be the cogyroline passing through the points $\mathbf{a}_0, \ldots, \mathbf{a}_n$ which, in turn, correspond to the cogyroline parameter t in increasing order, that is,

$$\mathbf{a}_k = (\mathbf{b} \boxminus \mathbf{a}) \otimes t_k \oplus \mathbf{a} \tag{6.229}$$

$k = 0, \ldots, n$, and

$$t_0 \le t_1 \le \ldots t_{n-1} \le t_n \tag{6.230}$$

For $n = 1$ the equality in (6.227) clearly holds. Let us assume, by induction, that (6.227) is valid for $n = i$. Then,

$$\begin{aligned} \sum_{\boxplus,\ k=1}^{i+1} \|\boxminus \mathbf{a}_{k-1} \boxplus \mathbf{a}_k\| &= \sum_{\boxplus,\ k=1}^{i} \|\boxminus \mathbf{a}_{k-1} \boxplus \mathbf{a}_k\| \boxplus \|\ominus \mathbf{a}_i \boxplus \mathbf{a}_{i+1}\| \\ &= \|\boxminus \mathbf{a}_0 \boxplus \mathbf{a}_i\| \boxplus \|\boxminus \mathbf{a}_i \boxplus \mathbf{a}_{i+1}\| \\ &= \|\ominus \mathbf{a}_0 \boxplus \mathbf{a}_{i+1}\| \end{aligned} \tag{6.231}$$

so that (6.227) holds for $n = i + 1$ as well.

The second equality in (6.231) follows from the induction assumption. To verify the third equality in (6.231) we note that the points $\mathbf{a}_0$, $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$ are given by

$$\begin{aligned} \mathbf{a}_0 &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_0 \oplus \mathbf{a} \\ \mathbf{a}_i &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_i \oplus \mathbf{a} \\ \mathbf{a}_{i+1} &= (\mathbf{b} \boxminus \mathbf{a}) \otimes t_{i+1} \oplus \mathbf{a} \end{aligned} \tag{6.232}$$

with cogyroline parameters satisfying

$$t_0 \le t_i \le t_{i+1} \tag{6.233}$$

so that the point $\mathbf{a}_i$ lies cobetween the points $\mathbf{a}_0$, and $\mathbf{a}_{i+1}$. Hence, the third equality in (6.231) follows from the cogyrotriangle equality in Theorem 6.74.

Hence, by induction, (6.227) is valid for all $n \geq 1$.

The proof of the theorem for the case when the gyroline parameters in (6.230) are in decreasing order is similar. □

Remark 6.79 (Cogyrogeodesics). *Owing to the analogy that Theorem 6.78 shares with Theorem 6.50, cogyrolines are also called cogyrogeodesics. One should, however, note that the analogies that gyrogeodesics and cogyrogeodesics share are incomplete since, unlike the gyrotriangle inequality, Theorem 6.9, the cogyrotriangle inequality, Theorem 6.11, is "corrected" by a gyration. Like the gyration corrections that are present in the gyroassociative and the gyrocommutative laws, the gyration correction that is present in the gyrotriangle inequality in Theorem 6.11 has useful geometric consequences.*

Three concrete examples of gyrovector spaces are presented in Secs. 6.12, 6.16, and 6.18.

6.12 Möbius Gyrovector Spaces

Möbius gyrogroups $(\mathbb{V}_s, \oplus_M)$ admit scalar multiplication $\otimes_M$, turning them into Möbius gyrovector spaces $(\mathbb{V}_s, \oplus_M, \otimes_M)$.

Definition 6.80 (Möbius Scalar Multiplication). *Let $(\mathbb{V}_s, \oplus_M)$ be a Möbius gyrogroup. The Möbius scalar multiplication $r\otimes_M \mathbf{v} = \mathbf{v}\otimes_M r$ in $\mathbb{V}_s$ is given by the equation*

$$\begin{aligned} r\otimes_M \mathbf{v} &= s\frac{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{s}\right)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s\tanh(r\tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{6.234}$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{V}_s$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes_M \mathbf{0} = \mathbf{0}$.

As an example we present the Möbius half,

$$\tfrac{1}{2}\otimes_M \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{6.235}$$

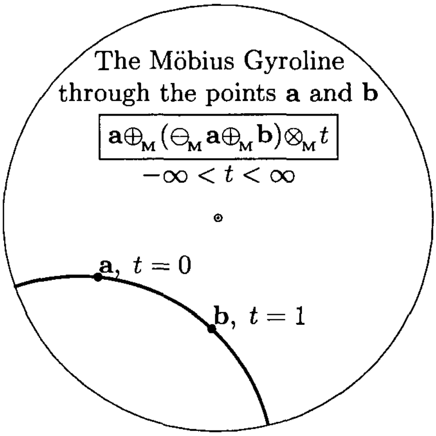


Fig. 6.1 The unique gyroline in a Möbius gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the Möbius gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_{s=1}$ is the real open unit disc, is shown graphically.

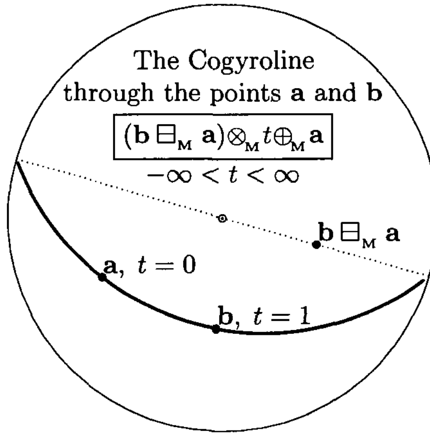


Fig. 6.2 The unique cogyroline in $(\mathbb{V}_s, \oplus_M, \otimes_M)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the Möbius gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_{s=1}$ is the real open unit disc, is shown graphically.

satisfying

$$\gamma_{(1/2)\otimes\mathbf{v}} = \sqrt{\frac{1+\gamma_{\mathbf{v}}}{2}} \tag{6.236}$$

where $\gamma_{\mathbf{v}}$ is the gamma factor (3.129).

Indeed, in accordance with the scalar associative law of gyrovector spaces,

$$\begin{aligned} 2\otimes_M(\tfrac{1}{2}\otimes_M\mathbf{v}) &= 2\otimes_M \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus_M \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \mathbf{v} \end{aligned} \tag{6.237}$$

The unique Möbius gyroline $L^g_{\mathbf{ab}}$ and cogyroline $L^c_{\mathbf{ab}}$ that pass through two given points $\mathbf{a}$ and $\mathbf{b}$ are represented by the equations

$$\begin{aligned} L^g_{\mathbf{ab}} &= \mathbf{a}\oplus_M(\ominus_M\mathbf{a}\oplus_M\mathbf{b})\otimes_M t \\ L^c_{\mathbf{ab}} &= (\mathbf{b} \boxminus_M \mathbf{a})\otimes_M t\oplus_M \mathbf{a} \end{aligned} \tag{6.238}$$

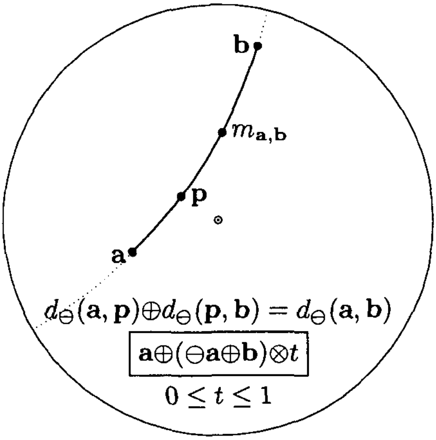


Fig. 6.3 The gyrosegment that links the two points $\mathbf{a}$ and $\mathbf{b}$ in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. $\mathbf{p}$ is a generic point between $\mathbf{a}$ and $\mathbf{b}$, and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$. See also Fig, 6.10.

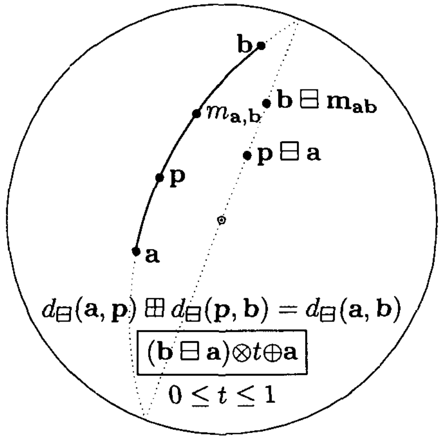


Fig. 6.4 The cogyrosegment that links the two points $\mathbf{a}$ and $\mathbf{b}$ in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. $\mathbf{p}$ is a generic point cobetween $\mathbf{a}$ and $\mathbf{b}$ and $m_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$. See also Fig, 6.11.

$t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$. Gyrolines in a Möbius gyrovector space coincide with the well-known geodesics of the Poincaré ball model of hyperbolic geometry, as we will prove in Sec. 7.3. Möbius gyrolines in the disc are Euclidean circular arcs that intersect the boundary of the disc orthogonally, Fig. 6.1. In contrast, Möbius cogyrolines in the disc are Euclidean circular arcs that intersect the boundary of the disc diametrically, that is, on the opposite sides of a diameter called the supporting gyrodiameter, Fig. 6.2.

The supporting gyrodiameter is both a gyroline and a cogyroline passing through the origin; see Remark 6.69. Let $\mathbf{d}$ be any point on the cogyroline, Fig. 6.2,

$$(\mathbf{b} \boxminus_M \mathbf{a})\otimes t \oplus_M \mathbf{a} \tag{6.239}$$

Then, by Theorem 6.66, the equation of its supporting gyrodiameter is

$$\{(\mathbf{b} \boxminus_M \mathbf{a})\otimes t \oplus_M \mathbf{a}\} \boxminus_M \mathbf{d} \tag{6.240}$$

$t \in \mathbb{R}$.

Figure 6.3 presents the gyrosegment $\mathbf{ab}$ that joins the points $\mathbf{a}$ and $\mathbf{b}$ in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus_M, \otimes)$ along with its gyromidpoint $\mathbf{m_{ab}}$, and a generic point $\mathbf{p}$ lying between $\mathbf{a}$ and $\mathbf{b}$. Since the points $\mathbf{a}, \mathbf{p}, \mathbf{b}$ are

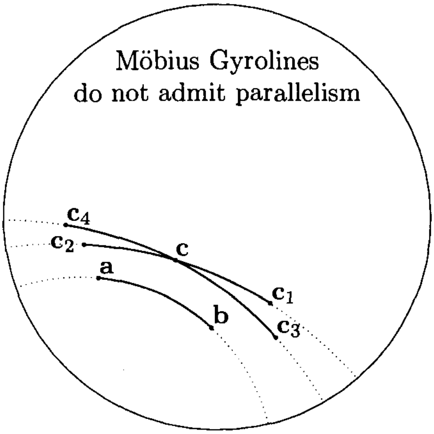


Fig. 6.5 Through the point $\mathbf{c}$, not on the gyroline $\mathbf{ab}$, there are infinitely many gyrolines, like $\mathbf{c}_1\mathbf{c}_2$ and $\mathbf{c}_3\mathbf{c}_4$, that do not intersect gyroline $\mathbf{ab}$. Hence, the Euclidean parallel postulate is not satisfied.

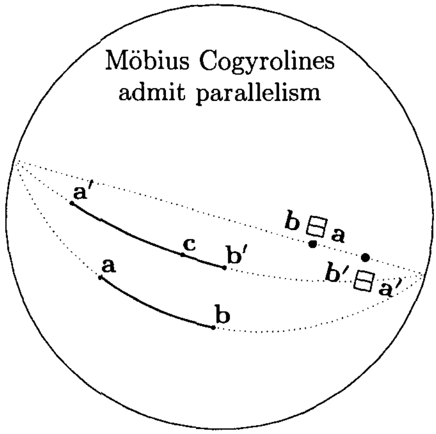


Fig. 6.6 Through the point $\mathbf{c}$, not on the cogyroline $\mathbf{ab}$, there is a unique cogyroline $\mathbf{a'b'}$ that does not intersect the cogyroline $\mathbf{ab}$. Hence, the Euclidean parallel postulate is satisfied.

gyrocollinear, they satisfy the gyrotriangle equality, Theorem 6.47, shown in the figure.

Figure 6.4 presents a cogyrosegment $\mathbf{ab}$ in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus_M, \otimes)$ along with its cogyromidpoint $\mathbf{m_{ab}}$, and a generic point $\mathbf{p}$ lying cobetween $\mathbf{a}$ and $\mathbf{b}$. Since the points $\mathbf{a}, \mathbf{p}, \mathbf{b}$ are cogyrocollinear, they satisfy the cogyrotriangle equality, Theorem 6.74, shown in the Figure. Moreover, since the points $\mathbf{a}, \mathbf{p}, \mathbf{m_{ab}}, \mathbf{b}$ lie on the cogyroline, the cogyrodifferences $\mathbf{p}\boxminus\mathbf{a}$ and $\mathbf{b} \boxminus \mathbf{m_{ab}}$, for instance, lie on the cogyroline supporting gyrodiameter, as expected from Theorem 6.68 and Remark 6.69.

6.13 Möbius Cogyroline Parallelism

Möbius gyrolines do not admit parallelism. Given a gyroline $L_0^g = \mathbf{ab}$ and a point $\mathbf{c}$ not on the gyroline, there exist infinitely many gyrolines that pass through the point $\mathbf{c}$ and do not intersect the gyroline L_0^g, two of which, $L_1^g = \mathbf{c}_1\mathbf{c}_2$ and $L_2^g = \mathbf{c}_3\mathbf{c}_4$, are shown in Fig. 6.5 for the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus_M, \otimes_M)$.

In contrast, cogyrolines do admit parallelism. Given a cogyroline $L_0^c = \mathbf{ab}$ and a point $\mathbf{c}$ not on the cogyroline, there exists a unique cogyroline that passes through the point $\mathbf{c}$ and does not intersect the cogyroline L_0^c.

It is the cogyroline $L_1^c = \mathbf{a}'\mathbf{b}'$ shown in Fig. 6.6 for the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus_M, \otimes_M)$.

We note that (i) the two parallel cogyrolines $L_0^c = \mathbf{ab}$ and $L_1^c = \mathbf{a}'\mathbf{b}'$ in Fig. 6.6 share their supporting gyrodiameters, and that (ii) their associated points $\mathbf{b}\boxminus\mathbf{a}$ and $\mathbf{b}'\boxminus\mathbf{a}'$ lie on the common supporting gyrodiameter. Hence, these points in $\mathbb{V}_s \subset \mathbb{V}$ represent two Euclidean vectors in $\mathbb{V}$ that are Euclidean parallel to the supporting gyrodiameter, so that there exists a real number $r \neq 0$ such that $\mathbf{b}' \boxminus \mathbf{a}' = r(\mathbf{b} \boxminus \mathbf{a})$. Equivalently, there exists a real number $\lambda \neq 0$ such that

$$\mathbf{b}' \boxminus \mathbf{a}' = \lambda \otimes (\mathbf{b} \boxminus \mathbf{a}) \tag{6.241}$$

as we see from Def. 6.80 of scalar multiplication. Hence, by Def. 6.63, the cogyrolines L_0^c and L_1^c are parallel.

6.14 Illustrating the Gyroline Gyration Transitive Law

In a Möbius gyrovector plane the Gyroline Gyration Transitive Law in Theorem 6.29 can be illustrated graphically in terms of the slide of a tangent line along a circular arc, Fig. 6.7.

Let

$$L = \mathbf{a} \oplus \mathbf{b} \otimes t \tag{6.242}$$

be a gyroline in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$, and let

$$\mathbf{p}(t) = \mathbf{a} \oplus \mathbf{b} \otimes t \tag{6.243}$$

be the generic point of the gyroline L parametrized by the gyroline parameter $t \in \mathbb{R}$.

Considering the gyroline parameter t as "time", the point $\mathbf{p}(t)$, $-\infty < t < \infty$, travels along the gyroline, reaching the point $\mathbf{a}$ at time $t = 0$ and the point $\mathbf{a} \oplus \mathbf{b}$ at time $t = 1$.

Finally, let

$$\mathbf{b}_t = \mathrm{gyr}[\mathbf{p}(t), \ominus \mathbf{a}]\mathbf{b} \tag{6.244}$$

be a gyration of $\mathbf{b}$ parametrized by the gyroline parameter t.

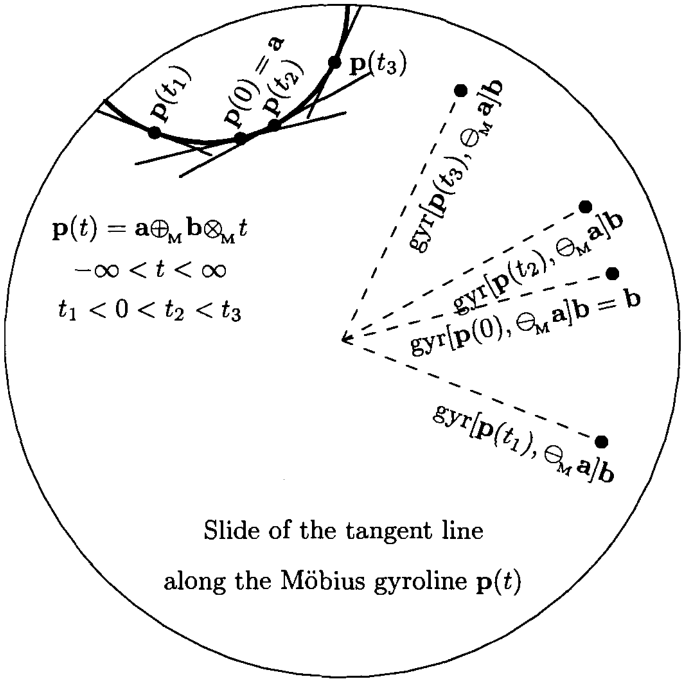


Fig. 6.7 The slide of the Euclidean tangent line along the Möbius gyroline. $\mathbf{p}(t_n) = \mathbf{a}\oplus_{\rm M}\mathbf{b}\otimes_{\rm M}t_n$, $t_n \in \mathbb{R}$, $n = 1, 2, 3$, $t_1 < 0 < t_2 < t_3$, and $\mathbf{p}(0)$ are four points on the gyroline $\mathbf{p}(t) = \mathbf{a}\oplus_{\rm M}\mathbf{b}\otimes_{\rm M}t$, parametrized by $t \in \mathbb{R}$. The Euclidean tangent line at any point $\mathbf{p}(t)$ of the gyroline is Euclidean parallel to the vector $\mathrm{gyr}[\mathbf{p}(t), \ominus_{\rm M}\mathbf{a}]\mathbf{b}$. Shown are the tangent lines at the four points of the gyroline, $\mathbf{p}(t_1), \mathbf{p}(0), \mathbf{p}(t_2), \mathbf{p}(t_3)$, and their corresponding Euclidean parallel vectors in the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus_{\rm M}, \otimes_{\rm M})$, which is the Poincaré disc model of hyperbolic geometry.

For $t = 0$ we have

$$\begin{aligned}\mathbf{b}_0 &= \mathrm{gyr}[\mathbf{p}(0), \ominus\mathbf{a}]\mathbf{b}\\ &= \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}]\mathbf{b}\\ &= \mathbf{b}\end{aligned} \tag{6.245}$$

Hence, initially, at "time" $t = 0$ the gyrated $\mathbf{b}$, $\mathbf{b}_t$, coincides with $\mathbf{b}$. Interpreting $\mathbf{b}$ as a Euclidean vector in $\mathbb{R}^2$ and the gyroline L as a Euclidean circular arc in $\mathbb{R}^2$, Fig. 6.7 shows that the Euclidean vector $\mathbf{b}$ is Euclidean parallel to the Euclidean tangent line of the Euclidean circular arc L at the point $\mathbf{p}(0) = \mathbf{a}$. In this sense we say that $\mathbf{b}$ is pointing in the direction of the gyroline (6.242), in full analogy with its Euclidean counterpart.

The gyration that gyrates the point

$$\mathrm{gyr}[\mathbf{p}(t_1), \ominus\mathbf{a}]\mathbf{b} \tag{6.246}$$

to the point

$$\mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{a}]\mathbf{b} \tag{6.247}$$

shown in Fig. 6.7, is clearly given by

$$\begin{aligned}\mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{a}]\mathrm{gyr}^{-1}[\mathbf{p}(t_1), \ominus\mathbf{a}] &= \mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{a}]\mathrm{gyr}[\ominus\mathbf{a}, \mathbf{p}(t_1)] \\ &= \mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{a}]\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{p}(t_1)] \\ &= \mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{p}(t_1)]\end{aligned} \tag{6.248}$$

In the chain of equations (6.248) we employ the Gyroline Gyration Transitive Law, Theorem 6.29, noting that the points $\mathbf{a}$, $\mathbf{p}(t_1)$ and $\mathbf{p}(t_2)$ are cogyrolinear, lying on the cogyroline L.

Similarly, the gyration that gyrates the point

$$\mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{a}]\mathbf{b} \tag{6.249}$$

to the point

$$\mathrm{gyr}[\mathbf{p}(t_3), \ominus\mathbf{a}]\mathbf{b} \tag{6.250}$$

shown in Fig. 6.7, is given by

$$\mathrm{gyr}[\mathbf{p}(t_3), \ominus\mathbf{p}(t_2)] \tag{6.251}$$

By the gyroline gyration transitive law, the composition of the gyroautomorphism (6.248) followed by the gyroautomorphism (6.251) is, again, a gyroautomorphism,

$$\mathrm{gyr}[\mathbf{p}(t_3), \ominus\mathbf{p}(t_2)]\mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{p}(t_1)] = \mathrm{gyr}[\mathbf{p}(t_3), \ominus\mathbf{p}(t_1)] \tag{6.252}$$

gyrating the point

$$\mathrm{gyr}[\mathbf{p}(t_1), \ominus\mathbf{a}]\mathbf{b} \tag{6.253}$$

to the point

$$\mathrm{gyr}[\mathbf{p}(t_3), \ominus\mathbf{a}]\mathbf{b} \tag{6.254}$$

shown in Fig. 6.7.

Noting that $\mathbf{a} = \mathbf{p}(0)$, gyroautomorphisms of the form $\mathrm{gyr}[\mathbf{p}(t_2), \ominus\mathbf{p}(t_1)]$ along with Fig. 6.7 illustrate symbolically and visually the gyroline gyration transitive law. The parametric gyration of the point $\mathbf{b}$,

$$\mathrm{gyr}[\mathbf{p}(t), \ominus\mathbf{p}(t_0)]\mathbf{b} \tag{6.255}$$

with the parameter $t \in \mathbb{R}$ and any fixed real number t_0, describes the slide of the Euclidean tangent line along the Euclidean circular arc L with the variation of t. The slide (6.255) of the tangent line is shown in Fig. 6.7 for four values, $t_1, 0, t_2, t_3$, of the parameter t in the special case when $t_0 = 0$, for which

$$\mathbf{p}(t_0) = \mathbf{p}(0) = \mathbf{a} \tag{6.256}$$

6.15 Turning the Möbius Gyrometric into the Poincaré Metric

A Möbius gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$ is a gyrometric space with gyrometric given by the Möbius gyrodistance function, Def. 6.8,

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} \ominus_M \mathbf{a}\| \tag{6.257}$$

satisfying the gyrotriangle inequality, Theorem 6.9, (3.133), and (3.147),

$$\begin{aligned}
d(\mathbf{a}, \mathbf{c}) &\le d(\mathbf{a}, \mathbf{b}) \oplus_M d(\mathbf{b}, \mathbf{c}) \\
&= \frac{d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})}{1 + \frac{1}{s^2} d(\mathbf{a}, \mathbf{b}) d(\mathbf{b}, \mathbf{c})} \\
&= s \frac{\tanh \phi_{\mathbf{b}\ominus_M \mathbf{a}} + \tanh \phi_{\mathbf{c}\ominus_M \mathbf{b}}}{1 + \tanh \phi_{\mathbf{b}\ominus_M \mathbf{a}} \tanh \phi_{\mathbf{c}\ominus_M \mathbf{b}}} \\
&= s \tanh(\phi_{\mathbf{b}\ominus_M \mathbf{a}} + \phi_{\mathbf{c}\ominus_M \mathbf{b}})
\end{aligned} \tag{6.258}$$

In the special case when the real inner product space $\mathbb{V}_s$ is realized by the complex open unit disc $\mathbb{D}$, (3.113), Möbius gyrodistance function (6.257) reduces to

$$d(a, b) = |a \ominus_M b| = \left| \frac{a - b}{1 - \bar{a}b} \right| \tag{6.259}$$

$a, b \in \mathbb{D}$, as we see from (3.115). Möbius gyrometric (6.259) of the complex open unit disc $\mathbb{D}$ leads to the well-known Poincaré metric of the ball, as we will see below.

Noting that, by (3.147),

$$d(\mathbf{a}, \mathbf{c}) = \|\mathbf{c} \ominus_{M} \mathbf{a}\| = s \tanh \phi_{\mathbf{c} \ominus_{M} \mathbf{a}} \tag{6.260}$$

etc., we have from (6.258),

$$\tanh \phi_{\mathbf{c} \ominus_{M} \mathbf{a}} \leq \tanh(\phi_{\mathbf{b} \ominus_{M} \mathbf{a}} + \phi_{\mathbf{c} \ominus_{M} \mathbf{b}}) \tag{6.261}$$

or, equivalently,

$$\phi_{\mathbf{c} \ominus_{M} \mathbf{a}} \leq \phi_{\mathbf{b} \ominus_{M} \mathbf{a}} + \phi_{\mathbf{c} \ominus_{M} \mathbf{b}} \tag{6.262}$$

where, by (3.147),

$$\phi_{\mathbf{b} \ominus_{M} \mathbf{a}} = \tanh^{-1} \frac{\|\mathbf{b} \ominus_{M} \mathbf{a}\|}{s} \tag{6.263}$$

is the rapidity of $\mathbf{b} \ominus_{M} \mathbf{a}$, etc.

Inequality (6.262) suggests the introduction of the Möbius distance function

$$\begin{aligned} h(\mathbf{a}, \mathbf{b}) &= \tanh^{-1} \frac{d(\mathbf{a}, \mathbf{b})}{s} \\ &= \tfrac{1}{2} \ln \tfrac{s + d(\mathbf{a}, \mathbf{b})}{s - d(\mathbf{a}, \mathbf{b})} \end{aligned} \tag{6.264}$$

Möbius distance function (6.264) turns the gyrotriangle inequality (6.258) into a corresponding triangle inequality (6.262) that, by (6.263) – (6.264), takes the form

$$h(\mathbf{a}, \mathbf{c}) \leq h(\mathbf{a}, \mathbf{b}) + h(\mathbf{b}, \mathbf{c}) \tag{6.265}$$

Accordingly, (6.264) takes the Möbius gyrometric $d(\mathbf{a}, \mathbf{b})$ into the Möbius metric $h(\mathbf{a}, \mathbf{b})$, the latter being a generalization of the well-known Poincaré distance function of the open complex unit disc [Krantz (1990), p. 53][Goebel and Reich (1984), pp. 65–66].

Möbius gyrometric (6.257) of a Möbius gyrovector space $(\mathbb{V}_s, \oplus_{M}, \otimes_{M})$ and its gyrotriangle inequality (6.258) are equivalent to Poincaré metric (6.264) of the Möbius gyrovector space and its triangle inequality (6.265). From the viewpoint of gyrovector spaces, the advantage of the gyrometric over the metric of Möbius gyrovector spaces rests on the analogies that the former shares with vector spaces. Owing to these analogies, the gyrometric

$\|\mathbf{b}\ominus_{\rm M}\mathbf{a}\|$, (6.257), and its associated gyrotriangle inequality on the first row of (6.258) appear natural from the perspective of inhabitants of the Möbius gyrovector space $(\mathbb{R}^3_s, \oplus_{\rm M}, \otimes_{\rm M})$ just as the metric $\|\mathbf{b}-\mathbf{a}\|$ and its associated triangle inequality appear natural from the perspective of inhabitants of the Euclidean vector space $(\mathbb{R}^3, +, \cdot)$ where we live.

Remark 6.81 *Like Möbius gyrometric, also Möbius cogyrometric can be expressed in terms of rapidities. This, however, will not give rise to a corresponding triangle inequality. Unlike Möbius gyrometric, Möbius cogyrometric does not obey the gyrotriangle inequality but, rather, the cogyrotriangle inequality, Theorem 6.11. The latter, in turn, involves a gyration correction just as the gyroassociative and the gyrocommutative laws involve gyration corrections. The presence of a gyration correction in the cogyrotriangle inequality does not allow its reduction to a triangle inequality. Thus, Möbius gyrometric is a gyroconcept that captures into gyroformalism the classical concept of the metric. As a result, for instance, the concept of gyrogeodesics in the Möbius gyrovector space $(\mathbb{V}_s, \oplus_M, \otimes_M)$ coincides with that of geodesics in the Poincaré ball model of hyperbolic geometry, as we will see in Sec. 7.3. Contrastingly, Möbius cogyrometric captures new objects called cogyrogeodesics, which do not have a classical counterpart. Interestingly, gyrogeodesics and cogyrogeodesics share remarkable duality symmetries that gyroformalism captures; see for instance, (i) Theorem 2.21, (ii) Theorems 2.29 – 2.30, and (iii) Theorems 6.29 and 6.62.*

6.16 Einstein Gyrovector Spaces

Einstein gyrogroups $(\mathbb{V}_s, \oplus_{\rm E})$ admit scalar multiplication $\otimes_{\rm E}$, turning them into Einstein gyrovector spaces $(\mathbb{V}_s, \oplus_{\rm E}, \otimes_{\rm E})$.

Definition 6.82 **(Einstein Scalar Multiplication).** *Let $(\mathbb{V}_s, \oplus_E)$ be a Möbius gyrogroup. The Möbius scalar multiplication $r\otimes_E\mathbf{v} = \mathbf{v}\otimes_E r$ in $\mathbb{V}_s$ is given by the equation*

$$\begin{aligned} r\otimes_E\mathbf{v} &= s\frac{(1+\|\mathbf{v}\|/s)^r - (1-\|\mathbf{v}\|/s)^r}{(1+\|\mathbf{v}\|/s)^r + (1-\|\mathbf{v}\|/s)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s\tanh(r\,\tanh^{-1}\frac{\|\mathbf{v}\|}{s})\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{6.266}$$

where $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{V}_s$, $\mathbf{v}\neq\mathbf{0}$; and $r\otimes_E\mathbf{0} = \mathbf{0}$.

Interestingly, the scalar multiplication that Möbius and Einstein addition admit coincide. This stems from the fact that for parallel vectors in the ball, Möbius addition and Einstein addition coincide as well.

Einstein scalar multiplication can also be written in terms of the gamma factor (3.129) as

$$r\otimes_{\mathrm{E}}\mathbf{v} = \frac{1-(\gamma_{\mathbf{v}}-\sqrt{\gamma_{\mathbf{v}}^2-1})^{2r}}{1+(\gamma_{\mathbf{v}}-\sqrt{\gamma_{\mathbf{v}}^2-1})^{2r}}\frac{\gamma_{\mathbf{v}}}{\sqrt{\gamma_{\mathbf{v}}^2-1}}\mathbf{v} \tag{6.267}$$

$\mathbf{v}\neq\mathbf{0}$.

As an example, the Einstein half is given by the equation

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \tag{6.268}$$

so that, accordingly,

$$\begin{aligned} 2\otimes(\tfrac{1}{2}\otimes\mathbf{v}) &= 2\otimes\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} \\ &= \mathbf{v} \end{aligned} \tag{6.269}$$

as expected from the scalar associative law of gyrovector spaces.

The gamma factor of $r\otimes\mathbf{v}$ is expressible in terms of the gamma factor of $\mathbf{v}$ by the identity

$$\gamma_{r\otimes_{\mathrm{E}}\mathbf{v}} = \frac{1}{2}\gamma_{\mathbf{v}}^r\left\{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r+\left(1-\frac{\|\mathbf{v}\|}{s}\right)^r\right\} \tag{6.270}$$

and hence, by (6.266),

$$\gamma_{r\otimes_{\mathrm{E}}\mathbf{v}}(r\otimes_{\mathrm{E}}\mathbf{v}) = \frac{1}{2}\gamma_{\mathbf{v}}^r\left\{\left(1+\frac{\|\mathbf{v}\|}{s}\right)^r-\left(1-\frac{\|\mathbf{v}\|}{s}\right)^r\right\}\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{6.271}$$

for $\mathbf{v}\neq\mathbf{0}$. The special case of $r=2$ is of particular interest,

$$\gamma_{2\otimes_{\mathrm{E}}\mathbf{v}}(2\otimes_{\mathrm{E}}\mathbf{v}) = 2\gamma_{\mathbf{v}}^2\mathbf{v} \tag{6.272}$$

Noting the identity

$$\frac{\|\mathbf{v}\|^2}{s^2} = \frac{\gamma_{\mathbf{v}}^2-1}{\gamma_{\mathbf{v}}^2} \tag{6.273}$$

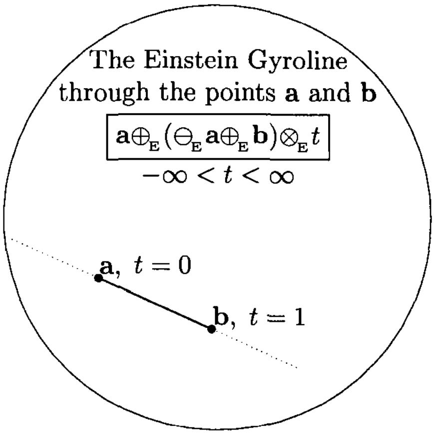


Fig. 6.8 The unique gyroline in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_E, \otimes_E)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the Einstein gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_{s=1}$ is the real open unit disc, is shown graphically.

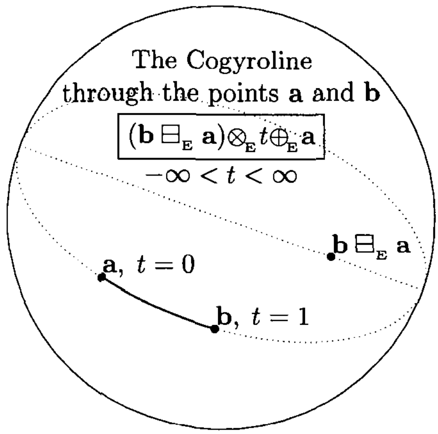


Fig. 6.9 The unique cogyroline in $(\mathbb{V}_s, \oplus_E, \otimes_E)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the Einstein gyrovector plane, when $\mathbb{V}_s = \mathbb{R}^2_{s=1}$ is the real open unit disc, is shown graphically.

we have from (6.266),

$$2\otimes_{E}\mathbf{v} = \frac{2\gamma_{\mathbf{v}}^2}{2\gamma_{\mathbf{v}}^2 - 1}\mathbf{v} \tag{6.274}$$

so that

$$\begin{aligned}\gamma_{2\otimes_{E}\mathbf{v}} &= 2\gamma_{\mathbf{v}}^2 - 1\\ &= \frac{1+\|\mathbf{v}\|^2/s^2}{1-\|\mathbf{v}\|^2/s^2}\end{aligned} \tag{6.275}$$

The unique Einstein gyroline $L^{g}_{\mathbf{ab}}$ and cogyroline $L^{c}_{\mathbf{ab}}$ that pass through two given points $\mathbf{a}$ and $\mathbf{b}$ are represented by the equations

$$\begin{aligned}L^{g}_{\mathbf{ab}} &= \mathbf{a}\oplus_{E}(\ominus_{E}\mathbf{a}\oplus_{E}\mathbf{b})\otimes_{E}t\\ L^{c}_{\mathbf{ab}} &= (\mathbf{b}\boxminus_{E}\mathbf{a})\otimes_{E}t\oplus_{E}\mathbf{a}\end{aligned} \tag{6.276}$$

$t\in\mathbb{R}$, in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_E, \otimes_E)$. Gyrolines in an Einstein gyrovector space coincide with the well-known geodesics of the Beltrami (also known as Klein) ball model of hyperbolic geometry, as we will prove in Sec. 7.5. Einstein gyrolines in the disc are Euclidean straight lines,

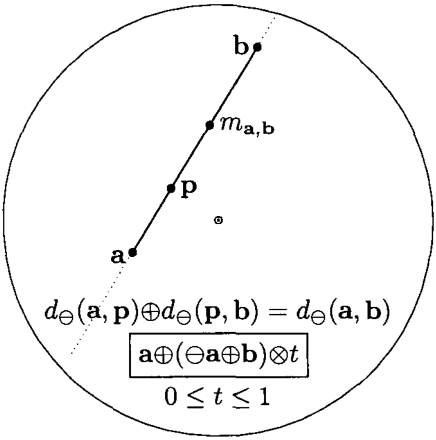


Fig. 6.10 The gyrosegment that links the two points **a** and **b** in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. **p** is a generic point between **a** and **b**, and $\mathbf{m}_{\mathbf{a},\mathbf{b}}$ is the gyromidpoint of **a** and **b**. See also Fig. 6.3.

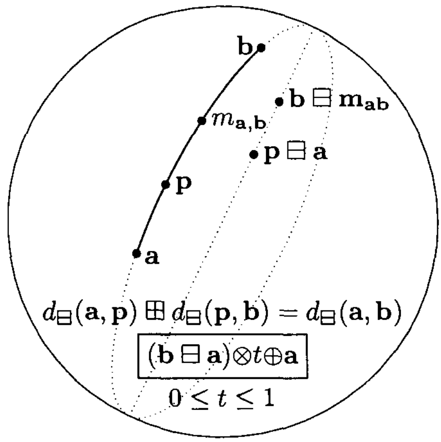


Fig. 6.11 The cogyrosegment that links the two points **a** and **b** in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. **p** is a generic point cobetween **a** and **b**, and $m_{\mathbf{a},\mathbf{b}}$ is the cogyromidpoint of **a** and **b**. See also Fig. 6.4.

Fig. 6.8. In contrast, Einstein cogyrolines in the disc are Euclidean elliptical arcs that intersect the boundary of the disc diametrically, that is, on the opposite sides of a diameter, that is, the supporting gyrodiameter, Fig. 6.9.

Figure 6.10 presents a gyrosegment **ab** in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_E, \otimes)$ along with its gyromidpoint $\mathbf{m}_{\mathbf{ab}}$, and a generic point **p** lying between **a** and **b**. Since the points $\mathbf{a}, \mathbf{p}, \mathbf{b}$ are gyrocollinear, they satisfy the gyrotriangle equality, Theorem 6.47, shown in the figure.

Figure 6.11 presents a cogyrosegment **ab** in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_E, \otimes)$ along with its cogyromidpoint $\mathbf{m}_{\mathbf{ab}}$, and a generic point **p** lying cobetween **a** and **b**. Since the points $\mathbf{a}, \mathbf{p}, \mathbf{b}$ are cogyrocollinear, they satisfy the cogyrotriangle equality, Theorem 6.74, shown in the figure. Moreover, since the points $\mathbf{a}, \mathbf{p}, \mathbf{m}_{\mathbf{ab}}, \mathbf{b}$ lie on the cogyroline, the cogyrodifferences $\mathbf{p} \boxminus \mathbf{a}$ and $\mathbf{b} \boxminus \mathbf{m}_{\mathbf{ab}}$, for instance, lie on the cogyroline supporting gyrodiameter, as expected from Theorem 6.68 and Remark 6.69.

6.17 Turning Einstein Gyrometric into a Metric

Since Einstein and Möbius addition of parallel vectors in $\mathbb{V}_s$ coincide, the present section is similar to Sec. 6.15.

An Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\text{E}}, \otimes_{\text{E}})$ is a gyrometric space with gyrometric given by the Einstein gyrodistance function, Def. 6.8,

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} \ominus_{\text{E}} \mathbf{a}\| \tag{6.277}$$

satisfying the gyrotriangle inequality, Theorem 6.9, (3.133) and (3.147),

$$\begin{aligned} d(\mathbf{a}, \mathbf{c}) &\le d(\mathbf{a}, \mathbf{b}) \oplus_{\text{E}} d(\mathbf{b}, \mathbf{c}) \\ &= \frac{d(\mathbf{a}, \mathbf{b}) + d(\mathbf{b}, \mathbf{c})}{1 + \frac{1}{s^2} d(\mathbf{a}, \mathbf{b}) d(\mathbf{b}, \mathbf{c})} \\ &= s \frac{\tanh \phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} + \tanh \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{b}}}{1 + \tanh \phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} \tanh \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{b}}} \\ &= s \tanh(\phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} + \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{b}}) \end{aligned} \tag{6.278}$$

Noting that, by (3.147),

$$d(\mathbf{a}, \mathbf{c}) = \|\mathbf{c} \ominus_{\text{E}} \mathbf{a}\| = s \tanh \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{a}} \tag{6.279}$$

etc., we have from (6.278),

$$\tanh \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{a}} \le \tanh(\phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} + \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{b}}) \tag{6.280}$$

or, equivalently,

$$\phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{a}} \le \phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} + \phi_{\mathbf{c} \ominus_{\text{E}} \mathbf{b}} \tag{6.281}$$

where, by (3.147),

$$\phi_{\mathbf{b} \ominus_{\text{E}} \mathbf{a}} = \tanh^{-1} \frac{\|\mathbf{b} \ominus_{\text{E}} \mathbf{a}\|}{s} \tag{6.282}$$

is the rapidity of $\mathbf{b} \ominus_{\text{E}} \mathbf{a}$, etc.

Inequality (6.281) suggests the introduction of the Einstein distance function

$$\begin{aligned} h(\mathbf{a}, \mathbf{b}) &= \tanh^{-1} \frac{d(\mathbf{a}, \mathbf{b})}{s} \\ &= \tfrac{1}{2} \ln \tfrac{s + d(\mathbf{a}, \mathbf{b})}{s - d(\mathbf{a}, \mathbf{b})} \end{aligned} \tag{6.283}$$

Einstein distance function (6.283) turns the gyrotriangle inequality (6.278) into a corresponding triangle inequality (6.281) that, by (6.282)–(6.283),

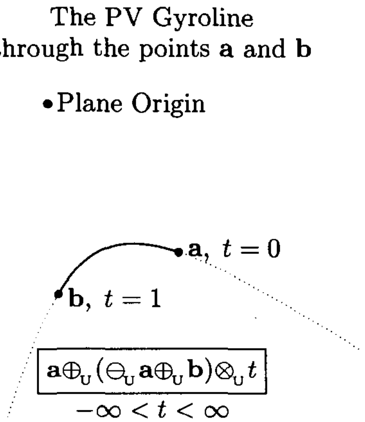


Fig. 6.12 The unique gyroline in a PV gyrovector space $(\mathbb{V}, \oplus_U, \otimes_U)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the PV gyrovector plane, when $\mathbb{V} = \mathbb{R}^2$ is the Euclidean plane, is shown. Interestingly, the gyroline asymptotes intersect at the origin.

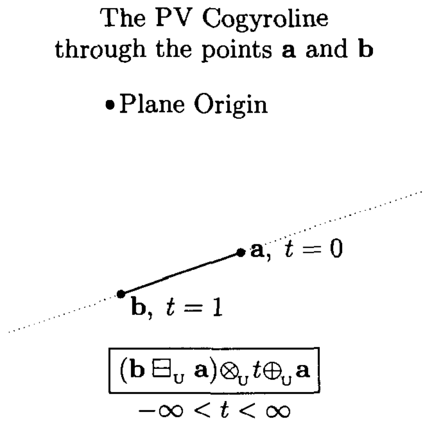


Fig. 6.13 The unique cogyroline in $(\mathbb{V}, \oplus_U, \otimes_U)$ through two given points $\mathbf{a}$ and $\mathbf{b}$. The case of the PV gyrovector plane, when $\mathbb{V} = \mathbb{R}^2$ is the Euclidean plane, is shown. Interestingly, the cogyroline is a Euclidean straight line.

takes the form

$$h(\mathbf{a}, \mathbf{c}) \le h(\mathbf{a}, \mathbf{b}) + h(\mathbf{b}, \mathbf{c}) \tag{6.284}$$

We thus see that the gyrometric (6.277) of an Einstein gyrovector space $(\mathbb{V}_s, \oplus_E, \otimes_E)$ and its gyrotriangle inequality (6.278) are equivalent to the metric (6.283) of the Einstein gyrovector space and its triangle inequality (6.284).

6.18 PV (Proper Velocity) Gyrovector Spaces

PV gyrogroups $(\mathbb{V}, \oplus_U)$ admit scalar multiplication $\otimes_U$, turning them into PV gyrovector spaces $(\mathbb{V}, \oplus_U, \otimes_U)$.

Definition 6.83 (PV Scalar Multiplication). *Let* $(\mathbb{V}, \oplus_E)$ *be a PV gyrogroup. The PV scalar multiplication* $r\otimes_U\mathbf{v} = \mathbf{v}\otimes_U r$ *in* $\mathbb{V}$ *is given by the* *equation*

$$r \otimes_U \mathbf{v} = \frac{s}{2} \left\{ \left(\sqrt{1 + \frac{\|\mathbf{v}\|^2}{s^2}} + \frac{\|\mathbf{v}\|}{s} \right)^r - \left(\sqrt{1 + \frac{\|\mathbf{v}\|^2}{s^2}} - \frac{\|\mathbf{v}\|}{s} \right)^r \right\} \frac{\mathbf{v}}{\|\mathbf{v}\|}$$
$$= s \sinh \left(r \sinh^{-1} \frac{\|\mathbf{v}\|}{s} \right) \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{6.285}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}$, $\mathbf{v} \neq \mathbf{0}$; *and* $r \otimes_U \mathbf{0} = \mathbf{0}$.

The unique PV gyroline $L^g_{\mathbf{ab}}$ and cogyroline $L^c_{\mathbf{ab}}$ that pass through two given points $\mathbf{a}$ and $\mathbf{b}$ are represented by the equations

$$\begin{aligned} L^g_{\mathbf{ab}} &= \mathbf{a} \oplus_U (\ominus_U \mathbf{a} \oplus_U \mathbf{b}) \otimes_U t \\ L^c_{\mathbf{ab}} &= (\mathbf{b} \boxminus_U \mathbf{a}) \otimes_U t \oplus_U \mathbf{a} \end{aligned} \tag{6.286}$$

$t \in \mathbb{R}$, in a PV gyrovector space $(\mathbb{V}, \oplus_U, \otimes_U)$. PV gyrolines in the space $\mathbb{V}$ are Euclidean hyperbolas with asymptotes that intersect at the origin of the space $\mathbb{V}$, Fig. 6.12. In contrast, PV cogyrolines in the space are Euclidean straight lines, Fig. 6.13.

Let $\mathbf{c}$ be a point *not on* the cogyroline $L^c_{\mathbf{ab}}$ that passes through the two given points $\mathbf{a}$ and $\mathbf{b}$ in a PV gyrovector space $(\mathbb{V}, \oplus_U, \otimes_U)$, Fig. 6.14. In order to find a point $\mathbf{d} \in \mathbb{V}$ such that the resulting cogyroline that passes through the points $\mathbf{c}$ and $\mathbf{d}$,

$$L^c_{\mathbf{cd}} = (\mathbf{d} \boxminus_U \mathbf{c}) \otimes_U t \oplus_U \mathbf{c} \tag{6.287}$$

is parallel to the given cogyroline $L^c_{\mathbf{ab}}$, we impose on $\mathbf{d}$ the condition

$$\mathbf{d} \boxminus_U \mathbf{c} = \mathbf{b} \boxminus_U \mathbf{a} \tag{6.288}$$

that follows from Def. 6.63 as the parallelism condition.

Solving (6.288) for $\mathbf{d}$ by a right cancellation we have

$$\mathbf{d} = (\mathbf{b} \boxminus_U \mathbf{a}) \oplus_U \mathbf{c} \tag{6.289}$$

thus determining the unique cogyroline $L^c_{\mathbf{cd}}$, in (6.287), that passes through the given point $\mathbf{c}$ and is parallel to the given cogyroline $L^c_{\mathbf{ab}}$.

Since cogyrolines in PV gyrovector spaces are Euclidean straight lines, parallelism in these gyrovector spaces coincides with Euclidean parallelism, as shown in Fig. 6.15 with the cogyrolines $L^c_{\mathbf{ab}}$ and $L^c_{\mathbf{cd}}$ of (6.286) and (6.287).

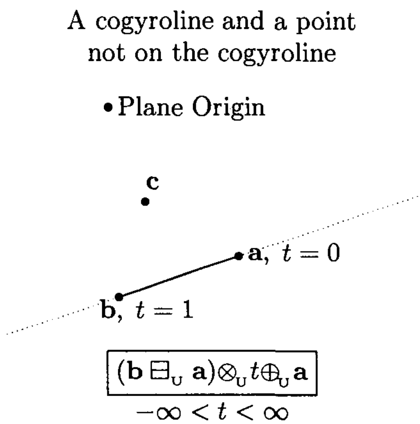


Fig. 6.14 A cogyroline $L^c_{\mathbf{ab}}$ and a point $\mathbf{c}$ not on the cogyroline in a PV gyrovector plane $(\mathbb{R}^2, \oplus_U, \otimes_U)$.

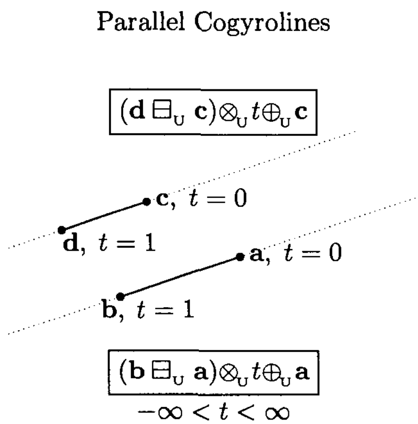


Fig. 6.15 The unique cogyroline $L^c_{\mathbf{cd}}$ that passes through the given point and is parallel to the given cogyroline in Fig. 6.14.

6.19 Gyrovector Space Isomorphism

Definition 6.84 (Gyrovector Space Isomorphism). *An isomorphism from a gyrovector space $(G_1, \oplus_1, \otimes_1)$ to a gyrovector space $(G_2, \oplus_2, \otimes_2)$ is a bijective map*

$$\phi_{21} : G_1 \to G_2, \qquad \mathbf{u}_1 \mapsto \phi_{21}\mathbf{u}_1 = \mathbf{u}_2, \qquad \mathbf{v}_1 \mapsto \phi_{21}\mathbf{v}_1 = \mathbf{v}_2$$

that preserves the vector space operation,

$$\phi_{21}(\mathbf{u}_1 \oplus_1 \mathbf{v}_1) = \phi_{21}\mathbf{u}_1 \oplus_2 \phi_{21}\mathbf{v}_1 = \mathbf{u}_2 \oplus_2 \mathbf{v}_2 \tag{6.290}$$

and scalar multiplication,

$$\phi_{21}(r \otimes_1 \mathbf{v}_1) = r \otimes_2 \phi_{21}\mathbf{v}_1 = r \otimes_2 \mathbf{v}_2 \tag{6.291}$$

and that keeps the inner product of unit gyrovectors invariant,

$$\frac{\phi_{21}\mathbf{u}}{\|\phi_{21}\mathbf{u}\|} \cdot \frac{\phi_{21}\mathbf{v}}{\|\phi_{21}\mathbf{v}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{6.292}$$

for all nonzero $\mathbf{u}, \mathbf{v} \in G_1$.

An isomorphism preserves gyrations and cooperation as well. To see this, let us recall the gyration applications in G_1 and G_2, Theorem 2.8

(10),

$$\begin{aligned} \mathrm{gyr}_1[\mathbf{u}_1,\mathbf{v}_1]\mathbf{w}_1 &= \ominus_1(\mathbf{u}_1\oplus_1\mathbf{v}_1)\oplus_1\{\mathbf{u}_1\oplus_1(\mathbf{v}_1\oplus_1\mathbf{w}_1)\} \\ \mathrm{gyr}_2[\mathbf{u}_2,\mathbf{v}_2]\mathbf{w}_2 &= \ominus_2(\mathbf{u}_2\oplus_2\mathbf{v}_2)\oplus_2\{\mathbf{u}_2\oplus_2(\mathbf{v}_2\oplus_2\mathbf{w}_2)\} \end{aligned} \tag{6.293}$$

We clearly have

$$\begin{aligned} \phi_{21}\mathrm{gyr}_1[\mathbf{u}_1,\mathbf{v}_1]\mathbf{w}_1 &= \phi_{21}[\ominus_1(\mathbf{u}_1\oplus_1\mathbf{v}_1)\oplus_1\{\mathbf{u}_1\oplus_1(\mathbf{v}_1\oplus_1\mathbf{w}_1)\}] \\ &= \ominus_2(\phi_{21}\mathbf{u}_1\oplus_2\phi_{21}\mathbf{v}_1)\oplus_2\{\phi_{21}\mathbf{u}_1\oplus_2(\phi_{21}\mathbf{v}_1\oplus_2\phi_{21}\mathbf{w}_1)\} \\ &= \ominus_2(\mathbf{u}_2\oplus_2\mathbf{v}_2)\oplus\{\mathbf{u}_2\oplus_2(\mathbf{v}_2\oplus_2\mathbf{w}_2)\} \\ &= \mathrm{gyr}_2[\mathbf{u}_2,\mathbf{v}_2]\mathbf{w}_2 \end{aligned} \tag{6.294}$$

and

$$\begin{aligned} \phi_{21}(\mathbf{u}_1 \boxplus_1 \mathbf{v}_1) &= \phi_{21}(\mathbf{u}_1\oplus_1\mathrm{gyr}_1[\mathbf{u}_1,\ominus_1\mathbf{v}_1]\mathbf{v}_1) \\ &= \phi_{21}\mathbf{u}_1\oplus_2\phi_{21}\mathrm{gyr}_1[\mathbf{u}_1,\ominus_1\mathbf{v}_1]\mathbf{v}_1 \\ &= \mathbf{u}_2\oplus_2\mathrm{gyr}_2[\mathbf{u}_2,\ominus_2\mathbf{v}_2]\mathbf{v}_2 \\ &= \mathbf{u}_2 \boxplus_2 \mathbf{u}_2 \end{aligned} \tag{6.295}$$

The isomorphism inverse to ϕ_{21} is denoted ϕ_{12}. The isomorphisms between the gyrovector spaces of Möbius, Einstein and PV are presented in Table 6.1.

Table 6.1 Gyrovector Space Isomorphisms.

Gyrogroup Isomorphism	Formula
$\phi_{\mathrm{UE}} : (\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}}) \to (\mathbb{V}, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}})$	$\phi_{\mathrm{UE}} : \mathbf{v} \mapsto \gamma_{\mathbf{v}}\mathbf{v}$
$\phi_{\mathrm{EU}} : (\mathbb{V}, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}}) \to (\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$	$\phi_{\mathrm{EU}} : \mathbf{v} \mapsto \beta_{\mathbf{v}}\mathbf{v}$
$\phi_{\mathrm{ME}} : (\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}}) \to (\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$	$\phi_{\mathrm{ME}} : \mathbf{v} \mapsto \frac{1}{2}\otimes_{\mathrm{E}}\mathbf{v}$
$\phi_{\mathrm{EM}} : (\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}) \to (\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$	$\phi_{\mathrm{EM}} : \mathbf{v} \mapsto 2\otimes_{\mathrm{M}}\mathbf{v}$
$\phi_{\mathrm{UM}} : (\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}) \to (\mathbb{V}, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}})$	$\phi_{\mathrm{UM}} : \mathbf{v} \mapsto 2\gamma_{\mathbf{v}}^2\mathbf{v}$
$\phi_{\mathrm{MU}} : (\mathbb{V}, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}}) \to (\mathbb{V}_s, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$	$\phi_{\mathrm{MU}} : \mathbf{v} \mapsto \frac{\beta_{\mathbf{v}}}{1+\beta_{\mathbf{v}}}\mathbf{v}$

The isomorphism between Einstein addition $\oplus_{\mathrm{E}}$ and Möbius addition $\oplus_{\mathrm{M}}$ in the ball $\mathbb{V}_c$ is surprisingly simple when expressed in the language of gyrovector spaces. As we see from Table 6.1, the relationship between $\oplus_{\mathrm{E}}$

and $\oplus_{_M}$ is given by the equations

$$\begin{aligned} \mathbf{u}\oplus_{_E}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{_M}\tfrac{1}{2}\otimes\mathbf{v}) \\ \mathbf{u}\oplus_{_M}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{_M}2\otimes\mathbf{v}) \end{aligned} \tag{6.296}$$

for all $\mathbf{u},\mathbf{v} \in \mathbb{V}_c$. The related connection between Möbius transformation and Lorentz transformation of Einstein's special theory of relativity was recognized by H. Liebmann in 1905 [Needham (1997), Chap. 3].

As an illustration, the derivation of the isomorphisms $\phi_{\rm UM}$ and $\phi_{\rm MU}$ of Table 6.1 from the isomorphisms $\phi_{\rm UE}$, $\phi_{\rm EU}$, $\phi_{\rm ME}$, and $\phi_{\rm EM}$ follows. Noting (6.272) and (6.235), and the definition of the factors β and γ in (3.159) and (3.129), we have

$$\phi_{\rm UM} = \phi_{\rm UE}\phi_{\rm EM} : \mathbf{v} \mapsto 2\otimes_{_E}\mathbf{v} \mapsto \gamma_{2\otimes_{_E}\mathbf{v}}2\otimes_{_E}\mathbf{v} = 2\gamma_{\mathbf{v}}^2\mathbf{v} \tag{6.297}$$

and

$$\phi_{\rm MU} = \phi_{\rm ME}\phi_{\rm EU} : \mathbf{v} \mapsto \beta_{\mathbf{v}}\mathbf{v} \mapsto \tfrac{1}{2}\otimes_{_E}(\beta_{\mathbf{v}}\mathbf{v}) = \frac{\gamma_{\beta_{\mathbf{v}}\mathbf{v}}}{1+\gamma_{\beta_{\mathbf{v}}\mathbf{v}}}\beta_{\mathbf{v}}\mathbf{v} = \frac{\beta_{\mathbf{v}}}{1+\beta_{\mathbf{v}}}\mathbf{v} \tag{6.298}$$

Definition 6.85 (Gyrovector Space Models). *We say that two isomorphic gyrovector spaces are two equivalent models of the abstract gyrovector space.*

Accordingly, we see from Table 6.1 that Einstein, Möbius, and PV gyrovector spaces are three equivalent models of the abstract gyrovector space. Indeed, we will find in Chap. 7 that the Einstein, Möbius, and PV gyrovector spaces provide the setting for three models of the hyperbolic geometry of Bolyai and Lobachevsky. These are, respectively, the Poincaré ball model, the Beltrami (or, Klein) ball model, and the PV space model.

6.20 Gyrotriangle Gyromedians and Gyrocentroids

The gyromidpoint $\mathbf{p}_{\mathbf{uv}}^m$ of two points $\mathbf{u}$ and $\mathbf{v}$ of the abstract gyrovector space $(G,\oplus,\otimes)$ is given by, (6.91),

$$\mathbf{p}_{\mathbf{uv}}^m = \tfrac{1}{2}\otimes(\mathbf{u} \boxplus \mathbf{v}) \tag{6.299}$$

Definition 6.86 (Gyrotriangle Gyromedians, Gyrocentroids). *The gyrosegment connecting the gyromidpoint of a side of a gyrotriangle with its opposite vertex, Fig. 6.16, is called a gyromedian. The point of*

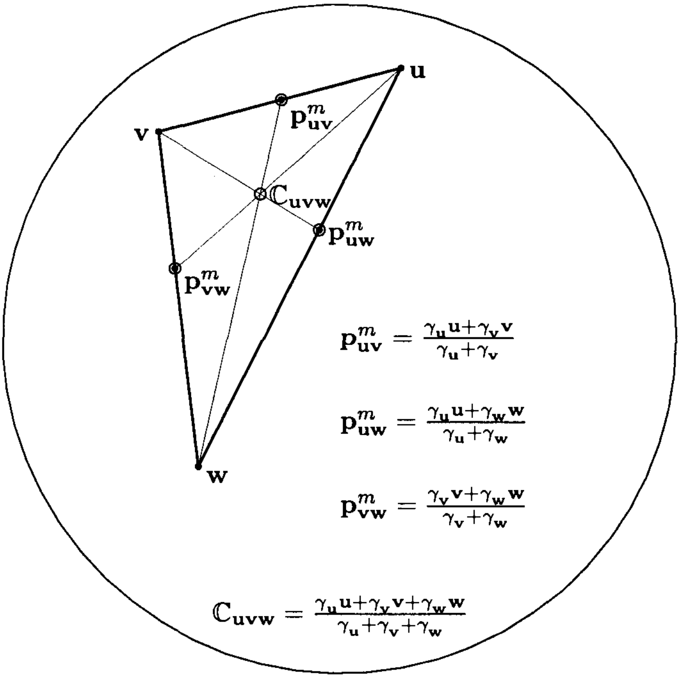


Fig. 6.16 The gyromidpoints of the three sides of a gyrotriangle **uvw** in the Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus_{\rm E}, \otimes_{\rm E})$ are shown along with its gyromedians and gyrocentroid. Interestingly, Einsteinian gyromidpoints and gyrocentroids have interpretation in relativistic mechanics, fully analogous to the interpretation of Euclidean midpoints and centroids in classical mechanics that one discovers in the vector space approach to Euclidean geometry [Hausner (1998)]; see Fig. 10.3.

concurrency of the three gyrotriangle gyromedians is called the gyrotriangle gyrocentroid.

We will now study the gyrotriangle gyromedians and gyrocentroid in Einstein, Möbius, and PV gyrovector spaces [Ungar (2004a)].

6.20.1 *In Einstein Gyrovector Spaces*

Einstein gyrovector spaces $(\mathbb{V}_c, \oplus_{\rm E}, \otimes_{\rm E})$ are particularly suitable for the study of gyromedians and gyrocentroids since gyrolines are Euclidean straight lines so that the calculation of points of intersection of gyrolines can be performed by methods of linear algebra. The resulting determination of gyrocentroids in Einstein gyrovector spaces can readily be translated into other isomorphic gyrovector spaces.

It follows from (6.299), by (3.156) and the scalar associative law of gyrovector spaces, that the gyromidpoint $\mathbf{p}^m_{\mathbf{uv}}$ in Einstein gyrovector spaces is given by the equation

$$\begin{aligned}\mathbf{p}^m_{\mathbf{uv}} &= \tfrac{1}{2}\otimes_{\mathrm{E}}(\mathbf{u}\boxplus_{\mathrm{E}}\mathbf{v})\\ &= \tfrac{1}{2}\otimes_{\mathrm{E}}\{2\otimes_{\mathrm{E}}\tfrac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}}\}\\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}}\end{aligned} \tag{6.300}$$

We have thus obtained the following

Theorem 6.87 **(The Einstein Gyromidpoint).** *Let* $\mathbf{u},\mathbf{v}\in\mathbb{V}_c$ *be any two points of an Einstein gyrovector space* $(\mathbb{V}_c,\oplus_{E},\otimes_{E})$. *The gyromidpoint* $\mathbf{p}^m_{\mathbf{uv}}$ *of the gyrosegment* $\mathbf{uv}$ *is given by the equation*

$$\mathbf{p}^m_{\mathbf{uv}} = \frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}} \tag{6.301}$$

The gyromidpoints of the three sides of a gyrotriangle $\mathbf{uvw}$ in the Einstein gyrovector plane $(\mathbb{R}^2_c,\oplus_{\mathrm{E}},\otimes_{\mathrm{E}})$, and its gyrocentroid are shown in Fig. 6.16. Gyromidpoints are gyrocovariant under left gyrotranslations, Theorem 6.36. Therefore a left gyrotranslation by $\mathbf{x}\in\mathbb{R}^2_s$ of the gyrosegment $\mathbf{uv}$ in Fig. 6.16 into the gyrosegment $(\ominus\mathbf{x}\oplus\mathbf{u})(\ominus\mathbf{x}\oplus\mathbf{v})$ in Fig. 6.17 does not distort the gyrosegment gyromidpoint. Hence, the gyromidpoint of the gyrotranslated gyrosegment $(\ominus\mathbf{x}\oplus\mathbf{u})(\ominus\mathbf{x}\oplus\mathbf{v})$, Fig. 6.17, is given by the equation

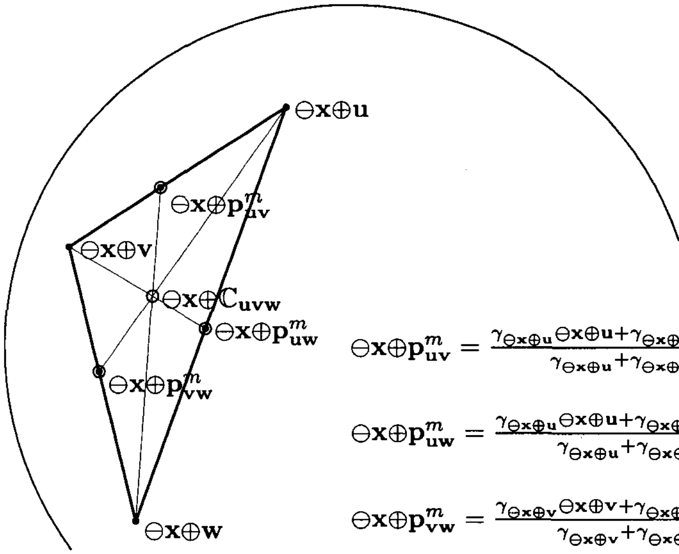
$$\ominus\mathbf{x}\oplus\mathbf{p}^m_{\mathbf{uv}} = \mathbf{p}^m_{(\ominus\mathbf{x}\oplus\mathbf{u})(\ominus\mathbf{x}\oplus\mathbf{v})} \tag{6.302}$$

thus uncovering the interesting identity

$$\ominus\mathbf{x}\oplus\frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}} = \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}} \tag{6.303}$$

in any Einstein gyrovector space.

Gyrolines in the Beltrami ball model of hyperbolic geometry, that is, in Einstein gyrovector spaces, are Euclidean straight lines. Hence, by elementary techniques of linear algebra, one can verify that the three gyromedians in Fig. 6.16 are concurrent, and that the point of concurrency, that is, the gyrocentroid $C_{\mathbf{uvw}}$ of gyrotriangle $\mathbf{uvw}$, is given by the elegant equation (6.304) that we place in the following



$$\ominus\mathbf{x}\oplus\mathbf{p}^m_{\mathbf{uv}} = \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}}$$

$$\ominus\mathbf{x}\oplus\mathbf{p}^m_{\mathbf{uw}} = \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}\ominus\mathbf{x}\oplus\mathbf{w}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}}$$

$$\ominus\mathbf{x}\oplus\mathbf{p}^m_{\mathbf{vw}} = \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}\ominus\mathbf{x}\oplus\mathbf{w}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}}$$

$$\ominus\mathbf{x}\oplus\mathbb{C}_{\mathbf{uvw}} = \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}\ominus\mathbf{x}\oplus\mathbf{w}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}+\gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}}$$

Fig. 6.17 A left gyrotranslation by $\mathbf{x}\in(\mathbb{R}^2_s,\oplus,\otimes)$ of the gyrotriangle **uvw** of Fig. 6.16 in an Einstein plane is shown. Gyromidpoints are gyrocovariant under left gyrotranslations, Theorem 6.36. Hence, a left gyrotranslation of a gyrotriangle does not distort its side-gyromidpoints and gyrocentroid. Accordingly, related formulas are left gyrotranslated gyrocovariantly.

Theorem 6.88 **(The Einstein Gyrotriangle Gyrocentroid).** *Let* $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}_c$ *be any three nongyrocollinear points of an Einstein gyrovector space* $(\mathbb{V}_c,\oplus_E,\otimes_E)$. *The gyrocentroid* $\mathbb{C}^m_{\mathbf{uvw}}$ *of the gyrotriangle* **uvw**, *Fig. 6.16, is given by the equation*

$$\mathbb{C}_{\mathbf{uvw}} = \frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}+\gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}}+\gamma_{\mathbf{v}}+\gamma_{\mathbf{w}}} \tag{6.304}$$

Gyromidpoints are gyrocovariant, Theorem 6.36. Therefore a left gyrotranslation by $\mathbf{x}\in\mathbb{R}^2_s$ of the gyrotriangle **uvw** in Fig. 6.16 into the gyrotriangle $(\ominus\mathbf{x}\oplus\mathbf{u})(\ominus\mathbf{x}\oplus\mathbf{v})(\ominus\mathbf{x}\oplus\mathbf{w})$ in Fig. 6.17 does not distort the gyrotriangle side gyromidpoints and gyrocentroid. Hence, the gyrocentroid of the gyrotranslated gyrotriangle $(\ominus\mathbf{x}\oplus\mathbf{u})(\ominus\mathbf{x}\oplus\mathbf{v})(\ominus\mathbf{x}\oplus\mathbf{w})$, Fig. 6.17, is given by the

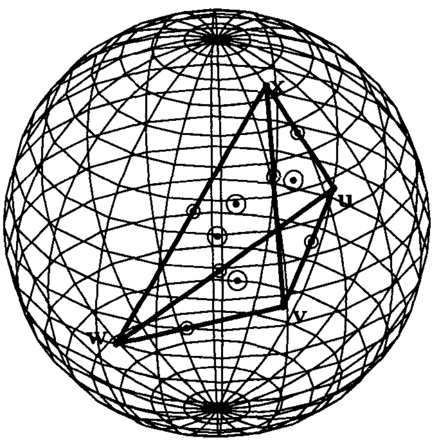


Fig. 6.18 The hyperbolic tetrahedron **uvwx**, that is, a gyrotetrahedron, is shown in the Einstein gyrovector space
$$\mathbb{R}_c^3 = (\mathbb{R}_c^3, \oplus_E, \otimes_E)$$
underlying the Beltrami ball model of hyperbolic geometry. The gyrotetrahedron **uvwx** is shown inside the c-ball $\mathbb{R}_c^3$ of the Euclidean 3-space $\mathbb{R}^3$ where it lives. The faces of the gyrotetrahedron are gyrotriangles.

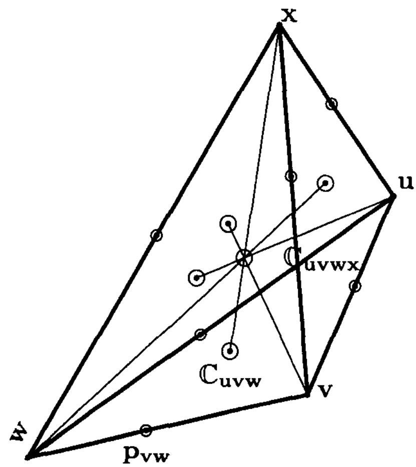


Fig. 6.19 Shown are also the gyromidpoints of the 6 sides and the gyrocentroids of the 4 faces. The gyroline joining a vertex of a gyrotetrahedron and the hyperbolic centroid, gyrocentroid, of the opposite face is called a gyrotetrahedron gyromedian. The four gyromedians of the gyrotetrahedron **uvwx** are concurrent. The point of concurrency is the gyrotetrahedron gyrocentroid $\mathbb{C}_{\mathbf{uvwx}}$.

equation

$$\begin{aligned}\ominus\mathbf{x}\oplus\mathbb{C}_{\mathbf{uvw}} &= \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}\ominus\mathbf{x}\oplus\mathbf{w}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{v}} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}} \\ &= \mathbb{C}_{\ominus\mathbf{x}\oplus\mathbf{u}\ominus\mathbf{x}\oplus\mathbf{v}\ominus\mathbf{x}\oplus\mathbf{w}}\end{aligned} \tag{6.305}$$

thus uncovering the interesting identity

$$\begin{aligned}&\ominus\mathbf{x}\oplus\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \\ &= \frac{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}}\ominus\mathbf{x}\oplus\mathbf{u} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{v}}\ominus\mathbf{x}\oplus\mathbf{v} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}\ominus\mathbf{x}\oplus\mathbf{w}}{\gamma_{\ominus\mathbf{x}\oplus\mathbf{u}} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{v}} + \gamma_{\ominus\mathbf{x}\oplus\mathbf{w}}}\end{aligned} \tag{6.306}$$

in any Einstein gyrovector space.

Euclidean triangle centroids have classical mechanical interpretation [Hausner (1998)]. Interestingly, we will see in Chap. 10 on Einstein's special theory of relativity that gyrocentroids in Einstein two and three-dimensional gyrovector spaces (or, equivalently, in the Beltrami disc and ball model of hyperbolic geometry) have analogous relativistic mechanical interpretation in which the gamma factor plays the role of the relativistic mass correction.

Further extension to n-dimensional gyrotetrahedrons in n-dimensional Einstein gyrovector spaces, $n \geq 3$, shown in Figs. 6.18 and 6.19 for $n = 3$, is obvious. Thus, for instance, the gyrocentroid $\mathbb{C}_{\mathbf{uvwx}}$ of the gyrotetrahedron $\mathbf{uvwx}$ in Fig. 6.19 is given by the equation

$$\mathbb{C}_{\mathbf{uvwx}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{x}}\mathbf{x}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{x}}} \tag{6.307}$$

6.20.2 *In Möbius Gyrovector Spaces*

Let

$$\begin{aligned} G_e &= (\mathbb{V}_c, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}}) \\ G_m &= (\mathbb{V}_c, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}}) \end{aligned} \tag{6.308}$$

be, respectively, the Einstein and the Möbius gyrovector spaces of the ball $\mathbb{V}_c$ of a real inner product space $\mathbb{V}$. They are gyrovector space isomorphic, with the isomorphism ϕ_{EM} and its inverse isomorphism ϕ_{ME} from G_m into G_e shown in Table 6.1. Accordingly, the correspondence between elements of G_m and G_e is given by the equations

$$\begin{aligned} \mathbf{v}_e &= 2\otimes_{\mathrm{M}}\mathbf{v}_m \\ \mathbf{v}_m &= \tfrac{1}{2}\otimes_{\mathrm{E}}\mathbf{v}_e \end{aligned} \tag{6.309}$$

$\mathbf{v}_e \in G_e$, $\mathbf{v}_m \in G_m$.

Hence, by (6.309) and the first identity in (6.275) we have

$$\gamma_{\mathbf{v}_e} = \gamma_{2\otimes_{\mathrm{M}}\mathbf{v}_m} = 2\gamma^2_{\mathbf{v}_m} - 1 \tag{6.310}$$

Similarly, following (6.309) and (6.272) we have

$$\gamma_{\mathbf{v}_e}\mathbf{v}_e = \gamma_{2\otimes_{\mathrm{M}}\mathbf{v}_m} 2\otimes_{\mathrm{M}}\mathbf{v}_m = 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m \tag{6.311}$$

Hence, by (6.301), (6.310) and (6.311) we have

$$\begin{aligned}
\mathbf{p}^m_{\mathbf{u}_e\mathbf{v}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e}} \\
&= \frac{2\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{(2\gamma^2_{\mathbf{u}_m} - 1) + (2\gamma^2_{\mathbf{v}_m} - 1)} \\
&= \frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} - 1}
\end{aligned} \tag{6.312}$$

so that

$$\begin{aligned}
\mathbf{p}^m_{\mathbf{u}_m\mathbf{v}_m} &= \tfrac{1}{2}\otimes_{\mathrm{E}}\mathbf{p}^m_{\mathbf{u}_e\mathbf{v}_e} \\
&= \tfrac{1}{2}\otimes_{\mathrm{E}}\frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} - 1}
\end{aligned} \tag{6.313}$$

We have thus obtained in (6.313) the following

Theorem 6.89 (The Möbius Gyromidpoint). *Let $\mathbf{u},\mathbf{v}\in\mathbb{V}_c$ be any two points of a Möbius gyrovector space $(\mathbb{V}_c,\oplus_M,\otimes_M)$. The gyromidpoint $\mathbf{p}^m_{\mathbf{uv}}$ of the gyrosegment $\mathbf{uv}$ joining the points $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{V}_c$ is given by the equation*

$$\mathbf{p}^m_{\mathbf{uv}} = \tfrac{1}{2}\otimes_M\frac{\gamma^2_{\mathbf{u}}\mathbf{u} + \gamma^2_{\mathbf{v}}\mathbf{v}}{\gamma^2_{\mathbf{u}} + \gamma^2_{\mathbf{v}} - 1} \tag{6.314}$$

Similar to (6.312) and (6.313) we have, by (6.304), (6.310) and (6.311),

$$\begin{aligned}
C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e + \gamma_{\mathbf{w}_e}\mathbf{w}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e} + \gamma_{\mathbf{w}_e}} \\
&= \frac{2\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + 2\gamma^2_{\mathbf{v}_m}\mathbf{v}_m + 2\gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{(2\gamma^2_{\mathbf{u}_m} - 1) + (2\gamma^2_{\mathbf{v}_m} - 1) + (2\gamma^2_{\mathbf{w}_m} - 1)} \\
&= \frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m + \gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} + \gamma^2_{\mathbf{w}_m} - \frac{3}{2}}
\end{aligned} \tag{6.315}$$

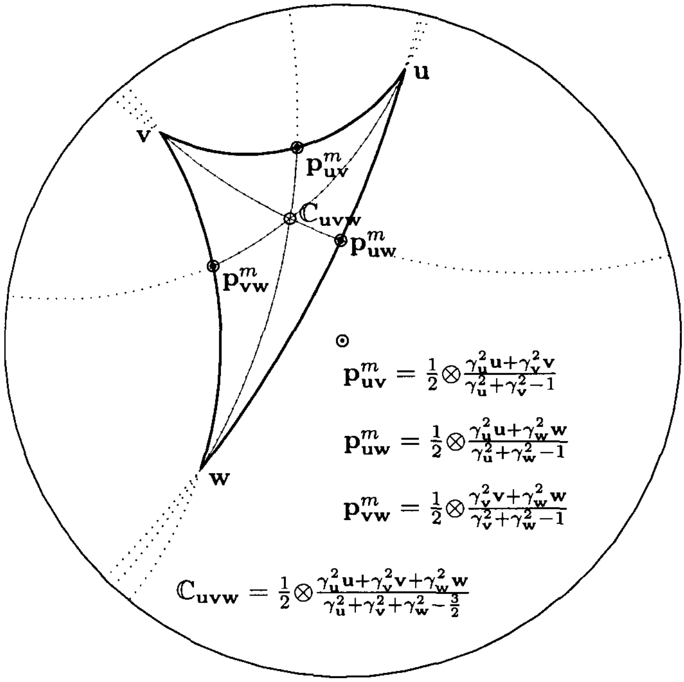


Fig. 6.20 A triangle $\mathbf{uvw}$ in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus_M, \otimes)$ is shown with the gyromidpoints $\mathbf{p_{uv}}$, $\mathbf{p_{uw}}$ and $\mathbf{p_{vw}}$ of its sides, and its gyromedians, and centroid $\mathbb{C}_{\mathbf{uvw}}$.

so that

$$
\begin{aligned}
C_{\mathbf{u}_m\mathbf{v}_m\mathbf{w}_m} &= \tfrac{1}{2}\otimes_M C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} \\
&= \tfrac{1}{2}\otimes_M \frac{\gamma^2_{\mathbf{u}_m}\mathbf{u}_m + \gamma^2_{\mathbf{v}_m}\mathbf{v}_m + \gamma^2_{\mathbf{w}_m}\mathbf{w}_m}{\gamma^2_{\mathbf{u}_m} + \gamma^2_{\mathbf{v}_m} + \gamma^2_{\mathbf{w}_m} - \frac{3}{2}}
\end{aligned}
\tag{6.316}
$$

for any $\mathbf{u}_m, \mathbf{v}_m, \mathbf{w}_m \in G_m$.

We have thus obtained in (6.316) the following

Theorem 6.90 **(The Möbius Gyrotriangle Gyrocentroid).** *Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}_c$ be any three nongyrocollinear points of a Möbius gyrovector space $(\mathbb{V}_c, \oplus_M, \otimes_M)$. The centroid $\mathbb{C}_{\mathbf{uvw}}$, Fig. 6.20, of the hyperbolic triangle $\mathbf{uvw}$ in $\mathbb{V}_c$ is given by the equation*

$$
C_{\mathbf{uvw}} = \tfrac{1}{2}\otimes_M \frac{\gamma^2_{\mathbf{u}}\mathbf{u} + \gamma^2_{\mathbf{v}}\mathbf{v} + \gamma^2_{\mathbf{w}}\mathbf{w}}{\gamma^2_{\mathbf{u}} + \gamma^2_{\mathbf{v}} + \gamma^2_{\mathbf{w}} - \frac{3}{2}}
\tag{6.317}
$$

The gyrotriangle gyrocentroid in the Möbius gyrovector space (that is, the hyperbolic triangle centroid in the Poincaré ball model) was also studied by O. Bottema [Bottema (1958)]. The gyrocentroid of the hyperbolic triangle $\mathbf{uvw}$ in the Poincaré disc model, as determined by (6.317), is shown in Fig. 6.20.

6.20.3 *In PV Gyrovector Spaces*

Let $G_e = (\mathbb{V}_c, \oplus_{\text{E}}, \otimes_{\text{E}})$ and $G_u = (\mathbb{V}, \oplus_{\text{U}}, \otimes_{\text{U}})$ be, respectively, the Einstein and the PV gyrovector spaces of the ball $\mathbb{V}_c$ of a real inner product space $\mathbb{V}$ and of the real inner product space $\mathbb{V}$. They are gyrovector space isomorphic, with the isomorphism ϕ_{EU} and its inverse isomorphism ϕ_{UE} from G_u into G_e shown in Table 6.1. Accordingly, the correspondence between elements of G_u and G_e is given by the equations

$$\begin{aligned} \mathbf{v}_e &= \beta_{\mathbf{v}_u} \mathbf{v}_u \\ \mathbf{v}_u &= \gamma_{\mathbf{v}_e} \mathbf{v}_e \end{aligned} \tag{6.318}$$

$\mathbf{v}_e \in G_e$, $\mathbf{v}_u \in G_u$, so that

$$\gamma_{\mathbf{v}_e} = \frac{1}{\beta_{\mathbf{v}_u}} \tag{6.319}$$

Hence, by (6.301) and (6.319) we have

$$\begin{aligned} \mathbf{p}^m_{\mathbf{u}_e\mathbf{v}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e}} \\ &= \frac{\mathbf{u}_u + \mathbf{v}_u}{\dfrac{1}{\beta_{\mathbf{u}_u}} + \dfrac{1}{\beta_{\mathbf{v}_u}}} \\ &= \frac{\beta_{\mathbf{u}_u}\beta_{\mathbf{v}_u}}{\beta_{\mathbf{u}_u} + \beta_{\mathbf{v}_u}}(\mathbf{u}_u + \mathbf{v}_u) \end{aligned} \tag{6.320}$$

so that, using the notation $\gamma(\mathbf{v}) = \gamma_{\mathbf{v}}$ when convenient, we have

$$\mathbf{p}^m_{\mathbf{u}_u\mathbf{v}_u} = \gamma(\mathbf{p}^m_{\mathbf{u}_e\mathbf{v}_e})\mathbf{p}^m_{\mathbf{u}_e\mathbf{v}_e} \tag{6.321}$$

Similarly, the gyrocentroid $C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e}$ of gyrotriangle $\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e$ in G_e is

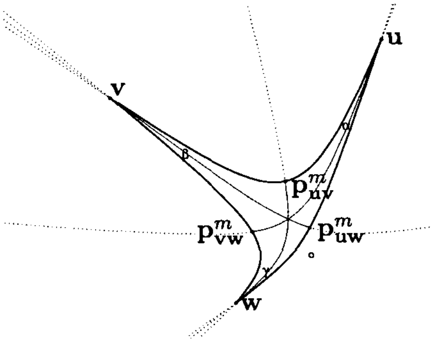


Fig. 6.21 Gyrotriangle gyromedians in a PV gyrovector space are concurrent. The gyrotriangle gyroangles satisfy
$\alpha + \beta + \gamma < \pi$

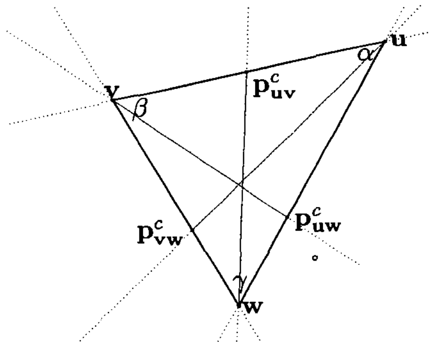


Fig. 6.22 Cogyrotriangle cogyromedians in a PV gyrovector space are not concurrent. The cogyrotriangle cogyroangles satisfy $\alpha + \beta + \gamma = \pi$

given in terms of corresponding points of G_u by the equation

$$
\begin{aligned}
C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} &= \frac{\gamma_{\mathbf{u}_e}\mathbf{u}_e + \gamma_{\mathbf{v}_e}\mathbf{v}_e + \gamma_{\mathbf{w}_e}\mathbf{w}_e}{\gamma_{\mathbf{u}_e} + \gamma_{\mathbf{v}_e} + \gamma_{\mathbf{w}_e}} \\
&= \frac{\mathbf{u}_u + \mathbf{v}_u + \mathbf{w}_u}{\dfrac{1}{\beta_{\mathbf{u}_u}} + \dfrac{1}{\beta_{\mathbf{v}_u}} + \dfrac{1}{\beta_{\mathbf{w}_u}}}
\end{aligned}
\tag{6.322}
$$

so that the gyrocentroid $C_{\mathbf{u}_u\mathbf{v}_u\mathbf{w}_u}$ of gyrotriangle $\mathbf{u}_u\mathbf{v}_u\mathbf{w}_u$ in G_u is given by the equation

$$
C_{\mathbf{u}_u\mathbf{v}_u\mathbf{w}_u} = \gamma(C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e})C_{\mathbf{u}_e\mathbf{v}_e\mathbf{w}_e} \tag{6.323}
$$

A gyrotriangle in the PV gyrovector plane $(\mathbb{R}^2, \oplus_{\text{U}}, \otimes_{\text{U}})$, along with its gyromedians and gyrocentroid, is shown in Fig. 6.21.

We may finally note that cogyrocentroids of cogyrotriangles do not exist since the cogyromedians of a cogyrotriangle are not concurrent as we see, for instance, in Fig. 6.22. The non-concurrency of the cogyrotriangle cogyromedians is "explained" by the bifurcation principle in the hyperbolic bifurcation diagram of Fig. 8.38.

6.21 Exercises

(1) Verify the last identity in (6.269).

(2) Prove the identity

$$\gamma_{3\otimes\mathbf{v}}(3\otimes\mathbf{v}) = (4\gamma_{\mathbf{v}}^3 - \gamma_{\mathbf{v}})\mathbf{v} \tag{6.324}$$

for any gyrovector $\mathbf{v}$ in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$.

(3) The Einstein Quarter. Prove the identity

$$\frac{1}{4}\otimes_{\mathrm{E}}\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1 + \sqrt{2(1+\gamma_{\mathbf{v}})}}\mathbf{v} \tag{6.325}$$

for any gyrovector $\mathbf{v}$ in an Einstein gyrovector space $(\mathbb{V}_s, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$. Hint: Use the Einstein half (6.268) successively, noting the scalar associative law of gyrovector spaces.

(4) Verify Theorem 6.45 by employing Theorem 3.8 and the gyrocommutative law.

(5) Identities (6.205) and (6.210) along with the gyrocommutative law suggest the identity

$$\mathrm{gyr}[\mathbf{p}_{\mathbf{ab}}^c, \mathbf{b}]\mathrm{gyr}[\mathbf{b}, \mathbf{a}] = \mathrm{gyr}[\mathbf{p}_{\mathbf{ab}}^c, \mathbf{a}] \tag{6.326}$$

(Why?) Verify the suggested gyroautomorphism identity (6.326).

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Chapter 7

Rudiments of Differential Geometry

In this chapter we uncover the link between gyrovector spaces embedded in the Euclidean n-space $\mathbb{R}^n$, $n \geq 2$, and differential geometry. Accordingly, we explore the differential geometry of Möbius gyrovector spaces $(\mathbb{R}^n_c, \oplus_M, \otimes_M)$, Einstein gyrovector spaces $(\mathbb{R}^n_c, \oplus_E, \otimes_E)$, and PV gyrovector spaces $(\mathbb{R}^n, \oplus_U, \otimes_U)$, where $\mathbb{R}^n_c$ is the c-ball of the Euclidean n-space,

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \ \|\mathbf{v}\| < c\} \tag{7.1}$$

We will find that the differential geometry of Möbius and Einstein gyrovector spaces reveals that Möbius gyrovector spaces coincide with the Poincaré ball model of hyperbolic geometry while Einstein gyrovector spaces coincide with the Beltrami (also known as the Klein) ball model of hyperbolic geometry. In contrast, PV gyrovector spaces seem to provide a new space (as opposed to ball) model of hyperbolic geometry.

In $\mathbb{R}^n$ we use the vector notation

$$\begin{aligned}
\mathbf{r} &= (x_1, x_2, \ldots, x_n) \\
d\mathbf{r} &= (dx_1, dx_2, \ldots, dx_n) \\
\mathbf{r}^2 &= \mathbf{r}\cdot\mathbf{r} = \|\mathbf{r}\|^2 = \sum_{i=1}^{n} x_i^2, \quad \mathbf{r}^4 = (\mathbf{r}^2)^2 \\
d\mathbf{r}^2 &= d\mathbf{r}\cdot d\mathbf{r} = \|d\mathbf{r}\|^2 = \sum_{i=1}^{n} dx_i^2 \\
\mathbf{r}\cdot d\mathbf{r} &= \sum_{i=1}^{n} x_i dx_i \\
(\mathbf{r}\times d\mathbf{r})^2 &= \mathbf{r}^2 d\mathbf{r}^2 - (\mathbf{r}\cdot d\mathbf{r})^2
\end{aligned} \tag{7.2}$$

noting that by the last equation in (7.2) $(\mathbf{r}\times d\mathbf{r})^2$ is defined in $\mathbb{R}^n$ for any dimension n.

7.1 The Riemannian Line Element of Euclidean Metric

To set the stage for the study of the gyroline and the cogyroline element of the gyrovector spaces $(\mathbb{R}^n_c, \oplus, \otimes)$ and $(\mathbb{R}^n, \oplus, \otimes)$ in Secs. 7.3–7.8 we begin with the study of the Riemannian line element ds^2 of the Euclidean vector space $\mathbb{R}^n$ with its standard metric given by the distance function

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{b} - \mathbf{a}\| \tag{7.3}$$

The *differential*

$$\Delta\mathbf{s} = (\mathbf{v} + \Delta\mathbf{v}) - \mathbf{v} = \Delta\mathbf{v} \tag{7.4}$$

has the norm

$$\|\Delta\mathbf{s}\| = d(\mathbf{v} + \Delta\mathbf{v}, \mathbf{v}) = \|(\mathbf{v} + \Delta\mathbf{v}) - \mathbf{v}\| = \|\Delta\mathbf{v}\| \tag{7.5}$$

The latter gives the distance between the two neighboring points $\mathbf{v}$ and $\mathbf{v} + \Delta\mathbf{v}$ in $\mathbb{R}^n$, where $\Delta\mathbf{v}$ is of sufficiently small length, $\|\Delta\mathbf{v}\| < \varepsilon$ for some $\varepsilon > 0$.

Let $\mathbf{v}, \Delta\mathbf{v} \in \mathbb{R}^n_c$ or $\mathbb{R}^n$ be represented by their components relative to rectangular Cartesian coordinates as $\mathbf{v} = (x_1, \ldots x_n)$ and $\Delta\mathbf{v} = (\Delta x_1, \ldots \Delta x_n)$. The differential $\Delta\mathbf{s}$ can be written as

$$\begin{aligned}\Delta\mathbf{s} = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=0} \Delta x_1 + \ \ldots \ + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=0} \Delta x_n \\ + \varepsilon_1\Delta x_1 + \ \ldots \ + \varepsilon_n\Delta x_n\end{aligned} \tag{7.6}$$

where $\varepsilon_1, \ldots, \varepsilon_n \to 0$ as $\varepsilon \to 0$.

We write (7.6) as

$$ds = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=0} dx_1 + \ \ldots \ + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=0} dx_n \tag{7.7}$$

and use the notation $ds^2 = \|ds\|^2$.

Since

$$\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_k} = (0, \ldots, 1, \ldots, 0) \tag{7.8}$$

(a 1 in the kth position), (7.7) gives

$$ds = (dx_1, \ldots, dx_n) \tag{7.9}$$

so that the Riemannian line element of the Euclidean n-space $\mathbb{R}^n$ with its standard metric (7.3) is

$$ds^2 = \sum_{i=1}^{n} dx_i^2 = d\mathbf{r}^2 \tag{7.10}$$

Following the calculation of the Riemannian line element (7.10) of the Euclidean n-space $\mathbb{R}^n$ with its metric given by the Euclidean distance function (7.3) the stage is set for the presentation of analogies in Sec. 7.2 and, subsequently, for the calculation of the

(1) gyroline element of each of the gyrovector spaces in Secs. 7.3–7.8 with their gyrometrics given by their respective gyrodistance functions; and the
(2) cogyroline element of each of the gyrovector spaces in Secs. 7.3–7.8 with their cogyrometrics given by their respective cogyrodistance functions.

7.2 The Gyroline and the Cogyroline Element

The gyrometric and the cogyrometric of a gyrovector space $(G, \oplus, \otimes)$ is given by its gyrodistance and cogyrodistance function

$$\begin{aligned} d_{\oplus}(\mathbf{b}\ominus\mathbf{a}) &= \|\mathbf{b}\ominus\mathbf{a}\| \\ d_{\boxplus}(\mathbf{b}\boxminus\mathbf{a}) &= \|\mathbf{b}\boxminus\mathbf{a}\| \end{aligned} \tag{7.11}$$

respectively, in full analogy with (7.3).

To determine the line element ds^2 of the n-dimensional Riemannian manifold which corresponds to a gyrovector space gyrometric and cogyrometric, we consider the *gyrodifferential* and the *cogyrodifferential* given, respectively, by the equations

$$\begin{aligned} \Delta\mathbf{s} &= (\mathbf{v} + \Delta\mathbf{v})\ominus\mathbf{v} \\ \Delta\mathbf{s} &= (\mathbf{v} + \Delta\mathbf{v})\boxminus\mathbf{v} \end{aligned} \tag{7.12}$$

in a gyrovector space $(G, \oplus, \otimes)$, where $G = \mathbb{R}_c^n$ or $G = \mathbb{R}^n$. The analogies that the gyrodifferential and the cogyrodifferential in (7.12) share with the differential in (7.4) are obvious.

The norm of the gyrodifferential and the cogyrodifferential in $\mathbb{R}^n$ gives, respectively, the gyrodistance and the cogyrodistance

$$\begin{aligned} \|\Delta\mathbf{s}\| &= d_{\oplus}(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}) = \|(\mathbf{v}+\Delta\mathbf{v})\ominus\mathbf{v}\| \\ \|\Delta\mathbf{s}\| &= d_{\boxplus}(\mathbf{v}+\Delta\mathbf{v},\mathbf{v}) = \|(\mathbf{v}+\Delta\mathbf{v})\boxminus\mathbf{v}\| \end{aligned} \tag{7.13}$$

between the two neighboring points $\mathbf{v}$ and $\mathbf{v}+\Delta\mathbf{v}$ of $\mathbb{R}^n_c$ or $\mathbb{R}^n$. Here $+$ is vector addition in $\mathbb{R}^n$, and $\Delta\mathbf{v}$ is an element of $\mathbb{R}^n_c$ or $\mathbb{R}^n$ of sufficiently small length, $\|\Delta\mathbf{v}\| < \varepsilon$ for some $\varepsilon > 0$, in full analogy with (7.5).

Let $\mathbf{v}, \Delta\mathbf{v} \in \mathbb{R}^n_c$ or $\mathbb{R}^n$ be represented by their components relative to rectangular Cartesian coordinates as $\mathbf{v} = (x_1, \dots x_n)$ and $\Delta\mathbf{v} = (\Delta x_1, \dots \Delta x_n)$. The differential $\Delta\mathbf{s}$ can be written as

$$\begin{aligned} \Delta\mathbf{s} = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=\mathbf{0}} \Delta x_1 + \ \dots\ + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=\mathbf{0}} \Delta x_n \\ + \varepsilon_1\Delta x_1 + \ \dots\ + \varepsilon_n\Delta x_n \end{aligned} \tag{7.14}$$

where $\varepsilon_1, \dots, \varepsilon_n \to 0$ as $\varepsilon \to 0$.

We write (7.14) as

$$d\mathbf{s} = \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_1}\right]_{\Delta\mathbf{v}=\mathbf{0}} dx_1 + \ \dots\ + \left[\frac{\partial\Delta\mathbf{s}}{\partial\Delta x_n}\right]_{\Delta\mathbf{v}=\mathbf{0}} dx_n \tag{7.15}$$

and use the notation $ds^2 = \|d\mathbf{s}\|^2$. Following the origin of $d\mathbf{s}$ from a gyrodifferential or a cogyrodifferential, we call $d\mathbf{s}$ the element of arc gyrolength or cogyrolength, and call $ds^2 = \|d\mathbf{s}\|^2$ the gyroline or cogyroline element, respectively. Each gyroline and cogyroline element forms a Riemannian line element.

For the sake of simplicity, further details are given explicitly for the special case of $n = 2$, but the generalization to any integer $n > 2$ is obvious, and will be presented without further details. In the special case when $n = 2$, (7.15) reduces to

$$\begin{aligned} d\mathbf{s} &= \left[\frac{\partial\Delta\mathbf{s}_{\mathrm{M}}}{\partial\Delta x_1}\right]_{\left\{\begin{smallmatrix}\Delta x_1 = 0\\ \Delta x_2 = 0\end{smallmatrix}\right\}} dx_1 \ + \ \left[\frac{\partial\Delta\mathbf{s}_{\mathrm{M}}}{\partial\Delta x_2}\right]_{\left\{\begin{smallmatrix}\Delta x_1 = 0\\ \Delta x_2 = 0\end{smallmatrix}\right\}} dx_2 \\ &= \mathbf{X}_1(x_1,x_2)dx_1 + \mathbf{X}_2(x_1,x_2)dx_2 \end{aligned} \tag{7.16}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}^2_c \to \mathbb{R}^2$ (where $G = \mathbb{R}^2_c$) or $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}^2 \to \mathbb{R}^2$ (where $G = \mathbb{R}^2$) are given by

$$\mathbf{X}_k(x_1, x_2) = \left[\frac{\partial \Delta \mathrm{s}}{\partial \Delta x_k}\right]_{\begin{Bmatrix} \Delta x_1 = 0 \\ \Delta x_2 = 0 \end{Bmatrix}} \tag{7.17}$$

$k = 1, 2$.

Following standard notation in differential geometry [Carmo (1976), p. 92], the metric coefficients of the gyrometric or cogyrometric of the gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$ or $(\mathbb{R}^2, \oplus, \otimes)$ in the Cartesian x_1x_2-coordinates are

$$\begin{aligned} E &= \mathbf{X}_1 \cdot \mathbf{X}_1 \\ F &= \mathbf{X}_1 \cdot \mathbf{X}_2 \\ G &= \mathbf{X}_2 \cdot \mathbf{X}_2 \end{aligned} \tag{7.18}$$

These metric coefficients give rise to the Riemannian line element

$$ds^2 = Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2 \tag{7.19}$$

suggesting the following two definitions.

Definition 7.1 (Gyrometric and Cogyrometric Coefficients). *The metric coefficients* (7.18) *that result from the gyrodistance (cogyrodistance)* (7.11) *in a gyrovector space* $(G, \oplus, \otimes)$ *are called the gyrometric (cogyrometric) coefficients of the gyrovector space.*

Definition 7.2 (Gyroline and Cogyroline Elements). *The Riemannian line element* (7.19) *that results from the gyrodistance (cogyrodistance)* (7.11) *in a gyrovector space* $(G, \oplus, \otimes)$ *is called the gyroline (cogyroline) element of the gyrovector space.*

The gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$ or $(\mathbb{R}^2, \oplus, \otimes)$, with its gyrometric or cogyrometric, results in a Riemannian line element ds^2. The latter, in turn, gives rise to the Riemannian surface $(\mathbb{R}^2_c, ds^2)$ or $(\mathbb{R}^2, ds^2)$. The Gaussian curvature K of this surface is given by the equation [McCleary (1994),

p. 149]

$$K = \frac{1}{(EG-F^2)^2}\left\{\det\begin{pmatrix} -\frac{1}{2}E_{22}+F_{12}-\frac{1}{2}G_{11} & \frac{1}{2}E_1 & F_1-\frac{1}{2}E_2 \\ F_2-\frac{1}{2}G_1 & E & F \\ \frac{1}{2}G_2 & F & G \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}E_2 & \frac{1}{2}G_1 \\ \frac{1}{2}E_2 & E & F \\ \frac{1}{2}G_1 & F & G \end{pmatrix}\right\} \tag{7.20}$$

where $E_1 = \partial E/\partial x_1$, $F_{12} = \partial^2 F/\partial x_1 \partial x_2$, etc.

In the special case of $F = 0$ (7.20) reduces to the equation

$$K = -\frac{1}{2\sqrt{EG}}\left\{\frac{\partial}{\partial x_2}\frac{\frac{\partial E}{\partial x_2}}{\sqrt{EG}} + \frac{\partial}{\partial x_1}\frac{\frac{\partial G}{\partial x_1}}{\sqrt{EG}}\right\} \tag{7.21}$$

$EG > 0$, [Carmo (1976), p. 237] [McCleary (1994), p. 155] [Oprea (1997), p. 105].

The following definition is natural along the line of Defs. 7.1 and 7.2.

Definition 7.3 **(Gyrocurvature and Cogyrocurvature).** *The Gaussian curvature of a gyrovector space with its gyroline (cogyroline) element is called a gyrocurvature (cogyrocurvature).*

Following Defs. 7.1, 7.2 and 7.3, any gyrovector space possesses (i) a gyrodistance and a cogyrodistance function; (ii) a gyrometric and a cogyrometric; (iii) gyrometric and cogyrometric coefficients; (iv) a gyroline and a cogyroline element; and (v) a gyrocurvature and a cogyrocurvature.

Concrete examples are presented in Secs. 7.3–7.8, and are summarized in Table 7.1 of Sec. 7.9.

7.3 The Gyroline Element of Möbius Gyrovector Spaces

In this section we uncover the Riemannian line element to which the gyrometric of the Möbius gyrovector space $(\mathbb{R}^n_c, \oplus_M, \otimes_M)$ gives rise.

Let us consider the gyrodifferential (7.12),

$$\begin{aligned}\Delta\mathbf{s}_{\mathrm{M}} &= (\mathbf{v}+\Delta\mathbf{v})\ominus_{\mathrm{M}}\mathbf{v}\\ &= \begin{pmatrix}x_1+\Delta x_1\\ x_2+\Delta x_2\end{pmatrix}\ominus_{\mathrm{M}}\begin{pmatrix}x_1\\ x_2\end{pmatrix}\end{aligned} \tag{7.22}$$

in the Möbius gyrovector plane $(\mathbb{R}_c^2, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$, where ambiguously, $+$ is the Euclidean addition in $\mathbb{R}^2$ and in $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned}d\mathbf{s}_{\mathrm{M}} &= \left[\frac{\partial\Delta\mathbf{s}_{\mathrm{M}}}{\partial\Delta x_1}\right]_{\left\{\begin{array}{l}\Delta x_1=0\\ \Delta x_2=0\end{array}\right\}} dx_1 \quad + \quad \left[\frac{\partial\Delta\mathbf{s}_{\mathrm{M}}}{\partial\Delta x_2}\right]_{\left\{\begin{array}{l}\Delta x_1=0\\ \Delta x_2=0\end{array}\right\}} dx_2\\ &= \mathbf{X}_1(x_1,x_2)dx_1 + \mathbf{X}_2(x_1,x_2)dx_2\end{aligned} \tag{7.23}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}_c^2 \to \mathbb{R}^2$, obtaining

$$\begin{aligned}\mathbf{X}_1(x_1,x_2) &= \frac{c^2}{c^2-r^2}(1,\, 0) \in \mathbb{R}_c^2\\ \mathbf{X}_2(x_1,x_2) &= \frac{c^2}{c^2-r^2}(0,\, 1) \in \mathbb{R}_c^2\end{aligned} \tag{7.24}$$

where $r^2 = x_1^2 + x_2^2$.

The gyrometric coefficients of the gyrometric of the Möbius gyrovector plane in the Cartesian x_1x_2-coordinates are therefore

$$\begin{aligned}E &= \mathbf{X}_1\cdot\mathbf{X}_1 = \frac{c^4}{(c^2-r^2)^2}\\ F &= \mathbf{X}_1\cdot\mathbf{X}_2 = 0\\ G &= \mathbf{X}_2\cdot\mathbf{X}_2 = \frac{c^4}{(c^2-r^2)^2}\end{aligned} \tag{7.25}$$

Hence, the gyroline element of the Möbius gyrovector plane $(\mathbb{R}_c^2, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$ is the Riemannian line element

$$\begin{aligned}ds_{\mathrm{M}}^2 &= \|d\mathbf{s}_{\mathrm{M}}\|^2\\ &= Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2\\ &= \frac{c^4}{(c^2-r^2)^2}(dx_1^2 + dx_2^2)\end{aligned} \tag{7.26}$$

known as the Poincaré metric. Thus, for instance, the Riemannian line element of the Poincaré disc or, equivalently, the Möbius gyrovector plane

$(\mathbb{R}^2_{c=2}, \oplus_{\text{M}}, \otimes_{\text{M}})$, is [McCleary (1994), p. 226]

$$ds^2_{\text{M}} = \frac{dx_1^2 + dx_2^2}{(1 - \frac{r^2}{4})^2} \tag{7.27}$$

An interesting elementary study of the Riemannian structure (7.26) in the context of the hyperbolic plane is presented in the introductory chapter of [Helgason (1984)]. The Riemannian line element ds^2_{M} is identified in [Farkas and Kra (1992), p. 216] as a Riemannian metric on the Riemann surface $\mathbb{D}_{c=1}$, where $\mathbb{D}_{c=1}$ is the Poincaré complex open unit disc.

Following Riemann [Stahl (1993), p. 73] we note that E, G and $EG - F^2 = EG$ are all positive in the open disc $\mathbb{R}^2_c$, so that the quadratic form (7.26) is *positive definite* [Kreyszig (1991), p. 84].

The gyrocurvature of the Möbius gyrovector plane is the Gaussian curvature K of the surface with the line element (7.26). It is a negative constant,

$$K = -\frac{4}{c^2} \tag{7.28}$$

as one can calculate from (7.21).

Extending (7.26) from $n = 2$ to $n \geq 2$ we have

$$\begin{aligned} ds^2_{\text{M}} &= \frac{c^4}{(c^2 - \mathbf{r}^2)^2} d\mathbf{r}^2 \\ &= \frac{d\mathbf{r}^2}{(1 + \frac{1}{4}K\mathbf{r}^2)^2} \end{aligned} \tag{7.29}$$

The Riemannian line element ds^2_{M} reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c \to \infty} ds^2_{\text{M}} = d\mathbf{r}^2 \tag{7.30}$$

as expected.

The study of the line element (7.29) by Riemann is described in [Coxeter (1998), p. 12]. In two dimensions it is known as the Poincaré metric of the Poincaré disc [McCleary (1994)]. We have thus established the following

Theorem 7.4 *The gyroline element of a Möbius gyrovector space* $(\mathbb{R}^n_c, \oplus_M, \otimes_M)$ *is given by the equation*

$$ds^2_{\text{M}} = \frac{c^4}{(c^2 - \mathbf{r}^2)^2} d\mathbf{r}^2 \tag{7.31}$$

and its gyrocurvature is given by the equation

$$K = -\frac{4}{c^2} \tag{7.32}$$

In particular, for $n = 2$ and $c = 1$ the gyroline element (7.31) *coincides with the Riemannian line element of the Poincaré disc model of hyperbolic geometry.*

A Riemannian metric g in $\mathbb{R}^n$ has the form

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j \tag{7.33}$$

Two Riemannian metrics g_1 and g_2 are said to be *conformal* to each other if there is a positive smooth (that is, infinitely differentiable) function $f : \mathbb{R}^n \to \mathbb{R}^{>0}$ such that $g_1 = fg_2$ [Lee (1997)], $\mathbb{R}^{>0} = \{r \in \mathbb{R} : r > 0\}$ being the positive ray of the real line $\mathbb{R}$.

The Riemannian metric of Möbius gyrovector spaces $(\mathbb{R}^n_c, \oplus_{\text{M}}, \otimes_{\text{M}})$, given by (7.29), is conformal to the Riemannian metric

$$d\mathbf{r}^2 = \sum_{i=1}^{n} dx_i^2 \tag{7.34}$$

of the Euclidean n-space $\mathbb{R}^n$. Hence, hyperbolic spheres in Möbius gyrovector spaces are also Euclidean spheres. However, Euclidean and hyperbolic sphere centers need not coincide, as shown in Fig. 7.1. The reason in terms of analogies that Euclidean and hyperbolic geometry share is clear. For any two tangent Euclidean circles with centers $\mathbf{a}, \mathbf{c}$ and concurrent point $\mathbf{b}$, the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are collinear. In full analogy, for any two tangent Möbius circles with centers $\mathbf{a}, \mathbf{c}$ and concurrent point $\mathbf{b}$, the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are gyrocollinear, as shown in Fig. 7.1.

7.4 The Cogyroline Element of Möbius Gyrovector Spaces

In this section we uncover the Riemannian line element to which the cogyrometric of the Möbius gyrovector space $(\mathbb{R}^n_c, \oplus_{\text{M}}, \otimes_{\text{M}})$ gives rise.

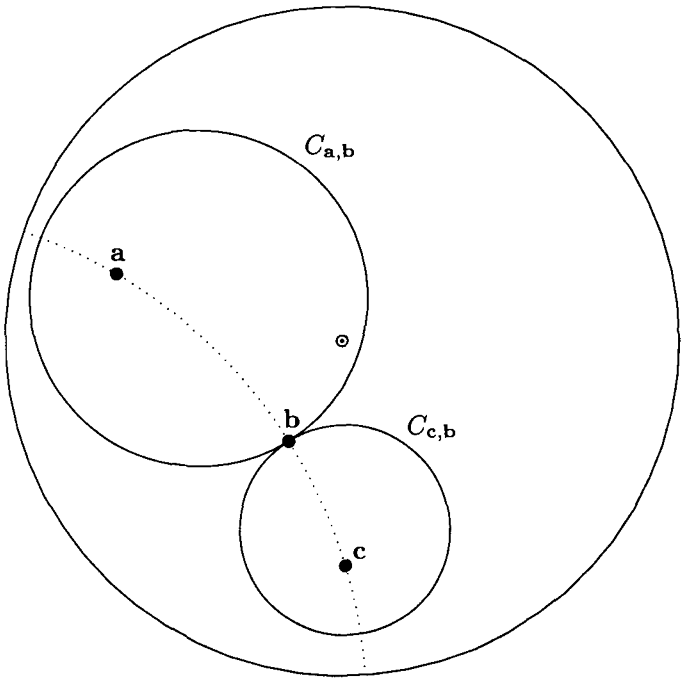


Fig. 7.1 The Möbius gyrocircle $C_{\mathbf{a},\mathbf{b}}$, given by the equation

$$\mathbf{a}\oplus_{\mathrm{M}} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} (\ominus_{\mathrm{M}}\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b}), \qquad 0 \le \theta < 2\pi,$$

is the hyperbolic circle in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$. It is a Euclidean circle with hyperbolic radius $\|\ominus_{\mathrm{M}}\mathbf{a}\oplus_{\mathrm{M}}\mathbf{b}\|$, hyperbolically centered at $\mathbf{a}$.
A second gyrocircle, $C_{\mathbf{c},\mathbf{b}}$, that intersects $C_{\mathbf{a},\mathbf{b}}$ at a single point, $\mathbf{b}$, is shown. In full analogy with Euclidean geometry, the two gyrocenters $\mathbf{a}$ and $\mathbf{c}$ and the concurrent point $\mathbf{b}$ are gyrocollinear.

Let us consider the cogyrodifferential (7.12),

$$\begin{aligned} \Delta\mathbf{s}_{\mathrm{CM}} &= (\mathbf{v}+\Delta\mathbf{v}) \boxminus_{\mathrm{M}} \mathbf{v} \\ &= \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \end{pmatrix} \boxminus_{\mathrm{M}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{7.35}$$

in the Möbius gyrovector plane $(\mathbb{R}^2_c, \oplus_{\mathrm{M}}, \otimes_{\mathrm{M}})$, where $+$ is the Euclidean addition in $\mathbb{R}^2$ and $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned} ds_{\text{CM}} &= \left[\frac{\partial \Delta \mathbf{s}_{\text{CM}}}{\partial \Delta x_1}\right]_{\left\{\begin{array}{l}\Delta x_1 = 0\\ \Delta x_2 = 0\end{array}\right\}} dx_1 \quad + \quad \left[\frac{\partial \Delta \mathbf{s}_{\text{CM}}}{\partial \Delta x_2}\right]_{\left\{\begin{array}{l}\Delta x_1 = 0\\ \Delta x_2 = 0\end{array}\right\}} dx_2 \\ &= \mathbf{X}_1(x_1, x_2)dx_1 + \mathbf{X}_2(x_1, x_2)dx_2 \end{aligned} \tag{7.36}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}_c^2 \to \mathbb{R}^2$, obtaining

$$\begin{aligned} \mathbf{X}_1(x_1, x_2) &= \frac{c^2}{c^4 - r^4}(c^2 + x_1^2 - x_2^2,\ 2x_1x_2) \in \mathbb{V}_c = \mathbb{R}_c^2 \\ \mathbf{X}_2(x_1, x_2) &= \frac{c^2}{c^4 - r^4}(2x_1x_2,\ c^2 + x_1^2 - x_2^2) \in \mathbb{V}_c = \mathbb{R}_c^2 \end{aligned} \tag{7.37}$$

where $r^2 = x_1^2 + x_2^2$.

The cogyrometric coefficients of the cogyrometric of the Möbius gyrovector plane in the Cartesian x_1x_2-coordinates are therefore

$$\begin{aligned} E &= \mathbf{X}_1 \cdot \mathbf{X}_1 = \frac{c^4}{(c^4 - r^4)^2} \{(c^2 + r^2)^2 - 4c^2 x_2^2\} \\ F &= \mathbf{X}_1 \cdot \mathbf{X}_2 = \frac{c^6}{(c^4 - r^4)^2} x_1 x_2 \\ G &= \mathbf{X}_2 \cdot \mathbf{X}_2 = \frac{c^4}{(c^4 - r^4)^2} \{(c^2 + r^2)^2 - 4c^2 x_1^2\} \end{aligned} \tag{7.38}$$

Hence, the cogyroline element of the Möbius gyrovector plane $(\mathbb{R}_c^2, \oplus_{\text{M}}, \otimes_{\text{M}})$ is the Riemannian line element

$$\begin{aligned} ds_{\text{CM}}^2 &= \|d\mathbf{s}_{\text{CM}}\|^2 \\ &= E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2 \\ &= \frac{c^4}{(c^4 - r^4)^2}\{(c^2 + r^2)^2 (dx_1^2 + dx_2^2) - 4c^2 (x_1 dx_2 - x_2 dx_1)^2\} \end{aligned} \tag{7.39}$$

Following Riemann [Stahl (1993), p. 73]), we note that E, G and

$$EG - F^2 = \frac{c^8}{(c^4 - r^4)^2} \tag{7.40}$$

are all positive in the open disc $\mathbb{R}_c^2$, so that the quadratic form (7.39) is *positive definite* [Kreyszig (1991), p. 84].

In vector notation the Riemannian line element (7.39), extended to n dimensions, takes the form

$$ds_{\mathrm{CM}}^2 = \frac{c^4}{(c^4 - \mathbf{r}^4)^2}\{(c^2 + \mathbf{r}^2)^2 d\mathbf{r}^2 - 4c^2(\mathbf{r}\times d\mathbf{r})^2\} \tag{7.41}$$

in Cartesian coordinates.

As expected, the Riemannian line element ds_{CM}^2 reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c\to\infty} ds_{\mathrm{CM}}^2 = d\mathbf{r}^2 \tag{7.42}$$

The cogyrocurvature of the Möbius gyrovector plane is the Gaussian curvature K of this surface. It is a positive variable,

$$K = \frac{8c^6}{(c^2 + \mathbf{r}^2)^4} \tag{7.43}$$

as one can calculate from (7.20). We have thus established the following

Theorem 7.5 *The cogyroline element of a Möbius gyrovector space* $(\mathbb{R}_c^n, \oplus_M, \otimes_M)$ *is given by the equation*

$$ds_{\mathrm{CM}}^2 = \frac{c^4}{(c^4 - \mathbf{r}^4)^2}\{(c^2 + \mathbf{r}^2)^2 d\mathbf{r}^2 - 4c^2(\mathbf{r}\times d\mathbf{r})^2\} \tag{7.44}$$

and its cogyrocurvature is given by the equation

$$K = \frac{8c^6}{(c^2 + \mathbf{r}^2)^4} \tag{7.45}$$

7.5 The Gyroline Element of Einstein Gyrovector Spaces

In this section we uncover the Riemannian line element to which the gyrometric of the Einstein gyrovector space $(\mathbb{R}_c^n, \oplus_E, \otimes_E)$ gives rise.

Let us consider the gyrodifferential (7.12),

$$\begin{aligned}\Delta \mathbf{s}_E &= (\mathbf{v} + \Delta\mathbf{v}) \ominus_E \mathbf{v} \\ &= \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \end{pmatrix} \ominus_E \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\end{aligned} \tag{7.46}$$

in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_{\rm E}, \otimes_{\rm E})$, where $+$ is the Euclidean addition in $\mathbb{R}^2$ and in $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned} d\mathbf{s}_{\rm E} &= \left[\frac{\partial \Delta \mathbf{s}_{\rm E}}{\partial \Delta x_1}\right]_{\left\{\begin{array}{l}\Delta x_1 = 0\\ \Delta x_2 = 0\end{array}\right\}} dx_1 \quad + \quad \left[\frac{\partial \Delta \mathbf{s}_{\rm E}}{\partial \Delta x_2}\right]_{\left\{\begin{array}{l}\Delta x_1 = 0\\ \Delta x_2 = 0\end{array}\right\}} dx_2 \\ &= \mathbf{X}_1(x_1, x_2)dx_1 + \mathbf{X}_2(x_1, x_2)dx_2 \end{aligned} \tag{7.47}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}^2_c \to \mathbb{R}^2$, obtaining

$$\begin{aligned} \mathbf{X}_1(x_1, x_2) &= c\left(\frac{1}{R} + \frac{x_1^2}{R^2(c+R)}, \quad \frac{x_1 x_2}{R^2(c+R)}\right) \\ \mathbf{X}_2(x_2, x_2) &= c\left(\frac{x_1 x_2}{R^2(c+R)}, \quad \frac{1}{R} + \frac{x_2^2}{R^2(c+R)}\right) \end{aligned} \tag{7.48}$$

where $R^2 = c^2 - r^2, \quad r^2 = x_1^2 + x_2^2$.

The gyrometric coefficients of the gyrometric of the Einstein gyrovector plane in the Cartesian x_1x_2-coordinates are therefore

$$\begin{aligned} \mathbf{X}_1 \cdot \mathbf{X}_1 &= E = c^2 \frac{c^2 - x_2^2}{(c^2 - r^2)^2} \\ \mathbf{X}_1 \cdot \mathbf{X}_2 &= F = c^2 \frac{x_1 x_2}{(c^2 - r^2)^2} \\ \mathbf{X}_2 \cdot \mathbf{X}_2 &= G = c^2 \frac{c^2 - x_1^2}{(c^2 - r^2)^2} \end{aligned} \tag{7.49}$$

Hence, the gyroline element of the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_{\rm E}, \otimes_{\rm E})$ is the Riemannian line element

$$\begin{aligned} ds_E^2 &= \|d\mathbf{s}_E\|^2 \\ &= E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2 \\ &= c^2 \frac{dx_1^2 + dx_2^2}{c^2 - r^2} + c^2 \frac{(x_1 dx_1 + x_2 dx_2)^2}{(c^2 - r^2)^2} . \end{aligned} \tag{7.50}$$

Following Riemann [Stahl (1993), p. 73], we note that E, G and

$$EG - F^2 = \frac{c^6}{(c^2 - r^2)^3} \tag{7.51}$$

$r^2 = x_1^2 + x_2^2$, are all positive in the open disc $\mathbb{R}^2_c$, so that the quadratic form (7.50) is *positive definite* [Kreyszig (1991), pp. 84–85].

The Riemannian line element ds_E^2 of Einstein gyrometric in the disc turns out to be the line element of the Beltrami (or Klein) disc model of hyperbolic geometry. The Beltrami line element is presented, for instance, in McCleary [McCleary (1994), p. 220], for $n = 2$, and in Cannon *et al* [Cannon, Floyd and Walter (1997), ds_K^2, p. 71], for $n \geq 2$. An account of the first fifty years of hyperbolic geometry that emphasizes the contributions of Beltrami, who prepared the background for Poincaré and Klein, is found in [Milnor (1982)].

The gyrocurvature of the Einstein gyrovector plane is the Gaussian curvature of the surface with the line element (7.50). It is a negative constant,

$$K = -\frac{1}{c^2} \tag{7.52}$$

as one can calculate from (7.20).

Extending (7.50) from $n = 2$ to $n \geq 2$ we have

$$ds_E^2 = \frac{c^2}{c^2 - \mathbf{r}^2} d\mathbf{r}^2 + \frac{c^2}{(c^2 - \mathbf{r}^2)^2} (\mathbf{r}\cdot d\mathbf{r})^2 \tag{7.53}$$

in Cartesian coordinates. As expected, the hyperbolic Riemannian line element (7.53) reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c \to \infty} ds_E^2 = d\mathbf{r}^2 . \tag{7.54}$$

Interestingly, the Beltrami-Riemannian line element (7.53) can be written as

$$\frac{1}{c^2} ds_{B_3}^2 = \frac{c^2 d\mathbf{r}^2 - (\mathbf{r} \times d\mathbf{r})^2}{(c^2 - \mathbf{r}^2)^2} \tag{7.55}$$

as noted by Fock [Fock (1964), p. 39].

The line element ds_E^2 in (7.50) is the line element of Einstein gyrometric. It turns out to be the metric that the Italian mathematician Eugenio Beltrami introduced in 1868 in order to study hyperbolic geometry by a Euclidean disc model, now known as the Beltrami disc [McCleary (1994), p. 220]. An English translation of his historically significant 1868 essay on the interpretation of non-Euclidean geometry is found in [Stillwell (1996)]. The significance of Beltrami's 1868 essay rests on the generally known fact that it was the first to offer a concrete interpretation of hyperbolic geometry by interpreting "straight lines" as geodesics on a surface of a constant negative curvature. Using the metric (7.50), Beltrami constructed a Euclidean disc model of the hyperbolic plane [McCleary (1994)] [Stillwell (1996)], which now bears his name.

We have thus established the following

Theorem 7.6 *The gyroline element of an Einstein gyrovector space* $(\mathbb{R}^n_c, \oplus_E, \otimes_E)$ *is given by the equation*

$$ds_E^2 = \frac{c^2}{c^2 - \mathbf{r}^2} d\mathbf{r}^2 + \frac{c^2}{(c^2 - \mathbf{r}^2)^2} (\mathbf{r} \cdot d\mathbf{r})^2 \tag{7.56}$$

and its gyrocurvature is given by the equation

$$K = -\frac{1}{c^2} \tag{7.57}$$

In particular, for $n = 2$ *and* $c = 1$ *the gyroline element* (7.56) *coincides with the Riemannian line element of the Beltrami disc model of hyperbolic geometry.*

7.6 The Cogyroline Element of Einstein Gyrovector Spaces

In this section we uncover the Riemannian line element to which the cogyrometric of the Einstein gyrovector space $(\mathbb{R}^n_c, \oplus_E, \otimes_E)$ gives rise.

Let us consider the cogyrodifferential (7.12),

$$\begin{aligned} \Delta \mathbf{s}_{CE} &= (\mathbf{v} + \Delta\mathbf{v}) \boxminus_E \mathbf{v} \\ &= \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \end{pmatrix} \boxminus_E \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned} \tag{7.58}$$

in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus_E, \otimes_E)$, where $+$ is the Euclidean addition in $\mathbb{R}^2$ and $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned} d\mathbf{s}_{CE} &= \left[\frac{\partial \Delta \mathbf{s}_{CE}}{\partial \Delta x_1}\right]_{\left\{\begin{matrix} \Delta x_1 = 0 \\ \Delta x_2 = 0 \end{matrix}\right\}} dx_1 \quad + \quad \left[\frac{\partial \Delta \mathbf{s}_{CE}}{\partial \Delta x_2}\right]_{\left\{\begin{matrix} \Delta x_1 = 0 \\ \Delta x_2 = 0 \end{matrix}\right\}} dx_2 \\ &= \mathbf{X}_1(x_1, x_2) dx_1 + \mathbf{X}_2(x_1, x_2) dx_2 \end{aligned} \tag{7.59}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}_c^2 \to \mathbb{R}^2$, obtaining

$$\begin{aligned}\mathbf{X}_1(x_1,x_2) &= \frac{1}{c^2-r^2}(c^2-r^2+x_1^2,\, x_1x_2)\\ \mathbf{X}_2(x_2,x_2) &= \frac{1}{c^2-r^2}(x_1x_2,\, c^2-r^2+x_2^2)\end{aligned} \tag{7.60}$$

where $r^2 = x_1^2 + x_2^2$.

The cogyrometric coefficients of the cogyrometric of the Einstein gyrovector plane in the Cartesian x_1x_2-coordinates are therefore

$$\begin{aligned}\mathbf{X}_1{\cdot}\mathbf{X}_1 &= E = 1 + \frac{2c^2-r^2}{(c^2-r^2)^2}x_1^2\\ \mathbf{X}_1{\cdot}\mathbf{X}_2 &= F = \frac{2c^2-r^2}{(c^2-r^2)^2}x_1x_2\\ \mathbf{X}_2{\cdot}\mathbf{X}_2 &= G = 1 + \frac{2c^2-r^2}{(c^2-r^2)^2}x_2^2\end{aligned} \tag{7.61}$$

Hence, the cogyroline element of the Einstein gyrovector plane $(\mathbb{R}_c^2, \oplus_{\text{E}}, \otimes_{\text{E}})$ is the Riemannian line element

$$\begin{aligned}ds_{CE}^2 &= \|d\mathbf{s}_{CE}\|^2\\ &= Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2\\ &= dx_1^2 + dx_2^2 + \frac{(2c^2-r^2)}{(c^2-r^2)^2}(x_1dx_1 + x_2dx_2)^2\end{aligned} \tag{7.62}$$

where $r^2 = x_1^2 + x_2^2$. In the limit of large c, $c \to \infty$, the Riemannian line element ds_{CE}^2 reduces to its Euclidean counterpart.

Following Riemann [Stahl (1993), p. 73], we note that E, G and

$$EG - F^2 = \frac{c^4}{(c^2-r^2)^2} \tag{7.63}$$

are all positive in the open disc $\mathbb{R}_c^2$, so that the quadratic form (7.62) is *positive definite* [Kreyszig (1991), p. 84].

The cogyrocurvature of the Einstein gyrovector plane is the Gaussian curvature K of the Riemannian surface $(\mathbb{D}_c, ds_{CE}^2)$. It is a positive variable,

$$K = 2\frac{c^2-r^2}{c^4} \tag{7.64}$$

as one can calculate from (7.20).

Extending (7.62) from $n = 2$ to $n \geq 2$ we have

$$ds^2_{CE} = d\mathbf{r}^2 + \frac{2c^2 - \mathbf{r}^2}{(c^2 - \mathbf{r}^2)^2}(\mathbf{r}\cdot d\mathbf{r})^2 \tag{7.65}$$

in Cartesian coordinates. As expected, the hyperbolic Riemannian line element reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c\to\infty} ds^2_{CE} = d\mathbf{r}^2 \tag{7.66}$$

We have thus established the following

Theorem 7.7 *The cogyroline element of an Einstein gyrovector space* $(\mathbb{R}^n_c, \oplus_E, \otimes_E)$ *is given by the equation*

$$ds^2_{CE} = d\mathbf{r}^2 + \frac{2c^2 - \mathbf{r}^2}{(c^2 - \mathbf{r}^2)^2}(\mathbf{r}\cdot d\mathbf{r})^2 \tag{7.67}$$

and its cogyrocurvature is given by the equation

$$K = 2\frac{c^2 - r^2}{c^4} \tag{7.68}$$

7.7 The Gyroline Element of PV Gyrovector Spaces

In this section we uncover the Riemannian line element to which the gyrometric of the PV gyrovector space $(\mathbb{R}^n, \oplus_U, \otimes_U)$ gives rise.

Let us consider the gyrodifferential (7.12),

$$\begin{aligned}\Delta\mathbf{s}_U &= (\mathbf{v} + \Delta\mathbf{v})\ominus_U\mathbf{v} \\ &= \begin{pmatrix} x_1 + \Delta x_1 \\ x_2 + \Delta x_2 \end{pmatrix} \ominus_U \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\end{aligned} \tag{7.69}$$

in the PV in gyrovector plane $(\mathbb{R}^2_c, \oplus_U, \otimes_U)$ where $+$ is the Euclidean addition in $\mathbb{R}^2$ and in $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned} d\mathbf{s}_U &= \left[\frac{\partial\Delta\mathbf{s}_U}{\partial\Delta x_1}\right]\begin{Bmatrix} \Delta x_1 = 0 \\ \Delta x_2 = 0 \end{Bmatrix} dx_1 \quad + \quad \left[\frac{\partial\Delta\mathbf{s}_U}{\partial\Delta x_2}\right]\begin{Bmatrix} \Delta x_1 = 0 \\ \Delta x_2 = 0 \end{Bmatrix} dx_2 \\ &= \mathbf{X}_1(x_1, x_2)dx_1 + \mathbf{X}_2(x_1, x_2)dx_2 \end{aligned} \tag{7.70}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}^2 \to \mathbb{R}^2$, obtaining

$$\mathbf{X}_1(x_1, x_2) = \frac{1}{c^2 + r^2 + c\sqrt{c^2 + r^2}}(c^2 + r^2 + c\sqrt{c^2 + r^2} - x_1^2,\ -x_1 x_2)$$

$$\mathbf{X}_2(x_1, x_2) = \frac{1}{c^2 + r^2 + c\sqrt{c^2 + r^2}}(-x_1 x_2,\ c^2 + r^2 + c\sqrt{c^2 + r^2} - x_2^2) \tag{7.71}$$

The gyrometric coefficients of the gyrometric of the PV gyrovector plane in the Cartesian $x_1 x_2$-coordinates are therefore

$$\begin{aligned} E &= \mathbf{X}_1 \cdot \mathbf{X}_1 = \frac{c^2 + x_2^2}{c^2 + r^2} \\ F &= \mathbf{X}_1 \cdot \mathbf{X}_2 = -\frac{x_1 x_2}{c^2 + r^2} \\ G &= \mathbf{X}_2 \cdot \mathbf{X}_2 = \frac{c^2 + x_1^2}{c^2 + r^2} \end{aligned} \tag{7.72}$$

Hence, the gyroline element of the PV gyrovector plane $(\mathbb{R}^2, \oplus_{\text{U}}, \otimes_{\text{U}})$ is the Riemannian line element

$$\begin{aligned} ds_{\text{U}}^2 &= \|d\mathbf{s}_{\text{U}}\|^2 \\ &= E dx_1^2 + 2F dx_1 dx_2 + G dx_2^2 \\ &= dx_1^2 + dx_2^2 - \frac{1}{c^2 + r^2}(x_1 dx_1 + x_2 dx_2)^2 \end{aligned} \tag{7.73}$$

where $r^2 = x_1^2 + x_2^2$.

Following Riemann [Stahl (1993), p. 73], we note that E, G and

$$EG - F^2 = \frac{c^2}{c^2 + r^2} \tag{7.74}$$

are all positive in the $\mathbb{R}^2$, so that the quadratic form (7.73) is positive definite.

The gyrocurvature of the PV gyrovector plane is the Gaussian curvature K of the surface with the line element (7.73). It is a negative constant,

$$K = -\frac{1}{c^2} \tag{7.75}$$

as one can calculate from (7.20).

Extending (7.73) from $n = 2$ to $n \geq 2$ we have

$$ds_{\text{U}}^2 = d\mathbf{r}^2 - \frac{1}{c^2 + \mathbf{r}^2}(\mathbf{r} \cdot d\mathbf{r})^2 \tag{7.76}$$

and, as expected, the hyperbolic Riemannian line element reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c\to\infty} ds_{\mathrm{U}}^2 = d\mathbf{r}^2 \tag{7.77}$$

We have thus established the following

Theorem 7.8 *The gyroline element of a PV gyrovector space* $(\mathbb{R}^n, \oplus_U, \otimes_U)$ *is given by the equation*

$$ds_{\mathrm{U}}^2 = d\mathbf{r}^2 - \frac{1}{c^2+\mathbf{r}^2}(\mathbf{r}\cdot d\mathbf{r})^2 \tag{7.78}$$

and its gyrocurvature is given by the equation

$$K = -\frac{1}{c^2} \tag{7.79}$$

7.8 The Cogyroline Element of PV Gyrovector Spaces

In this section we uncover the Riemannian line element to which the cogyrometric of the PV gyrovector space $(\mathbb{R}^n, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}})$ gives rise.

Let us consider the cogyrodifferential (7.12),

$$\begin{aligned}\Delta\mathbf{s}_{CU} &= (\mathbf{v}+\Delta\mathbf{v}) \boxminus_{\mathrm{U}} \mathbf{v} \\ &= \begin{pmatrix} x_1+\Delta x_1 \\ x_2+\Delta x_2 \end{pmatrix} \boxminus_{\mathrm{U}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\end{aligned} \tag{7.80}$$

in the PV gyrovector plane $(\mathbb{R}_c^2, \oplus_{\mathrm{E}}, \otimes_{\mathrm{E}})$, where $+$ is the Euclidean addition in $\mathbb{R}^2$ and $\mathbb{R}$. To calculate $\mathbf{X}_1$ and $\mathbf{X}_2$ we have

$$\begin{aligned} d\mathbf{s}_{CU} &= \left[\frac{\partial\Delta\mathbf{s}_{CU}}{\partial\Delta x_1}\right]_{\left\{\begin{smallmatrix}\Delta x_1=0\\ \Delta x_2=0\end{smallmatrix}\right\}} dx_1 \quad + \quad \left[\frac{\partial\Delta\mathbf{s}_{CU}}{\partial\Delta x_2}\right]_{\left\{\begin{smallmatrix}\Delta x_1=0\\ \Delta x_2=0\end{smallmatrix}\right\}} dx_2 \\ &= \mathbf{X}_1(x_1,x_2)dx_1 + \mathbf{X}_2(x_1,x_2)dx_2 \end{aligned} \tag{7.81}$$

where $\mathbf{X}_1, \mathbf{X}_2 : \mathbb{R}_c^2 \to \mathbb{R}^2$, obtaining

$$\begin{aligned}\mathbf{X}_1(x_1, x_2) &= \frac{c^2}{c^2+r^2}(1,\, 0)\\ \mathbf{X}_2(x_1, x_2) &= \frac{c^2}{c^2+r^2}(0,\, 1)\end{aligned} \tag{7.82}$$

The cogyrometric coefficients of the cogyrometric of the PV gyrovector plane in the Cartesian x_1x_2-coordinates are therefore

$$\begin{aligned}E &= \mathbf{X}_1\cdot\mathbf{X}_1 = \frac{c^4}{(c^2+r^2)^2}\\ F &= \mathbf{X}_1\cdot\mathbf{X}_2 = 0\\ G &= \mathbf{X}_2\cdot\mathbf{X}_2 = \frac{c^4}{(c^2+r^2)^2}\end{aligned} \tag{7.83}$$

Hence, the cogyroline element of the PV gyrovector plane $(\mathbb{R}^2, \oplus_{\mathrm{U}}, \otimes_{\mathrm{U}})$ is the Riemannian line element

$$\begin{aligned}ds_{CU}^2 &= \|d\mathbf{s}_{CU}\|^2\\ &= Edx_1^2 + 2Fdx_1dx_2 + Gdx_2^2\\ &= \frac{c^4}{(c^2+r^2)^2}(dx_1^2 + dx_2^2)\end{aligned} \tag{7.84}$$

where $r^2 = x_1^2 + x_2^2$. In vector notation, (7.2), the Riemannian line element (7.84), extended to n dimensions, takes the form

$$ds_{CU}^2 = \frac{c^4}{(c^2+\mathbf{r}^2)^2}d\mathbf{r}^2 \tag{7.85}$$

and, as expected, the hyperbolic Riemannian line element reduces to its Euclidean counterpart in the limit of large c,

$$\lim_{c\to\infty} ds_{CU}^2 = d\mathbf{r}^2 \tag{7.86}$$

The metric (7.85) has the form $ds^2 = \lambda(\mathbf{r})d\mathbf{r}^2$, $\lambda(\mathbf{r}) > 0$, giving rise to an *isothermal* Riemannian surface $(\mathbb{R}^2, ds_{CU}^2)$ [Carmo (1976)].

The Riemannian metric ds_{CU}^2 in (7.84) is similar to the Riemannian metric ds_{M}^2 in (7.26). It is described in [Farkas and Kra (1992), p. 214], as a Riemannian metric on the Riemann surface M, M being the entire complex plane $\mathbb{C} \cup \{\infty\}$.

The cogyrocurvature of the PV gyrovector plane is the Gaussian curvature K of this surface. It is a positive constant,

$$K = \frac{16}{c^2} \tag{7.87}$$

as one can calculate from (7.20). We have thus established the following

Theorem 7.9 *The cogyroline element of a PV gyrovector space* $(\mathbb{R}^n, \oplus_U, \otimes_U)$ *is given by the equation*

$$ds_{CU}^2 = \frac{c^4}{(c^2 + \mathbf{r}^2)^2} d\mathbf{r}^2 \tag{7.88}$$

and its cogyrocurvature is given by the equation

$$K = \frac{16}{c^2} \tag{7.89}$$

7.9 Table of Riemannian Line Elements

The three analytic models of hyperbolic geometry that we study in this book are governed by the mutually isomorphic gyrovector spaces of Einstein, Möbius, and PV. They are all differentiable manifolds with a Riemannian metric. The Riemannian metric ds^2 on a Euclidean space $\mathbb{V} = \mathbb{R}^n$, or on its ball $\mathbb{V}_c = \mathbb{R}^n_c$, is a function that assigns at each point $\mathbf{x} \in \mathbb{V}$, or $\mathbb{V}_c$, a positive definite symmetric inner product on the tangent space at $\mathbf{x}$, varying differentiably with $\mathbf{x}$. Having studied the Riemannian line elements ds^2 in several gyrovector spaces, we we now summarize the results in Table 7.1.

Interestingly, the table shows that the Gaussian curvatures of the Riemannian gyrometrics of the gyrovector spaces of Einstein, Möbius and PV are negative, inversely proportional to the square of their free parameter c.

In contrast, the Gaussian curvatures of the Riemannian cogyrometrics are positive. The gyrometric and cogyrometric of a Euclidean space coincide, being reduced to the standard Euclidean metric. Accordingly, the Gaussian curvature of the Riemannian metric of the Euclidean space is zero, the only real number that equals its own negative.

In modern terms, hyperbolic geometry is the study of manifolds with Riemannian metrics with constant negative curvature. However, we see from Table 7.1 that in classical hyperbolic geometry, that is, the hyperbolic geometry of Bolyai and Lobachevsky, constant negative curvatures and variable positive ones are inseparable.

Table 7.1 The Riemannian line element, ds^2, in vector notation, (7.2), for gyrovector space gyrometrics $\|\mathbf{b}\ominus\mathbf{a}\|$ and cogyrometrics $\|\mathbf{b}\boxminus\mathbf{a}\|$. In the special case when a gyrovector space is a Euclidean vector space, the gyrovector space gyrometric and cogyrometric jointly reduce to the vector space metric. Interestingly, speaking gyrolanguage, the gyrocurvatures are negative, the cogyrocurvatures are positive, and the curvature is zero.

Metric Gyrometric Cogyrometric	Gyrovector Space	Riemannian Line Element	Gaussian Curvature
Metric $\|\mathbf{b}-\mathbf{a}\|$	Euclidean vector space $\mathbb{R}^n$	$ds^2_{euc} = d\mathbf{r}^2$ $\mathbf{r}^2 \geq 0$ Line Element	$K=0$ Curvature zero
Gyrometric $\|\mathbf{b}\ominus_E\mathbf{a}\|$	Einstein gyrovector space $\mathbb{R}^n_c$	$ds^2_E = \frac{c^2}{c^2-\mathbf{r}^2}d\mathbf{r}^2$ $+\frac{c^2}{(c^2-\mathbf{r}^2)^2}(\mathbf{r}\cdot d\mathbf{r})^2$ $\mathbf{r}^2 < c^2$ (Beltrami Model) Gyroline Element	$K=-\frac{1}{c^2}$ Gyrocurvature negative
Cogyrometric $\|\mathbf{b}\boxminus_E\mathbf{a}\|$	Einstein gyrovector space $\mathbb{R}^n_c$	$ds^2_{CE} = d\mathbf{r}^2$ $+\frac{2c^2-\mathbf{r}^2}{(c^2-\mathbf{r}^2)^2}(\mathbf{r}\cdot d\mathbf{r})^2$ $\mathbf{r}^2 < c^2$ Cogyroline Element	$K=2\frac{c^2-\mathbf{r}^2}{c^4}$ Cogyrocurvature positive
Gyrometric $\|\mathbf{b}\ominus_M\mathbf{a}\|$	Möbius gyrovector space $\mathbb{R}^n_c$	$ds^2_M = \frac{c^4}{(c^2-\mathbf{r}^2)^2}d\mathbf{r}^2$ $\mathbf{r}^2 < c^2$ (Poincaré Model) Gyroline Element	$K=-\frac{4}{c^2}$ Gyrocurvature negative
Cogyrometric $\|\mathbf{b}\boxminus_M\mathbf{a}\|$	Möbius gyrovector space $\mathbb{R}^n_c$	$ds^2_{CM} = \frac{c^4}{(c^4-\mathbf{r}^4)^2}$ $\{(c^2+\mathbf{r}^2)^2d\mathbf{r}^2 - 4c^2(\mathbf{r}\times d\mathbf{r})^2\}$ $\mathbf{r}^2 < c^2$ Cogyroline Element	$K=\frac{8c^6}{(c^2+\mathbf{r}^2)^4}$ Cogyrocurvature positive
Gyrometric $\|\mathbf{b}\ominus_U\mathbf{a}\|$	PV gyrovector space $\mathbb{R}^n$	$ds^2_U = d\mathbf{r}^2 - \frac{1}{c^2+\mathbf{r}^2}(\mathbf{r}\cdot d\mathbf{r})^2$ $\mathbf{r}^2 \geq 0$ (PV Model) Gyroline Element	$K=-\frac{1}{c^2}$ Gyrocurvature negative
Cogyrometric $\|\mathbf{b}\boxminus_U\mathbf{a}\|$	PV gyrovector space $\mathbb{R}^n$	$ds^2_{CU} = \frac{c^4}{(c^2+\mathbf{r}^2)^2}d\mathbf{r}^2$ $\mathbf{r}^2 \geq 0$ Cogyroline Element	$K=\frac{16}{c^2}$ Cogyrocurvature positive

Chapter 8

Gyrotrigonometry

Gyrotrigonometry is the study of how the sides and gyroangles of a gyrotriangle are related to each other, acting as a computational gyrogeometry. Gyrogeometry, in turn, is the geometry of gyrovector spaces. Since gyrovector spaces include vector spaces as a special case, gyrotrigonometry unifies Euclidean and hyperbolic trigonometry in the same way that gyrogeometry unifies Euclidean and hyperbolic geometry. Before embarking on gyrotrigonometry we must introduce the notion of the gyroangle and the study of the Pythagorean theorem of right-gyroangled gyrotriangles in gyrovector spaces.

8.1 Gyroangles

Definition 8.1 **(Unit Gyrovectors).** *Let* $\ominus\mathbf{a}\oplus\mathbf{b}$ *be a nonzero gyrovector in a gyrovector space* $(G, \oplus, \otimes)$. *Its gyrolength is* $\|\ominus\mathbf{a}\oplus\mathbf{b}\|$ *and its associated gyrovector*

$$\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|} \tag{8.1}$$

is called a unit gyrovector.

Unit gyrovectors represent "gyrodirections". A gyroangle is, accordingly, a relation between two gyrodirections.

Definition 8.2 **(The Gyrocosine Function And Gyroangles, I).** *Let* $\ominus\mathbf{a}\oplus\mathbf{b}$ *and* $\ominus\mathbf{a}\oplus\mathbf{c}$ *be two nonzero rooted gyrovectors rooted at a common point* $\mathbf{a}$ *in a gyrovector space* $(G, \oplus, \otimes)$. *The gyrocosine of the measure of the gyroangle* α, $0 \le \alpha \le \pi$, *that the two rooted gyrovectors generate is* *given by the equation*

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|} \tag{8.2}$$

The gyroangle α *in* (8.2) *is denoted by* $\alpha = \angle\mathbf{bac}$ *or, equivalently,* $\alpha = \angle\mathbf{cab}$. *Two gyroangles are congruent if they have the same measure.*

Theorem 8.3 *The measure of a gyroangle is model independent.*

Proof. Following Def. 6.85 let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be two isomorphic gyrovector spaces with isomorphism $\phi: G_1 \to G_2$. Furthermore, let α_1 be a gyroangle in G_1 given by

$$\cos\alpha_1 = \frac{\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1}{\|\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1\|}\cdot\frac{\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1}{\|\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1\|} \tag{8.3}$$

and let α_2 be the isomorphic gyroangle in G_2,

$$\cos\alpha_2 = \frac{\ominus_2\mathbf{a}_2\oplus_2\mathbf{b}_2}{\|\ominus_2\mathbf{a}_2\oplus_2\mathbf{b}_2\|}\cdot\frac{\ominus_2\mathbf{a}_2\oplus_2\mathbf{c}_2}{\|\ominus_2\mathbf{a}_2\oplus_2\mathbf{c}_2\|} \tag{8.4}$$

where $\mathbf{a}_2 = \phi(\mathbf{a}_1)$, $\mathbf{b}_2 = \phi(\mathbf{b}_1)$, and $\mathbf{c}_2 = \phi(\mathbf{c}_1)$.

Then, by (6.290) and (6.292) we have

$$\begin{aligned}
\cos\alpha_2 &= \frac{\ominus_2\mathbf{a}_2\oplus_2\mathbf{b}_2}{\|\ominus_2\mathbf{a}_2\oplus_2\mathbf{b}_2\|}\cdot\frac{\ominus_2\mathbf{a}_2\oplus_2\mathbf{c}_2}{\|\ominus_2\mathbf{a}_2\oplus_2\mathbf{c}_2\|} \\
&= \frac{\ominus_2\phi(\mathbf{a}_1)\oplus_2\phi(\mathbf{b}_1)}{\|\ominus_2\phi(\mathbf{a}_1)\oplus_2\phi(\mathbf{b}_1)\|}\cdot\frac{\ominus_2\phi(\mathbf{a}_1)\oplus_2\phi(\mathbf{c}_1)}{\|\ominus_2\phi(\mathbf{a}_1)\oplus_2\phi(\mathbf{c}_1)\|} \\
&= \frac{\phi(\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1)}{\|\phi(\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1)\|}\cdot\frac{\phi(\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1)}{\|\phi(\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1)\|} \\
&= \frac{\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1}{\|\ominus_1\mathbf{a}_1\oplus_1\mathbf{b}_1\|}\cdot\frac{\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1}{\|\ominus_1\mathbf{a}_1\oplus_1\mathbf{c}_1\|} \\
&= \cos\alpha_1
\end{aligned} \tag{8.5}$$

so that α_1 and α_2 have the same measure, $\alpha_1 = \alpha_2$. □

Möbius gyrovector spaces are conformal to corresponding Euclidean spaces, as explained in Sec. 7.3, p. 220. Accordingly, it is convenient to visualize gyroangles in the Möbius gyrovector plane, Fig. 8.1, where the measure of the gyroangle between two intersecting gyrolines is equal to the measure of the Euclidean angle between corresponding intersecting tangent lines.

To calculate the gyrocosine of a gyroangle α generated by two intersecting gyrolines, Fig. 8.1, we place on the two intersecting gyrolines two rooted gyrovectors with a common tail at the point of intersection. In Fig. 8.1 we, accordingly, place the two nonzero gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$ and $\ominus\mathbf{a}\oplus\mathbf{c}$ or, equivalently, $\ominus\mathbf{a}\oplus\mathbf{b}'$ and $\ominus\mathbf{a}\oplus\mathbf{c}'$ so that the measure of the gyroangle α in Fig. 8.1 is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|} \tag{8.6}$$

or, equivalently, by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}'}{\|\ominus\mathbf{a}\oplus\mathbf{b}'\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}'}{\|\ominus\mathbf{a}\oplus\mathbf{c}'\|} \tag{8.7}$$

In the following theorem we show that, as expected, (8.6) and (8.7) give the same gyroangle measure for the gyroangle α.

Definition 8.4 **(Gyrorays).** *Let* $\mathbf{o}$ *and* $\mathbf{p}$ *be any two distinct points of a gyrovector space* $(G,\oplus,\otimes)$. *A gyroray with origin* $\mathbf{o}$ *containing the point* $\mathbf{p}$ *is the set* L *of points in* G *given by*

$$L = \mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.8}$$

$t\in\mathbb{R}^{\geq 0}$.

Theorem 8.5 *Let*

$$\begin{aligned} L_{\mathbf{ab}} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t \\ L_{\mathbf{ac}} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t \end{aligned} \tag{8.9}$$

$t\in\mathbb{R}^{\geq 0}$, *be two gyrorays emanating from a point* $\mathbf{a}$ *in a gyrovector space* $(G,\oplus,\otimes)$, *and let* $\mathbf{b}'$ *and* $\mathbf{c}'$ *be any points, other than* $\mathbf{a}$, *lying on* $L_{\mathbf{ab}}$ *and* $L_{\mathbf{ac}}$, *respectively, Fig. 8.1. Furthermore, let* α *be the gyroangle between the two gyrorays, expressed in terms of* $\mathbf{b}'$ *and* $\mathbf{c}'$,

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}'}{\|\ominus\mathbf{a}\oplus\mathbf{b}'\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}'}{\|\ominus\mathbf{a}\oplus\mathbf{c}'\|} \tag{8.10}$$

Then, α *is independent of the choice of the points* $\mathbf{b}'$ *and* $\mathbf{c}'$ *on their respective gyrorays.*

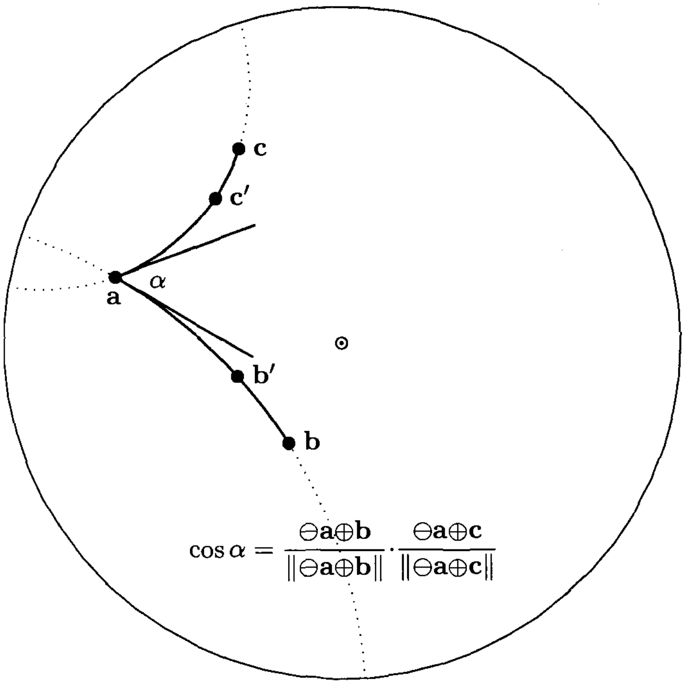


Fig. 8.1 A Möbius gyroangle α generated by two intersecting Möbius geodesic rays (gyrorays). Its measure equals the measure of the Euclidean angle generated by corresponding intersecting tangent lines.

Proof. Since the points $\mathbf{b}'$ and $\mathbf{c}'$ lie, respectively, on the gyrorays $L_{\mathbf{ab}}$ and $L_{\mathbf{ac}}$, and since they are different from $\mathbf{a}$, they are given by the equations

$$\begin{aligned}\mathbf{b}' &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1 \\ \mathbf{c}' &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_2\end{aligned} \tag{8.11}$$

for some $t_1, t_2 > 0$. Hence,

$$\begin{aligned}\ominus\mathbf{a}\oplus\mathbf{b}' &= (\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1 \\ \ominus\mathbf{a}\oplus\mathbf{c}' &= (\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_2\end{aligned} \tag{8.12}$$

so that, by (8.10), (8.12), and the scaling property $(V4)$ of gyrovector spaces, we have

$$\begin{aligned}\cos\alpha &= \frac{\ominus\mathbf{a}\oplus\mathbf{b}'}{\|\ominus\mathbf{a}\oplus\mathbf{b}'\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}'}{\|\ominus\mathbf{a}\oplus\mathbf{c}'\|}\\ &= \frac{(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1}{\|(\ominus\mathbf{a}\oplus\mathbf{b})\otimes t_1\|}\cdot\frac{(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_2}{\|(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_2\|}\\ &= \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}\end{aligned} \tag{8.13}$$

Hence, $\cos\alpha$ is independent of the choice of the points $\mathbf{b}'$ and $\mathbf{c}'$ on their respective gyrorays, Fig. 8.1. □

Theorem 8.6 *Gyroangles are invariant under gyrovector space motions.*

Proof. We have to show that the gyroangle $\alpha = \angle\mathbf{bac}$ for any points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of a gyrovector space $(G, \oplus, \otimes)$ is invariant under the gyrovector space motions, Def. 6.6. Equivalently, we have to show that

$$\angle\mathbf{bac} = \angle(\tau\mathbf{b})(\tau\mathbf{a})(\tau\mathbf{c}) \tag{8.14}$$

and

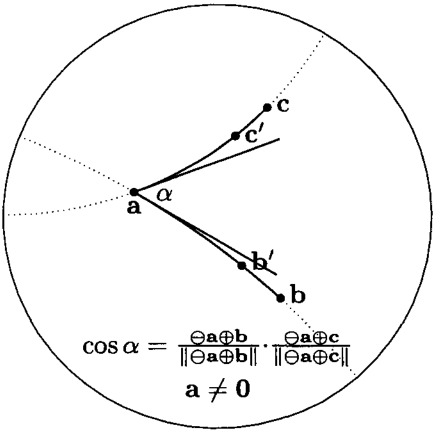
$$\angle\mathbf{bac} = \angle(\mathbf{x}\oplus\mathbf{b})(\mathbf{x}\oplus\mathbf{a})(\mathbf{x}\oplus\mathbf{c}) \tag{8.15}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{x} \in G$ and all $\tau \in Aut(G)$.

Employing (6.7) we have

$$\begin{aligned}\cos\angle(\tau\mathbf{b})(\tau\mathbf{a})(\tau\mathbf{c}) &= \frac{\ominus\tau\mathbf{a}\oplus\tau\mathbf{b}}{\|\ominus\tau\mathbf{a}\oplus\tau\mathbf{b}\|}\cdot\frac{\ominus\tau\mathbf{a}\oplus\tau\mathbf{c}}{\|\ominus\tau\mathbf{a}\oplus\tau\mathbf{c}\|}\\ &= \frac{\tau(\ominus\mathbf{a}\oplus\mathbf{b})}{\|\tau(\ominus\mathbf{a}\oplus\mathbf{b})\|}\cdot\frac{\tau(\ominus\mathbf{a}\oplus\mathbf{c})}{\|\tau(\ominus\mathbf{a}\oplus\mathbf{c})\|}\\ &= \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}\\ &= \cos\angle\mathbf{bac}\end{aligned} \tag{8.16}$$

since automorphisms preserve the inner product and the norm, thus verifying (8.14).



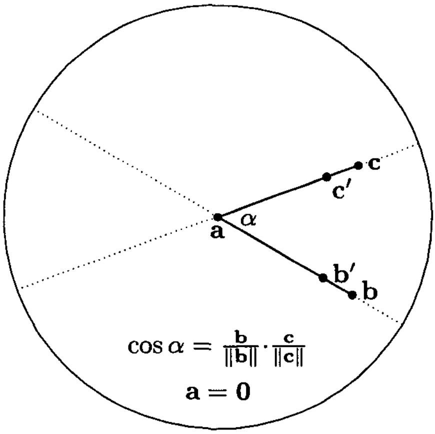


Fig. 8.2 Left gyrotranslations keep gyroangles invariant by Theorem 8.6. Two successive left gyrotranslations of the gyroangle α of Fig. 8.1 towards the origin of the Möbius disc are shown in Figs. 8.2 and 8.3. The measure of the gyroangle α between two gyrolines equals the Euclidean measure of the angle between corresponding Euclidean tangent lines.
With vertex $\mathbf{a} \neq \mathbf{0}$, the gyroangle α has a gyrocosine, denoted $\cos\alpha$.

Fig. 8.3 At the Euclidean origin of the Möbius disc the gyrolines that generate a gyroangle α coincide with their respective Euclidean tangent lines, and the gyroangle α coincides with its Euclidean counterpart. At the Euclidean origin of a Möbius disc (or ball), thus, the concepts of gyroangles and angles coincide.
With vertex $\mathbf{a} = \mathbf{0}$, the gyroangle α becomes an angle, and its gyrocosine becomes a cosine.

Noting the Gyrotranslation Theorem 3.13, we have

$$\begin{aligned}\cos\angle(\mathbf{x}\oplus\mathbf{b})(\mathbf{x}\oplus\mathbf{a})(\mathbf{x}\oplus\mathbf{c}) &= \frac{\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{b})}{\|\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{b})\|}\cdot\frac{\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{c})}{\|\ominus(\mathbf{x}\oplus\mathbf{a})\oplus(\mathbf{x}\oplus\mathbf{c})\|}\\ &= \frac{\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})}{\|\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})\|}\cdot\frac{\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})}{\|\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})\|}\\ &= \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}\\ &= \cos\angle\mathbf{bac}\end{aligned} \tag{8.17}$$

since gyroautomorphisms preserve the inner product and the norm, thus verifying (8.15). □

Figures 8.2 and 8.3, along with Theorem 8.6, show that gyroangles behave like angles so that, in particular, gyroangles add up to 2π.

Theorem 8.7 *If a point* $\mathbf{b}$ *lies between two distinct points* $\mathbf{a}$ *and* $\mathbf{c}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *then*

$$\angle\mathbf{abc} = \pi \tag{8.18}$$

Proof. The three points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are gyrocollinear, by Lemma 6.25 and Def. 6.22.

By Lemmas 6.25 and 6.26 we have

$$\begin{aligned} \mathbf{b} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0 \\ \mathbf{b} &= \mathbf{c}\oplus(\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0) \end{aligned} \tag{8.19}$$

for some $0 < t_0 < 1$. Hence, by left cancellations,

$$\begin{aligned} \ominus\mathbf{a}\oplus\mathbf{b} &= (\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0 \\ \ominus\mathbf{c}\oplus\mathbf{b} &= (\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0) \end{aligned} \tag{8.20}$$

Therefore, by the gyrocommutative law, the invariance of the inner product under gyrations, the scaling property of gyrovector spaces, Identity (6.82), and the automorphic inverse property we have the chain of equations

$$\begin{aligned} \cos\angle\mathbf{abc} &= \frac{\ominus\mathbf{b}\oplus\mathbf{a}}{\|\ominus\mathbf{b}\oplus\mathbf{a}\|}\cdot\frac{\ominus\mathbf{b}\oplus\mathbf{c}}{\|\ominus\mathbf{b}\oplus\mathbf{c}\|} \\ &= \frac{\mathrm{gyr}[\ominus\mathbf{b},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})}{\|\mathrm{gyr}[\ominus\mathbf{b},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{b})\|}\cdot\frac{\mathrm{gyr}[\ominus\mathbf{b},\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{b})}{\|\mathrm{gyr}[\ominus\mathbf{b},\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{b})\|} \\ &= \frac{\mathrm{gyr}[\ominus\mathbf{b},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0}{\|\mathrm{gyr}[\ominus\mathbf{b},\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0\|}\cdot\frac{\mathrm{gyr}[\ominus\mathbf{b},\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0)}{\|\mathrm{gyr}[\ominus\mathbf{b},\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0)\|} \\ &= \frac{(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0}{\|(\ominus\mathbf{a}\oplus\mathbf{c})\otimes t_0\|}\cdot\frac{\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b},\ominus\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0)}{\|\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b},\ominus\mathbf{c}](\ominus\mathbf{c}\oplus\mathbf{a})\otimes(1-t_0)\|} \\ &= \ominus\frac{(\ominus\mathbf{a}\oplus\mathbf{c})}{\|(\ominus\mathbf{a}\oplus\mathbf{c})\|}\cdot\frac{\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b},\ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c},\ominus\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})}{\|\mathrm{gyr}[\mathbf{a},\ominus\mathbf{b}]\mathrm{gyr}[\mathbf{b},\ominus\mathbf{c}]\mathrm{gyr}[\mathbf{c},\ominus\mathbf{a}](\ominus\mathbf{a}\oplus\mathbf{c})\|} \\ &= \ominus\frac{(\ominus\mathbf{a}\oplus\mathbf{c})}{\|(\ominus\mathbf{a}\oplus\mathbf{c})\|}\cdot\frac{(\ominus\mathbf{a}\oplus\mathbf{c})}{\|(\ominus\mathbf{a}\oplus\mathbf{c})\|} \\ &= -1 \\ &= \cos\pi \end{aligned} \tag{8.21}$$

thus verifying (8.18). $\square$

Left gyrotranslating the points of the Möbius gyrovector plane in Fig. 8.1 by $\ominus\mathbf{a}$ amounts to gyrotranslating the gyroangle α to the origin of the gyroplane. This gyrotranslation, in turn, keeps the gyroangle invariant according to Theorem 8.6. Once at the origin of its Möbius gyrovector space, a gyroangle measure coincides with its Euclidean measure and its generating gyrolines reduce to generating Euclidean straight lines, as shown in Figs. 8.2 and 8.3 for the Möbius gyrovector plane.

An extension of Def. 8.2 to the gyroangle between two gyrovectors that need not be rooted at a common point is natural, giving rise to the following

Definition 8.8 (Gyroangles Between Gyrovectors, II). *Let $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ be two nonzero gyrovectors in a gyrovector space $(G,\oplus,\otimes)$. The gyrocosine of the gyroangle α, $0\le\alpha\le\pi$, between the two gyrovectors,*

$$\alpha = \angle(\ominus\mathbf{a}_1\oplus\mathbf{b}_1)(\ominus\mathbf{a}_2\oplus\mathbf{b}_2) \tag{8.22}$$

is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\ominus\mathbf{a}_2\oplus\mathbf{b}_2}{\|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|} \tag{8.23}$$

Definition 8.9 (Gyrovector Parallelism and Perpendicularity). *Two gyrovectors are parallel (perpendicular) if the gyroangle α between them satisfies $\cos\alpha = 1$, that is, $\alpha = 0$ ($\cos\alpha = 0$, that is, $\alpha = \pi/2$, resp.).*

Theorem 8.10 *Gyrovector translations keep gyroangles between gyrovectors invariant.*

Proof. Gyrovector translations, Def. 5.6 and Theorem 5.7, do not modify the value of gyrovectors and, hence, keep (8.23) invariant. □

In order to visualize the gyroangle of Def. 8.8 in a Möbius gyrovector plane one may gyrovector translate the two rooted gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ that generate a gyroangle in (8.23) into new ones that possess a common tail, as shown in Fig. 8.4.

Thus, the notion of the gyrovector translation of gyrovectors allows, in Def. 8.8, to complete the analogy that the gyroangle of Def. 8.2 shares with its Euclidean counterpart. We can now define the gyroangle between gyrorays as well.

The gyroangle between any gyrovector and itself vanishes, Def. 8.8, so that any gyrovector is parallel to itself. Hence, by Theorem 8.10, successive

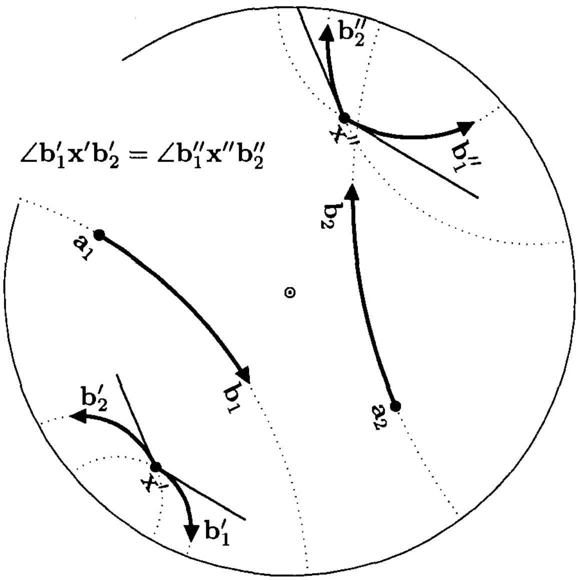


Fig. 8.4 Gyrovector translation allows the visualization of the gyroangle generated by two rooted gyrovectors that have distinct tails, $\mathbf{a}_1$ and $\mathbf{a}_2$. The gyroangle is visually revealed by gyrovector translating the rooted gyrovectors into new rooted gyrovectors with a common tail $\mathbf{x}$. The two cases of $\mathbf{x} = \mathbf{x}'$ and $\mathbf{x} = \mathbf{x}''$ in the Möbius gyrovector plane are shown. Thus

$$\ominus\mathbf{a}_1\oplus\mathbf{b}_1 = \ominus\mathbf{x}'\oplus\mathbf{b}'_1 = \ominus\mathbf{x}''\oplus\mathbf{b}''_1$$
$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2 = \ominus\mathbf{x}'\oplus\mathbf{b}'_2 = \ominus\mathbf{x}''\oplus\mathbf{b}''_2$$

The measure of the included gyroangle between two gyrovectors that share a common tail in a Möbius gyrovector space equals the measure of the included angle between two tangent rays at the common tail. These tangent rays are, therefore, shown.

gyrovector translations of any given gyrovector form a family of gyrovectors that are parallel to each other. The gyroray analog, a family of parallel gyrorays, is shown in Fig. 8.8.

Theorem 8.11 *Let $\ominus\mathbf{a}\oplus\mathbf{b}$, $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ be three nonzero gyrovectors in a gyrovector space $(G, \oplus, \otimes)$. The gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ are parallel if and only if*

$$\angle(\ominus\mathbf{a}\oplus\mathbf{b})(\ominus\mathbf{a}_1\oplus\mathbf{b}_1) = \angle(\ominus\mathbf{a}\oplus\mathbf{b})(\ominus\mathbf{a}_2\oplus\mathbf{b}_2) \tag{8.24}$$

Proof. If (8.24) holds then, by Def. 8.8,

$$\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}=\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}_2\oplus\mathbf{b}_2}{\|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|} \tag{8.25}$$

implying

$$\frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\left(\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}-\frac{\ominus\mathbf{a}_2\oplus\mathbf{b}_2}{\|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|}\right)=0 \tag{8.26}$$

in the carrier $\mathbb{V}$ of G. Owing to the positive definiteness of the inner product in the carrier vector space $\mathbb{V}$, Def. 6.2, (8.26) implies

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2=\lambda(\ominus\mathbf{a}_1\oplus\mathbf{b}_1) \tag{8.27}$$

for some $\lambda>0$ in the carrier vector space $\mathbb{V}$.

Hence, the gyroangle α between the gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ is given by the equation

$$\begin{aligned}\cos\alpha&=\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\ominus\mathbf{a}_2\oplus\mathbf{b}_2}{\|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|}\\&=\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\lambda(\ominus\mathbf{a}_1\oplus\mathbf{b}_1)}{\|\lambda(\ominus\mathbf{a}_1\oplus\mathbf{b}_1)\|}\\&=\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\\&=1\end{aligned} \tag{8.28}$$

implying $\alpha=0$ so that the two gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ are parallel.

Conversely, if the two gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ are parallel then the gyroangle between them vanishes,

$$\cos\alpha=\frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\ominus\mathbf{a}_2\oplus\mathbf{b}_2}{\|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|}=0 \tag{8.29}$$

implying

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2=\lambda(\ominus\mathbf{a}_1\oplus\mathbf{b}_1) \tag{8.30}$$

for some $\lambda>0$ in the carrier vector space $\mathbb{V}$. The latter, in turn, implies (8.25), which is equivalent to (8.24). □

Definition 8.12 (Gyroray Carriers of Gyrovectors, Gyroangles Between Gyrorays, III). *Let L be a gyroray with origin $\mathbf{o}$,*

$$L: \qquad \mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.31}$$

$t\in\mathbb{R}^{\geq 0}$, containing the point $\mathbf{a}$ in a gyrovector space $(G,\oplus,\otimes)$. Then, the gyrovector $\ominus\mathbf{o}\oplus\mathbf{a}$ lies on the gyroray L and, equivalently, the gyroray L carries (or, is the carrier of) the gyrovector $\ominus\mathbf{o}\oplus\mathbf{a}$.

Let L_1 and L_2 be two gyrorays carrying, respectively, the gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$ and $\ominus\mathbf{c}\oplus\mathbf{d}$. Then the gyroangle α between the gyrorays L_1 and L_2 is given by the gyroangle between the two gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$ and $\ominus\mathbf{c}\oplus\mathbf{d}$, that is,

$$\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{c}\oplus\mathbf{d}}{\|\ominus\mathbf{c}\oplus\mathbf{d}\|} \tag{8.32}$$

It follows from Def. 8.12 that the origin of a gyroray coincides with the tail of any gyrovector that it contains.

In terms of Def. 8.12, Theorem 8.5 can be stated as follows.

Theorem 8.13 *The gyroangle between two gyrorays is independent of the choice of the gyrovectors lying on the gyrorays.*

Owing to the presence of gyrations, the gyroangle between two gyrorays that lie on the same gyroline in the same direction does not vanish in general, as the following theorem demonstrates.

Theorem 8.14 *Let*

$$\begin{aligned} L_1: &\qquad \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t \\ L_2: &\qquad \mathbf{o}_2\oplus(\ominus\mathbf{o}_2\oplus\mathbf{p}_2)\otimes s \end{aligned} \tag{8.33}$$

$s,t\in\mathbb{R}^{\geq 0}$, be two gyrorays in a gyrovector space $(G,\oplus,\otimes)$ such that L_2 is contained in L_1, Fig. 8.6. Then, the gyroangle α between L_1 and L_2 is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{o}_1\oplus\mathbf{p}_1}{\|\ominus\mathbf{o}_1\oplus\mathbf{p}_1\|}\cdot\frac{\mathrm{gyr}[\mathbf{o}_2,\ominus\mathbf{o}_1](\ominus\mathbf{o}_1\oplus\mathbf{p}_1)}{\|\ominus\mathbf{o}_1\oplus\mathbf{p}_1\|} \tag{8.34}$$

Proof. Since L_2 is contained in L_1, the points $\mathbf{o}_2$ and $\mathbf{p}_2$ of L_2 lie on L_1. Hence, there exist real numbers $t_1, t_2 > 0$, $t_2 - t_1 > 0$, such that

$$\begin{aligned} \mathbf{o}_2 &= \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_1 \\ \mathbf{p}_2 &= \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_2 \end{aligned} \tag{8.35}$$

Hence, (i) by the Gyrotranslation Theorem 3.13, (ii) the first equation in (8.35) and the scalar distributive law $(V2)$ of gyrovector spaces, (iii) Theorem 2.30, and (iv) axiom $(V5)$ of gyrovector spaces, we have

$$\begin{aligned}\ominus\mathbf{o}_2\oplus\mathbf{p}_2 &= \ominus\{\mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_1\}\oplus\{\mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_2\}\\ &= \mathrm{gyr}[\mathbf{o}_1,(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_1]\{\ominus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t_2\}\\ &= \mathrm{gyr}[\mathbf{o}_1,\ominus\mathbf{o}_1\oplus\mathbf{o}_2](\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes(-t_1+t_2)\\ &= \mathrm{gyr}[\mathbf{o}_2,\ominus\mathbf{o}_1](\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes(-t_1+t_2)\\ &= \{\mathrm{gyr}[\mathbf{o}_2,\ominus\mathbf{o}_1](\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\}\otimes(-t_1+t_2)\end{aligned} \tag{8.36}$$

The gyroangle α between the gyrorays L_1 and L_2 is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{o}_1\oplus\mathbf{p}_1}{\|\ominus\mathbf{o}_1\oplus\mathbf{p}_1\|}\cdot\frac{\ominus\mathbf{o}_2\oplus\mathbf{p}_2}{\|\ominus\mathbf{o}_2\oplus\mathbf{p}_2\|} \tag{8.37}$$

Finally, substituting $\ominus\mathbf{o}_2\oplus\mathbf{p}_2$ from (8.36) in (8.37), and noting axiom $(V4)$ of gyrovector spaces and that gyrations preserve the norm, we have (8.34). □

Theorem 8.15 *Let*

$$\begin{aligned} L_1 : &\quad \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t\\ L_2 : &\quad \mathbf{o}_2\oplus(\ominus\mathbf{o}_2\oplus\mathbf{p}_2)\otimes s\end{aligned} \tag{8.38}$$

$s,t\in\mathbb{R}^{\ge 0}$, be two cogyrolinear gyrorays (that is, two gyrorays lying on the same gyroline) in a gyrovector space $(G,\oplus,\otimes)$, non of which contains the other one, Fig. 8.7. Then, the gyroangle α between L_1 and L_2 is given by the equation

$$\cos\alpha = -\frac{\ominus\mathbf{o}_1\oplus\mathbf{p}_1}{\|\ominus\mathbf{o}_1\oplus\mathbf{p}_1\|}\cdot\frac{\mathrm{gyr}[\mathbf{o}_2,\ominus\mathbf{o}_1](\ominus\mathbf{o}_1\oplus\mathbf{p}_1)}{\|\ominus\mathbf{o}_1\oplus\mathbf{p}_1\|} \tag{8.39}$$

Proof. The proof is identical with that of Theorem 8.14 with one exception. Here $t_2-t_1<0$ (rather than $t_2-t_1>0$ in Theorem 8.14), resulting in the negative sign in (8.39). □

8.2 Gyrovector Translation of Gyrorays

Definition 8.16 **(Gyrovector Translation Of Gyrorays).** *Let*

$$\mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.40}$$

$t \in \mathbb{R}^{\geq 0}$, *be a gyroray in a gyrovector space* $(G, \oplus, \otimes)$, *represented by its origin*, $\mathbf{o}$, *and any point* $\mathbf{p}$ *that it contains. Furthermore, let* $\ominus\mathbf{o}'\oplus\mathbf{p}'$ *be the gyrovector translation by* $\mathbf{t}$ *of gyrovector* $\ominus\mathbf{o}\oplus\mathbf{p}$, *Def. 5.6. Then, the gyroray*

$$\mathbf{o}'\oplus(\ominus\mathbf{o}'\oplus\mathbf{p}')\otimes t \tag{8.41}$$

$t\in\mathbb{R}^{\geq 0}$, *is said to be the gyrovector translation by* $\mathbf{t}$ *of the gyroray* (8.40).

Lemma 8.17 *The gyrovector translation of the gyroray*

$$\mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.42}$$

by a gyrovector $\mathbf{t}$ *in a gyrovector space* $(G, \oplus, \otimes)$ *results in the gyroray*

$$\mathrm{gyr}[\mathbf{o}, \mathbf{t}]\{[\mathbf{t}\oplus\mathbf{o}]\oplus(\ominus[\mathbf{t}\oplus\mathbf{o}]\oplus[\mathbf{t}\oplus\mathbf{p}])\otimes t\} \tag{8.43}$$

(which is obtainable from the original gyroray (8.42) *by a left gyrotranslation of the points* $\mathbf{o}$ *and* $\mathbf{p}$ *by* $\mathbf{t}$, *followed by a gyration* $\mathrm{gyr}[\mathbf{o}, \mathbf{t}]$*).*

Furthermore, (8.43) *can be written as*

$$(\mathbf{o}\oplus\mathbf{t})\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.44}$$

(which is obtainable from the original gyroray (8.42) *by a right gyrotranslation of the first* $\mathbf{o}$ *by* $\mathbf{t}$*).*

Proof. By Def. 8.16 the gyrovector translation of gyroray (8.42) by $\mathbf{t}$ is given by (8.41), in which $\mathbf{o}'$ and $\mathbf{p}'$ are determined by Def. 5.6,

$$\begin{aligned} \mathbf{o}' &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{o}) \\ \mathbf{p}' &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{p}) \end{aligned} \tag{8.45}$$

Hence, by (8.41), (8.45), the Gyrotranslation Theorem 3.13, the gyration inversive symmetry (2.93), and the gyrocommutative law, we have

$$\begin{aligned} \mathbf{o}'\oplus(\ominus\mathbf{o}'\oplus\mathbf{p}')\otimes t &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{o})\oplus\{\ominus\mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{o})\oplus\mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{p})\}\otimes t \\ &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}]\{[\mathbf{t}\oplus\mathbf{o}]\oplus(\ominus[\mathbf{t}\oplus\mathbf{o}]\oplus[\mathbf{t}\oplus\mathbf{p}])\otimes t\} \\ &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}]\{[\mathbf{t}\oplus\mathbf{o}]\oplus\mathrm{gyr}[\mathbf{t}, \mathbf{o}](\ominus\mathbf{o}\oplus\mathbf{p})\otimes t\} \\ &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}](\mathbf{t}\oplus\mathbf{o})\oplus\mathrm{gyr}[\mathbf{o}, \mathbf{t}]\mathrm{gyr}[\mathbf{t}, \mathbf{o}](\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \\ &= (\mathbf{o}\oplus\mathbf{t})\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \end{aligned} \tag{8.46}$$

$t\in\mathbb{R}^{\geq 0}$, thus verifying both (8.43) and (8.44). □

We may note that it follows from Lemma 8.17 that

(1) while the original gyroray (8.42) has origin $\mathbf{o}$, corresponding to $t = 0$, and contains the point $\mathbf{p}$, corresponding to $t = 1$,
(2) the gyrovector translated gyroray (8.44) by $\mathbf{t}$ has origin $\mathbf{o}\oplus\mathbf{t}$, corresponding to $t = 0$, and contains the point

$$(\mathbf{o}\oplus\mathbf{t})\oplus(\ominus\mathbf{o}\oplus\mathbf{p}) = \mathrm{gyr}[\mathbf{o},\mathbf{t}](\mathbf{t}\oplus\mathbf{p}) \tag{8.47}$$

corresponding to $t = 1$.

Interestingly, the origin of the gyrovector translated gyroray by $\mathbf{t}$ is right gyrotranslated by $\mathbf{t}$ from $\mathbf{o}$ to $\mathbf{o}\oplus\mathbf{t}$, as we see from the passage from (8.42) to (8.44) that the gyrovector translation by $\mathbf{t}$ generates.

Theorem 8.18 *A gyrovector translation by* $\mathbf{t}$ *of a gyroray*

$$\mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes t \tag{8.48}$$

$t\in\mathbb{R}^{\geq 0}$, *with origin* $\mathbf{o}$ *in a gyrovector space* $(G,\oplus,\otimes)$ *is independent of the selection of the point* $\mathbf{p}$ *on the gyroray.*

Proof. Let $\mathbf{b}$ be any point, different from $\mathbf{o}$, on the gyroray (8.48). Then, there exists a positive number r such that

$$\mathbf{b} = \mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes r \tag{8.49}$$

so that by left gyroassociativity and a left cancellation,

$$\begin{aligned} \ominus\mathbf{o}\oplus\mathbf{b} &= \ominus\mathbf{o}\oplus\{\mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes r\} \\ &= (\ominus\mathbf{o}\oplus\mathbf{p})\otimes r \end{aligned} \tag{8.50}$$

The gyrovector translation by $\mathbf{t}$ of the gyroray

$$\mathbf{o}\oplus(\ominus\mathbf{o}\oplus\mathbf{b})\otimes t \tag{8.51}$$

$t\in\mathbb{R}^{\geq 0}$, given by Lemma 8.17, is shown in (8.52) below, where we further manipulate it by means of (8.50) and the scalar associative law $(V3)$ of gyrovector spaces.

$$\begin{aligned} \mathbf{o}'\oplus(\ominus\mathbf{o}'\oplus\mathbf{b}')\otimes t &= (\mathbf{o}\oplus\mathbf{t})\oplus(\ominus\mathbf{o}\oplus\mathbf{b})\otimes t \\ &= (\mathbf{o}\oplus\mathbf{t})\oplus((\ominus\mathbf{o}\oplus\mathbf{p})\otimes r)\otimes t \\ &= (\mathbf{o}\oplus\mathbf{t})\oplus(\ominus\mathbf{o}\oplus\mathbf{p})\otimes(rt) \end{aligned} \tag{8.52}$$

$rt\in\mathbb{R}^{\geq 0}$. Comparing (8.52) with the extreme right hand side of the result (8.46) of Lemma 8.17 we see that the choice of the point $\mathbf{b}$ on the gyroray, in (8.51), instead of the point $\mathbf{p}$ on the gyroray, in (8.48), does not modify the gyrovector translated gyroray. Rather, it only reparametrizes the gyroray, replacing the positive parameter t by another positive parameter, rt, where r is a positive number that depends on $\mathbf{b}$. □

Theorem 8.19 *Gyrovector translations of gyrorays keep gyroangles between gyrorays invariant.*

Proof. By Def. 8.12, gyroangles between gyrorays are given by gyroangles between gyrovectors that respectively lie on the gyrorays.

Similarly, by Def. 8.16, gyrovector translations of gyrorays are given by gyrovector translations of gyrovectors that respectively lie on the gyrorays.

But, by Theorem 8.10, gyrovector translations keep gyroangles between gyrovectors invariant. Hence, gyrovector translations keep gyroangles between gyrorays invariant as well. □

It is instructive to present a second, direct proof of Theorem 8.19.

Proof. A second, direct proof of Theorem 8.19: Let

$$\begin{aligned} L_1 : &\quad \mathbf{o}_1 \oplus (\ominus \mathbf{o}_1 \oplus \mathbf{p}_1) \otimes s \\ L_2 : &\quad \mathbf{o}_2 \oplus (\ominus \mathbf{o}_2 \oplus \mathbf{p}_2) \otimes t \end{aligned} \tag{8.53}$$

$s, t \in \mathbb{R}^{\geq 0}$, be two gyrorays in a gyrovector space $(G, \oplus, \otimes)$, Fig. 8.5. They carry, respectively, the gyrovectors

$$\begin{aligned} &\ominus \mathbf{o}_1 \oplus \mathbf{p}_1 \\ &\ominus \mathbf{o}_2 \oplus \mathbf{p}_2 \end{aligned} \tag{8.54}$$

so that their gyroangle α is given by the equation, Def. 8.12,

$$\cos\alpha = \frac{\ominus \mathbf{o}_1 \oplus \mathbf{p}_1}{\|\ominus \mathbf{o}_1 \oplus \mathbf{p}_1\|} \cdot \frac{\ominus \mathbf{o}_2 \oplus \mathbf{p}_2}{\|\ominus \mathbf{o}_2 \oplus \mathbf{p}_2\|} \tag{8.55}$$

Let us simultaneously gyrovector translate the gyrorays L_1 and L_2 by $\mathbf{t}$, obtaining by Lemma 8.17,

$$\begin{aligned} L_1' : &\quad \mathrm{gyr}[\mathbf{o}_1, \mathbf{t}]\{[\mathbf{t} \oplus \mathbf{o}_1] \oplus (\ominus [\mathbf{t} \oplus \mathbf{o}_1] \oplus [\mathbf{t} \oplus \mathbf{p}_1]) \otimes s\} \\ L_2' : &\quad \mathrm{gyr}[\mathbf{o}_2, \mathbf{t}]\{[\mathbf{t} \oplus \mathbf{o}_2] \oplus (\ominus [\mathbf{t} \oplus \mathbf{o}_2] \oplus [\mathbf{t} \oplus \mathbf{p}_2]) \otimes t\} \end{aligned} \tag{8.56}$$

The gyrorays L_1' and L_2' carry, respectively, the following gyrovectors, each of which is manipulated by the Gyrotranslation Theorem 3.13 and the gyration inversive symmetry (2.93),

$$\begin{aligned} \mathrm{gyr}[\mathbf{o}_1, \mathbf{t}]\{\ominus(\mathbf{t} \oplus \mathbf{o}_1) \oplus (\mathbf{t} \oplus \mathbf{p}_1)\} &= \ominus \mathbf{o}_1 \oplus \mathbf{p}_1 \\ \mathrm{gyr}[\mathbf{o}_2, \mathbf{t}]\{\ominus(\mathbf{t} \oplus \mathbf{o}_2) \oplus (\mathbf{t} \oplus \mathbf{p}_2)\} &= \ominus \mathbf{o}_2 \oplus \mathbf{p}_2 \end{aligned} \tag{8.57}$$

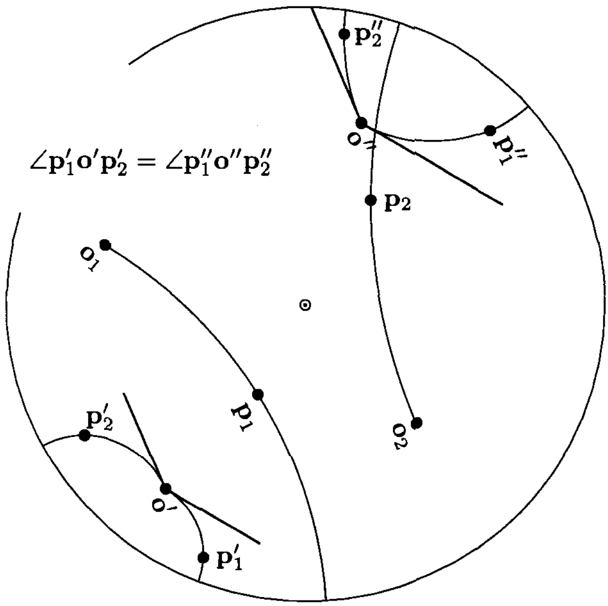


Fig. 8.5 It follows from Theorem 8.19 that in a gyrovector space one can gyrovector translate each of the two gyrorays $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$ to a new position where they share a common origin so that their gyroangle can be visualized as the gyroangle between two gyrorays that emanate from the same point. Since, by Theorem 8.19, gyrovector translations keep gyroangles between gyrorays invariant, the position of the common origin can be selected arbitrarily.
Accordingly, in this figure the two gyrorays $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$ are gyrovector translated (i) to the common origin $\mathbf{o}'$ and (ii) to the common origin $\mathbf{o}''$. As expected, the gyroangle $\angle\mathbf{p}_1'\mathbf{o}'\mathbf{p}_2'$ of the gyrovector translated gyrorays $\mathbf{o}'\mathbf{p}_1'$ and $\mathbf{o}'\mathbf{p}_2'$ and the gyroangle $\angle\mathbf{p}_1''\mathbf{o}''\mathbf{p}_2''$ of the gyrovector translated gyrorays $\mathbf{o}''\mathbf{p}_1''$ and $\mathbf{o}''\mathbf{p}_2''$ are equal.
The measure of the included gyroangle between two gyrorays that share a common origin in a Möbius gyrovector space equals the measure of the included angle between two tangent rays at the common origin. These tangent rays are, therefore, shown.

Hence, like the original gyrorays L_1 and L_2, also their simultaneously gyrovector translated gyrorays L_1' and L_2' carry, respectively, the gyrovectors (8.54). The gyroangle between L_1' and L_2' is therefore given by (8.55) as desired. □

In Fig. 8.5 two gyrorays

$$\begin{aligned} \mathbf{o}_1\mathbf{p}_1 &= \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t \\ \mathbf{o}_2\mathbf{p}_2 &= \mathbf{o}_2\oplus(\ominus\mathbf{o}_2\oplus\mathbf{p}_2)\otimes t \end{aligned} \tag{8.58}$$

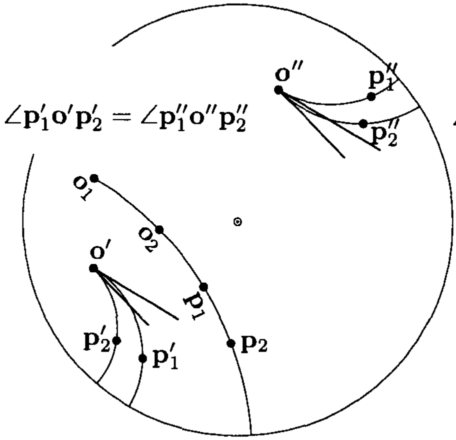


Fig. 8.6 This figure presents a special case of Fig. 8.5, illustrating Theorems 8.14 and 8.19 in the Möbius gyrovector plane. A gyroray, $\mathbf{o}_1\mathbf{p}_1$, containing another gyroray, $\mathbf{o}_2\mathbf{p}_2$, is shown. In order to visualize the gyroangle included between the gyrorays $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$, these gyrorays are gyrovector translated to the common origin $\mathbf{o}'$, or to the common origin $\mathbf{o}''$. As expected from Theorem 8.19, the gyroangle at the common origin $\mathbf{o}'$ equals the gyroangle at the common origin $\mathbf{o}''$.

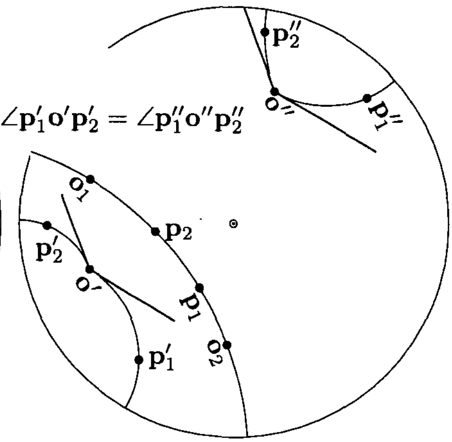


Fig. 8.7 This figure presents a special case of Fig. 8.5, illustrating Theorems 8.15 and 8.19 in the Möbius gyrovector plane. Two gyrocollinear gyroray, $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$, non of which contains the other, are shown. In order to visualize the gyroangle included between the gyrorays $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$, these gyrorays are gyrovector translated to the common origin $\mathbf{o}'$, or to the common origin $\mathbf{o}''$. As expected from Theorem 8.19, the gyroangle at the common origin $\mathbf{o}'$ equals the gyroangle at the common origin $\mathbf{o}''$.

$t\in\mathbb{R}^{\geq 0}$, with distinct origins $\mathbf{o}_1$ and $\mathbf{o}_2$ are gyrovector translated by $\mathbf{t}_1$ and $\mathbf{t}_2$, respectively, to a new, common origin $\mathbf{o}'$ (and, similarly, $\mathbf{o}''$) resulting in the gyrorays, (8.44),

$$\begin{aligned}\mathbf{o}'\mathbf{p}'_1 &= (\mathbf{o}_1\oplus\mathbf{t}_1)\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t\\ \mathbf{o}'\mathbf{p}'_2 &= (\mathbf{o}_2\oplus\mathbf{t}_2)\oplus(\ominus\mathbf{o}_2\oplus\mathbf{p}_2)\otimes t\end{aligned} \tag{8.59}$$

$t\in\mathbb{R}^{\geq 0}$, that have a common origin $\mathbf{o}'$, where

$$\mathbf{o}' = \mathbf{o}_1\oplus\mathbf{t}_1 = \mathbf{o}_2\oplus\mathbf{t}_2 \tag{8.60}$$

Clearly, to validate (8.60) we must select

$$\mathbf{t}_k = \ominus\mathbf{o}_k\oplus\mathbf{o}' \tag{8.61}$$

$k = 1, 2$.

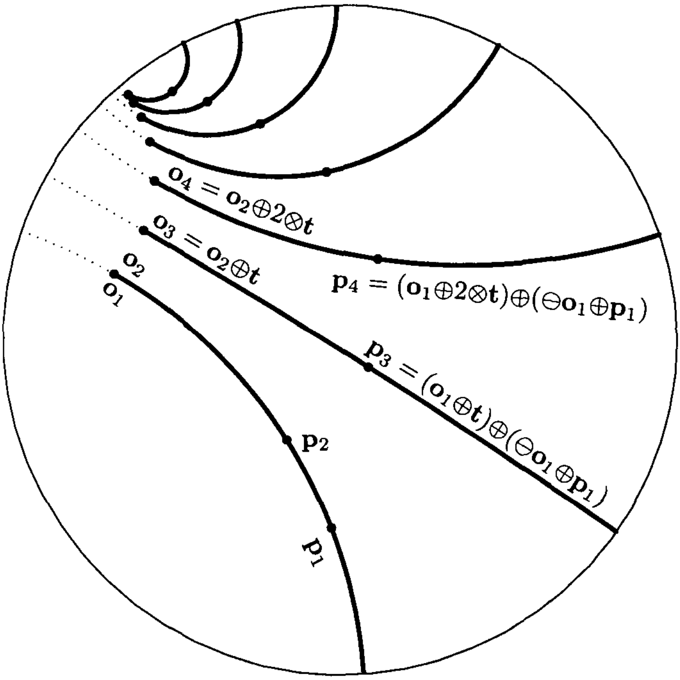


Fig. 8.8 Two gyrorays are parallel if their included gyroangle vanishes. The two gyrorays $\mathbf{o}_1\mathbf{p}_1$ and $\mathbf{o}_2\mathbf{p}_2$, $\mathbf{o}_1 = \mathbf{o}_2$, in the figure contain each other so that their included gyroangle vanishes and, hence, they are parallel. By Theorem 8.19, successive gyrovector translations of gyroray $\mathbf{o}_2\mathbf{p}_2$ give a family P of gyrorays that remain parallel to gyroray $\mathbf{o}_1\mathbf{p}_1$ and, hence, are parallel to each other. Several gyrovector translations of gyroray $\mathbf{o}_2\mathbf{p}_2$ by $k\otimes\mathbf{t}$, $k = 1, 2, \ldots$, are shown in the Möbius gyrovector plane. All the gyrorays in this figure are, thus, parallel to each other.

The gyrorays (8.59) contain, respectively, the points, (8.47),

$$
\begin{aligned}
\mathbf{p}_1' &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}_1](\mathbf{t}_1 \oplus \mathbf{p}_1) \\
\mathbf{p}_2' &= \mathrm{gyr}[\mathbf{o}, \mathbf{t}_2](\mathbf{t}_2 \oplus \mathbf{p}_2)
\end{aligned}
\tag{8.62}
$$

Selecting a different common origin, $\mathbf{o}''$, in Fig. 8.5 results in the gyrorays $\mathbf{o}''\mathbf{p}_1''$ and $\mathbf{o}''\mathbf{p}_2''$. By Theorem 8.19, the resulting gyroangles at the origins $\mathbf{o}'$ and $\mathbf{o}''$ are equal, as shown graphically in Fig. 8.5.

Two special cases when the gyrorays (8.58) in Fig. 8.5 are gyrocollinear, studied in Theorems 8.14 and 8.15, are illustrated in Figs. 8.6 and 8.7. It is clear from Theorems 8.14, 8.15 and 8.19, and their illustration in these figures that, owing to the presence of a gyration, gyrorays on the two senses

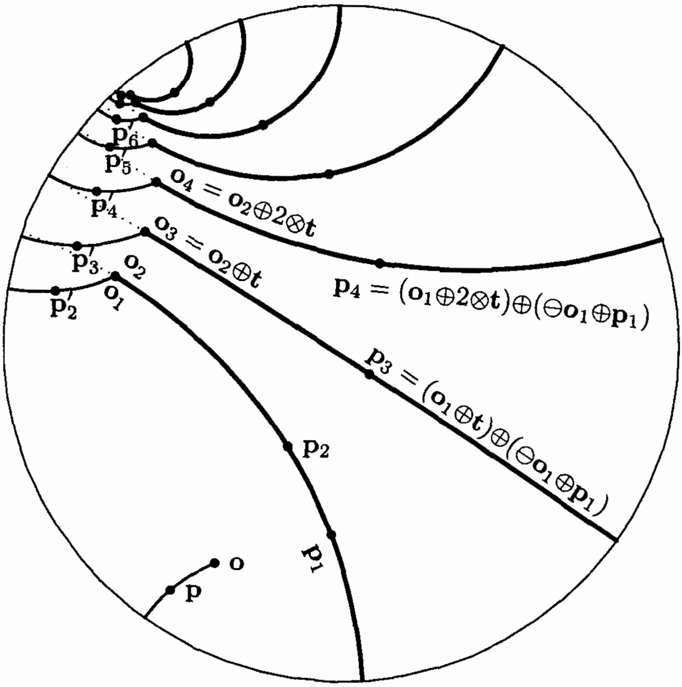


Fig. 8.9 The family P of parallel gyrorays in Fig. 8.8 is shown along with an additional, non-parallel gyroray **op**. In analogy with Euclidean geometry, the gyroray **op** meets each of the gyrorays in the family P at the same gyroangle. Thus, $\angle \mathbf{p}'_2\mathbf{o}_2\mathbf{p}_2 = \angle \mathbf{p}'_3\mathbf{o}_3\mathbf{p}_3 = \angle \mathbf{p}'_4\mathbf{o}_4\mathbf{p}_4 = \angle \mathbf{p}'_5\mathbf{o}_5\mathbf{p}_5$, etc.

of the gyroline respond differently to gyrovector translations. Indeed, only when the gyration in (8.34) and (8.39) is trivial, these two equations reduce to $\cos\alpha = \pm 1$, that is, $\alpha = 0$ and $\alpha = \pi$, respectively. As a result, gyrorays admit parallelism, as we will see in Sec. 8.3, while gyrolines do not.

8.3 Gyrorays Parallelism and Perpendicularity

Definition 8.20 (Gyroray Parallelism and Perpendicularity). *Two gyrorays are parallel if the included gyroangle between them vanishes, and two gyrorays are perpendicular (orthogonal) if the included gyroangle between them is $\pi/2$, Fig. 8.10.*

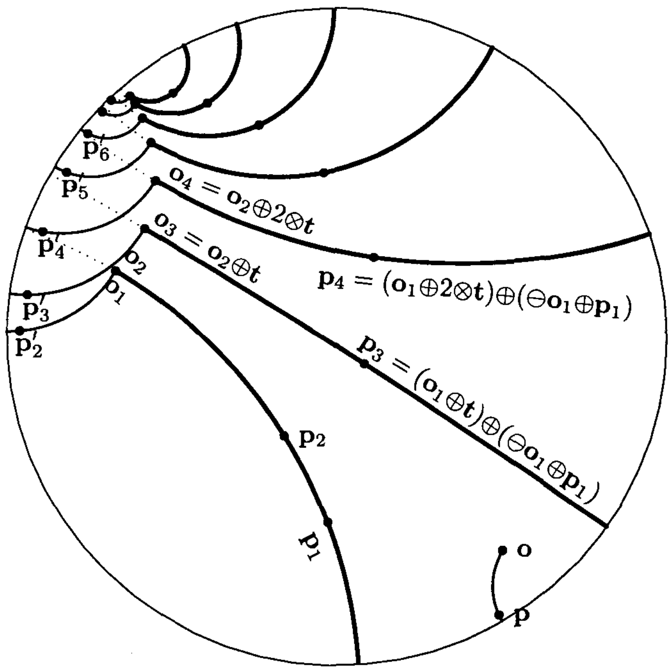


Fig. 8.10 The family P of parallel gyrorays in Fig. 8.8 is shown along with an additional, perpendicular gyroray **op**. In analogy with Euclidean geometry, the gyroray **op** meets each of the gyrorays in the family P at a right gyroangle $\pi/2$. Thus, $\pi/2 = \angle\mathbf{p}'_2\mathbf{o}_2\mathbf{p}_2 = \angle\mathbf{p}'_3\mathbf{o}_3\mathbf{p}_3 = \angle\mathbf{p}'_4\mathbf{o}_4\mathbf{p}_4 = \angle\mathbf{p}'_5\mathbf{o}_5\mathbf{p}_5$, etc; see Theorem 8.22 for perpendicular gyrorays.

The generation of a family of parallel gyrorays by successive gyrovector translations in a Möbius gyrovector space is shown in Fig. 8.8. Owing to the presence of gyrations, parallelism in gyrorays cannot be extended to gyrolines. Gyrorays in the two senses of the gyroline respond differently to gyrovector translations if a non-trivial gyration is involved, as we see from Theorems 8.14 and 8.15 and from their illustration in Figs. 8.6 and 8.7.

The analogies that parallelism in gyrorays shares with parallelism in rays are further enhanced by Theorem 8.21 and illustrated in Fig. 8.9. In the same way that any family of parallel rays is intersected by a non-parallel ray in equal angles, any family of parallel gyrorays is met by a non-parallel gyroray in equal gyroangles.

Theorem 8.21 *Let* $\mathbf{op}$, $\mathbf{o}_1\mathbf{p}_1$, *and* $\mathbf{o}_2\mathbf{p}_2$ *be three gyrorays in a gyrovector space* $(G, \oplus, \otimes)$. *The gyrorays* $\mathbf{o}_1\mathbf{p}_1$ *and* $\mathbf{o}_2\mathbf{p}_2$ *are parallel if and only if*

$$\angle(\mathbf{op})(\mathbf{o}_1\mathbf{p}_1) = \angle(\mathbf{op})(\mathbf{o}_2\mathbf{p}_2) \tag{8.63}$$

Proof. The Theorem follows immediately from Theorem 8.11 since, by Def. 8.12, gyroangles between gyrorays are given by gyroangles between gyrovectors that respectively lie on the gyrorays. □

Theorem 8.22 *Let*

$$\begin{aligned} L_1 : &\quad \mathbf{o}_1\oplus(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\otimes t \\ L_2 : &\quad \mathbf{o}_2\oplus(\ominus\mathbf{o}_2\oplus\mathbf{p}_2)\otimes t \end{aligned} \tag{8.64}$$

$t\in\mathbb{R}^{\geq 0}$, *be two gyrorays in a gyrovector space* $(G, \oplus, \otimes)$. *They are perpendicular if and only if*

$$(\ominus\mathbf{o}_1\oplus\mathbf{p}_1)\cdot(\ominus\mathbf{o}_2\oplus\mathbf{p}_2) = 0 \tag{8.65}$$

Proof. Gyrorays L_1 and L_2 carry, respectively, the gyrovectors $\ominus\mathbf{o}_1\oplus\mathbf{p}_1$ and $\ominus\mathbf{o}_2\oplus\mathbf{p}_2$. Hence, by Def. 8.12, the gyroangle between gyrorays L_1 and L_2 equals the gyroangle between gyrovectors $\ominus\mathbf{o}_1\oplus\mathbf{p}_1$ and $\ominus\mathbf{o}_2\oplus\mathbf{p}_2$. The latter is $\pi/2$ if and only if (8.65) is satisfied. □

We thus see that gyroray parallelism and perpendicularity is fully analogous to ray parallelism and perpendicularity. Owing to the presence of a gyration, a disanalogy emerges. Parallelism between gyrorays cannot be extended to lines.

8.4 Gyrotrigonometry in Möbius Gyrovector Spaces

Definition 8.23 **(Gyrotriangles).** *A gyrotriangle* ABC *in a gyrovector space* $(G, \oplus, \otimes)$ *is a gyrovector space object formed by the three points* $A, B, C \in G$, *called the vertices of the gyrotriangle, and the segments* AB, AC *and* BC, *called the sides of the gyrotriangle. These are, respectively, the sides opposite to the vertices* C, B *and* A. *The gyrotriangle sides generate the three gyrotriangle gyroangles* α, β, *and* γ *at the respective vertices* A, B *and* C, *Fig. 8.11.*

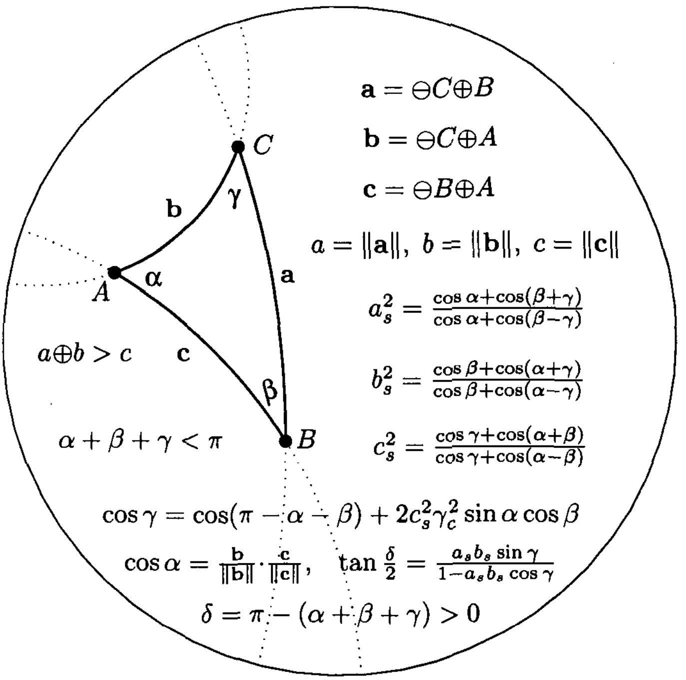


Fig. 8.11 A Möbius gyrotriangle ABC in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}_s^2, \oplus, \otimes)$ is shown. Its sides are formed by gyrovectors that link its vertices, in full analogy with Euclidean triangles. Its hyperbolic side lengths, a, b, c, are uniquely determined in (8.158), p. 280, by its gyroangles. The gyrotriangle gyroangle sum is less than π. Here, $a_s = a/s$, etc. Note that in the limit of large s, $s \to \infty$, the $\cos\gamma$ equation reduces to $\cos\gamma = \cos(\pi - \alpha - \beta)$ so that $\alpha + \beta + \gamma = \pi$, implying that both sides of each of the squared side gyrolength equations, shown in the figure and listed in (8.158), vanishes.

Definition 8.24 (Congruent Gyrotriangles). *Two gyrotriangles are congruent if their vertices can be paired so that (i) all pairs of corresponding sides are congruent and (ii) all pairs of corresponding gyroangles are congruent.*

The six elements of the gyrotriangle are the gyrolengths of its three sides, and the measure of its three gyroangles. The purpose of gyrotrigonometry is to deduce relations among the gyrotriangle elements. As we will see in the sequel, when three of the gyrotriangle elements are given, gyrotrigonometry allows the remaining three to be determined.

Theorem 8.25 (The Law of Gyrocosines in Möbius Gyrovector Spaces). *Let ABC be a gyrotriangle in a Möbius gyrovector space*

$(\mathbb{V}_s, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{V}_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_s$ and side gyrolengths $a, b, c \in (-s, s)$, Fig. 8.11,

$$\begin{aligned}
\mathbf{a} &= \ominus C \oplus B, & a &= \|\mathbf{a}\| \\
\mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\| \\
\mathbf{c} &= \ominus B \oplus A, & c &= \|\mathbf{c}\|
\end{aligned} \tag{8.66}$$

and with gyroangles α, β and γ at the vertices A, B and C, Fig. 8.11. Then

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s} \ominus \frac{1}{s} \frac{2\beta_a^2 a \beta_b^2 b \cos\gamma}{1 - \frac{2}{s^2}\beta_a^2 a \beta_b^2 b \cos\gamma} \tag{8.67}$$

where β_a is the beta factor

$$\beta_a = \frac{1}{\sqrt{1 + \dfrac{a^2}{s^2}}} \tag{8.68}$$

Furthermore, the law of gyrocosines (8.67) can be written, equivalently, as

$$c_s^2 = \frac{a_s^2 + b_s^2 - 2a_s b_s \cos\gamma}{1 + a_s^2 b_s^2 - 2a_s b_s \cos\gamma} \tag{8.69}$$

and as

$$\cos\gamma = \frac{a_s^2 + b_s^2 - c_s^2 - a_s^2 b_s^2 c_s^2}{2a_s b_s (1 - c_s^2)} \tag{8.70}$$

where $a_s = \frac{a}{s}$, etc.

Proof. By (8.66), Fig. 8.11, and the Gyrotranslation Theorem 3.13 we have

$$\begin{aligned}
\ominus\mathbf{a}\oplus\mathbf{b} &= \ominus(\ominus C \oplus B) \oplus (\ominus C \oplus A) \\
&= \mathrm{gyr}[\ominus C, B](\ominus B \oplus A) \\
&= \mathrm{gyr}[\ominus C, B]\mathbf{c}
\end{aligned} \tag{8.71}$$

so that $\|\ominus\mathbf{a}\oplus\mathbf{b}\| = \|\mathbf{c}\|$ and, accordingly, $\gamma_{\mathbf{c}} = \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}$. Hence, by the gamma identity (3.128) we have

$$\begin{aligned}
\gamma_{\mathbf{c}}^2 &= \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}^2 \\
&= \gamma_{\mathbf{a}}^2 \gamma_{\mathbf{b}}^2 \left(1 - \frac{2}{s^2}\mathbf{a}\cdot\mathbf{b} + \frac{1}{s^4}\|\mathbf{a}\|^2\|\mathbf{b}\|^2\right) \\
&= \gamma_{\mathbf{a}}^2 \gamma_{\mathbf{b}}^2 \left(1 - \frac{2}{s^2} ab\cos\gamma + \frac{1}{s^4} a^2 b^2\right)
\end{aligned} \tag{8.72}$$

Identity (8.72), in turn, is equivalent to each of the identities (8.67), (8.69), and (8.70). □

Remark 8.26 *Identities (8.69) and (8.70) involve addition, rather than gyroaddition. Accordingly, in these identities one may assume $s = 1$ without loss of generality. The more general case of $s > 0$ can readily be recovered from the special case of $s = 1$. This is, however, not the case in Identity (8.67) since it involve gyroadditions which, in turn, depend implicitly on the positive parameter s.*

Remark 8.27 *We may note that the Möbius addition $\oplus$ in (8.66) is a gyrocommutative gyrogroup operation in the Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$. In contrast, the Möbius addition $\oplus$ in (8.67) is a commutative group operation in the Möbius group $(\mathbb{I}, \oplus)$, $\mathbb{I}$ being the open interval $\mathbb{I} = (-s, s)$. Thus,*

$$a \oplus b = \frac{a+b}{1+\frac{ab}{s^2}} \tag{8.73}$$

in $\mathbb{I}$.

Remark 8.28 *In the limit of large s, $s \to \infty$, (8.69) reduces to the law of cosines,*

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \tag{8.74}$$

which further reduces to the Euclidean Pythagorean identity

$$c^2 = a^2 + b^2 \tag{8.75}$$

when $\gamma = \pi/2$.

Remark 8.29 *It is interesting to compare the law of gyrocosines (8.67) in the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$ with the standard hyperbolic law of cosines in the Poincaré disc. The latter is given by the identity [Varičak (1910b)]*

$$\cosh c' = \cosh a' \cosh b' - \sinh a' \sinh b' \cos\gamma \tag{8.76}$$

(see, for instance, [Schwerdtfeger (1962), p. 154] and [Pierseaux (2004), p. 68]), which is equivalent to (8.67) when $s = 1$ and when

$$a' = log\frac{1+a}{1-a}\,, \qquad b' = log\frac{1+b}{1-b}\,, \qquad c' = log\frac{1+c}{1-c} \tag{8.77}$$

In the special case when $\gamma = \pi/2$ we have $\cos\gamma = 0$, and the standard hyperbolic law of cosines (8.76) *reduces to the standard hyperbolic Pythagorean theorem*

$$\cosh c' = \cosh a' \cosh b' \tag{8.78}$$

in the Poincaré disc (see, for instance, [Schwerdtfeger (1962), p. 151]). Unlike results in gyrogeometry, neither formula (8.76) *nor formula* (8.78) *shares visual analogies with its Euclidean counterpart.*

The law of gyrocosines (8.67) is an identity in the Möbius vector space $(\mathbb{I}, \oplus, \otimes)$. To solve it for $\cos\gamma$ we use the notation

$$\begin{aligned} P_{abc} &= \frac{a^2}{s} \oplus \frac{b^2}{s} \ominus \frac{c^2}{s} \\ Q_{ab} &= 2\beta_a^2 a \beta_b^2 b \end{aligned} \tag{8.79}$$

so that (8.67) can be written as

$$P_{abc} = \frac{1}{s}\frac{Q_{ab}\cos\gamma}{1 - \frac{1}{s^2}Q_{ab}\cos\gamma} \tag{8.80}$$

implying

$$\cos\gamma = \frac{sP_{abc}}{(1 + \frac{1}{s}P_{abc})Q_{ab}} \tag{8.81}$$

Similarly, by cyclic permutations of the gyroangles and sides of gyrotriangle ABC, Fig. 8.11, we have

$$\cos\alpha = \frac{sP_{bca}}{(1 + \frac{1}{s}P_{bca})Q_{bc}} \tag{8.82}$$

and

$$\cos\beta = \frac{sP_{cab}}{(1 + \frac{1}{s}P_{cab})Q_{ca}} \tag{8.83}$$

Theorem 8.30 **(Side – Side – Side (SSS)).** *If, in two gyrotriangles, three sides of one are congruent to three sides of the other, then the two gyrotriangles are congruent.*

Proof. It follows from the law of gyrocosines, Theorem 8.25, that the three side gyrolengths of a gyrotriangle determine the measure of its three gyroangles. Hence, by Def. 8.24, the two gyrotriangles are congruent. □

Theorem 8.31 **(Side–gyroAngle–Side (SAS)).** *If, in two gyrotriangles, two sides and the included gyroangle of one, are congruent to two sides and the included gyroangle of the other, then the gyrotriangles are congruent.*

Proof. It follows from the law of gyrocosines, Theorem 8.25, that the side gyrolengths of two sides of a gyrotriangle and the included gyroangle determine the third side gyrolength. Hence, by SSS congruency (Theorem 8.30), the two gyrotriangles are congruent. □

The Pythagorean theorem has a long history [Veljan (2000)]. It plays an important role in trigonometry, giving rise to the elementary trigonometric functions $\sin\alpha$, $\cos\alpha$, etc. In the special case of $\gamma = \pi/2$, corresponding to a right gyroangled gyrotriangle, Fig. 8.12, the law of gyrocosines is of particular interest, giving rise (i) to the hyperbolic Pythagorean theorem in the Poincaré ball model of hyperbolic geometry, and (ii) to the elementary gyrotrigonometric functions $\sin\alpha$, $\cos\alpha$, etc.

The resulting hyperbolic Pythagorean theorem 8.32 shares with its Euclidean counterpart visual analogies, shown in Fig. 8.12.

Theorem 8.32 **(The Möbius Hyperbolic Pythagorean Theorem).** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{V}_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_s$ and side gyrolengths $a, b, c \in (-s, s)$,*

$$\begin{aligned} \mathbf{a} &= \ominus B \oplus C, & a &= \|\mathbf{a}\| \\ \mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\| \\ \mathbf{c} &= \ominus A \oplus B, & c &= \|\mathbf{c}\| \end{aligned} \tag{8.84}$$

and with gyroangles α, β and γ at the vertices A, B and C. If $\gamma = \pi/2$, Fig. 8.12, then

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s} \tag{8.85}$$

Proof. The hyperbolic Pythagorean identity (8.85) follows from the law of gyrocosines (8.67) with $\gamma = \pi/2$. □

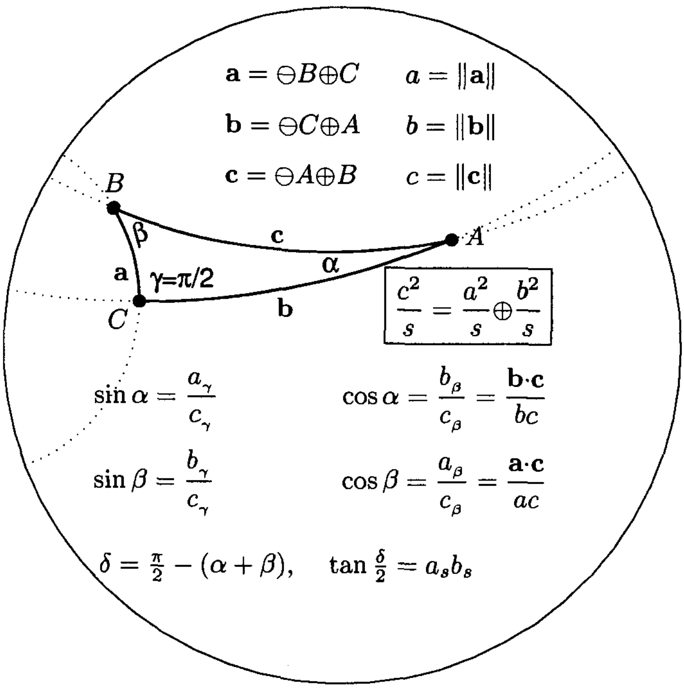


Fig. 8.12 Gyrotrigonometry in the Poincaré Model. A Möbius right gyroangled gyrotriangle ABC in the Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ is shown for the special case of the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$. Its sides, formed by the gyrovectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ that join its vertices A, B and C, have gyrolengths a, b and c, respectively. They satisfy the Möbius hyperbolic Pythagorean identity (8.85). Its acute gyroangles α and β satisfy gyrotrigonometric identities analogous to their trigonometric counterparts, where a_γ, a_β, b_γ, b_β, c_γ, c_β are related to a, b, c by (8.93). The right gyroangled gyrotriangle defect δ is calculated in (8.127), giving rise to the remarkably simple and elegant result $\tan(\delta/2) = a_s b_s$, where $a_s = a/s$, $b_s = b/s$ and $c_s = c/s$.

Remark 8.33 (The Euclidean Pythagorean Identity). *In the limit of large s, $s \to \infty$, the Möbius Hyperbolic Pythagorean Identity* (8.85),

$$\frac{a^2}{s} \oplus \frac{b^2}{s} = \frac{c^2}{s} \tag{8.86}$$

in Möbius gyrovector spaces $(\mathbb{R}_s^n, \oplus, \otimes)$ reduces to the Euclidean Pythagorean Identity

$$a^2 + b^2 = c^2 \tag{8.87}$$

in Euclidean spaces $(\mathbb{R}^n, +, \cdot)$; see Fig. 8.12 and Remark 8.28.

The gyroangles α and β in (8.82) – (8.83) that correspond to $\gamma = \pi/2$ in (8.81), shown in Fig. 8.12, are of particular interest. When $\gamma = \pi/2$ we have $\cos\gamma = 0$ and, hence, by (8.81), $P_{abc} = 0$ implying, by (8.79), the hyperbolic Pythagorean identity (8.67). The latter, in turn, implies

$$\begin{aligned}
P_{bca} &= \frac{b^2}{s}\oplus\frac{c^2}{s}\ominus\frac{a^2}{s} \\
&= \frac{b^2}{s}\oplus(\frac{a^2}{s}\oplus\frac{b^2}{s})\ominus\frac{a^2}{s} \\
&= 2\otimes\frac{b^2}{s} \\
&= \frac{2b^2/s}{1+b^4/s^4}
\end{aligned} \tag{8.88}$$

Finally, by substituting (8.88) in (8.82) and straightforward algebra, we have

$$\cos\alpha = \frac{\beta_b^2 b}{\beta_c^2 c} = \frac{b_\beta}{c_\beta} \tag{8.89}$$

Similarly, we also have

$$\cos\beta = \frac{\beta_a^2 a}{\beta_c^2 c} = \frac{a_\beta}{c_\beta} \tag{8.90}$$

as shown in Fig. 8.12, where we use the notation in (8.91) below.

As suggested in (8.89) – (8.90), we introduce the notation

$$\begin{aligned}
\mathbf{a}_\gamma &= \gamma_{\mathbf{a}}^2\mathbf{a} = \frac{\mathbf{a}}{1-\|\mathbf{a}\|^2/s^2} = \frac{\mathbf{a}}{1-a^2/s^2} \\
\mathbf{a}_\beta &= \beta_{\mathbf{a}}^2\mathbf{a} = \frac{\mathbf{a}}{1+\|\mathbf{a}\|^2/s^2} = \frac{\mathbf{a}}{1+a^2/s^2}
\end{aligned} \tag{8.91}$$

for $\mathbf{a}\in\mathbb{V}_s$, where $\gamma_{\mathbf{a}}$ and $\beta_{\mathbf{a}}$ are the gamma and the beta factors given by the pair of similar equations

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1-\dfrac{\|\mathbf{v}\|^2}{s^2}}} \qquad \text{and} \qquad \beta_{\mathbf{v}} = \frac{1}{\sqrt{1+\dfrac{\|\mathbf{v}\|^2}{s^2}}} \tag{8.92}$$

for any $\mathbf{v}\in\mathbb{V}_s$. We call $\mathbf{a}_\beta$ and $\mathbf{a}_\gamma$, respectively, the gamma and the beta corrections of $\mathbf{a}$.

Taking magnitudes in (8.91), we have

$$
\begin{aligned}
\|\mathbf{a}_\gamma\| &= \|\mathbf{a}\|_\gamma = \gamma_{\mathbf{a}}^2\|\mathbf{a}\| = \frac{\|\mathbf{a}\|}{1-\|\mathbf{a}\|^2/s^2} = \frac{a}{1-a^2/s^2} = a_\gamma \\
\|\mathbf{a}_\beta\| &= \|\mathbf{a}\|_\beta = \beta_{\mathbf{a}}^2\|\mathbf{a}\| = \frac{\|\mathbf{a}\|}{1+\|\mathbf{a}\|^2/s^2} = \frac{a}{1+a^2/s^2} = a_\beta
\end{aligned}
\tag{8.93}
$$

for $\mathbf{a} \in \mathbb{V}_s$, Clearly, $a_\gamma \in [0,\infty)$ and $a_\beta \in [0, s/2)$.

Inverting the equations in (8.91) and (8.93) we have,

$$
\begin{aligned}
\mathbf{a} &= \frac{2\mathbf{a}_\gamma}{1+\sqrt{1+(2\|\mathbf{a}_\gamma\|)^2/s^2}} \\
\mathbf{a} &= \frac{2\mathbf{a}_\beta}{1+\sqrt{1-(2\|\mathbf{a}_\beta\|)^2/s^2}}
\end{aligned}
\tag{8.94}
$$

and hence

$$
\begin{aligned}
\|\mathbf{a}\| &= \frac{2\|\mathbf{a}\|_\gamma}{1+\sqrt{1+(2\|\mathbf{a}\|_\gamma)^2/s^2}} \\
\|\mathbf{a}\| &= \frac{2\|\mathbf{a}\|_\beta}{1+\sqrt{1-(2\|\mathbf{a}\|_\beta)^2/s^2}}
\end{aligned}
\tag{8.95}
$$

or, equivalently,

$$
\begin{aligned}
a &= \frac{2a_\gamma}{1+\sqrt{1+(2a_\gamma)^2/s^2}} \\
a &= \frac{2a_\beta}{1+\sqrt{1-(2a_\beta)^2/s^2}}
\end{aligned}
\tag{8.96}
$$

Theorem 8.34 *Let a, b, c be the side gyrolengths of a right gyroangled gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.12. Then*

$$
\begin{aligned}
\left(\frac{a_\gamma}{c_\gamma}\right)^2 + \left(\frac{b_\beta}{c_\beta}\right)^2 &= 1 \\
\left(\frac{a_\beta}{c_\beta}\right)^2 + \left(\frac{b_\gamma}{c_\gamma}\right)^2 &= 1
\end{aligned}
\tag{8.97}
$$

Proof. The hyperbolic Pythagorean identity (8.85), expressed in terms of ordinary addition rather than Möbius addition, takes the form

$$\frac{c^2}{s} = \frac{a^2}{s} \oplus \frac{b^2}{s} = \frac{1}{s} \frac{a^2 + b^2}{1 + \frac{a^2 b^2}{s^4}} \tag{8.98}$$

so that

$$\frac{1}{1 + \frac{a^2 b^2}{s^4}} \left\{ \left(\frac{a}{s}\right)^2 + \left(\frac{b}{s}\right)^2 \right\} = \left(\frac{c}{s}\right)^2 \tag{8.99}$$

Expressing a, b, c in terms of their gamma and beta corrections by the identities in (8.96) for a and by similar identities for b and c, and substituting these appropriately in (8.99) give the desired identities, (8.97), of the theorem. □

The identities of Theorem 8.34 can be written, by means of (8.89)–(8.90), as

$$\begin{aligned} \left(\frac{a_\gamma}{c_\gamma}\right)^2 + \cos^2 \alpha &= 1 \\ \cos^2 \beta + \left(\frac{b_\gamma}{c_\gamma}\right)^2 &= 1 \end{aligned} \tag{8.100}$$

where α and β are the two acute gyroangles of a right gyroangled gyrotriangle, Fig. 8.12. The identities in (8.100) suggest the following

Definition 8.35 **(The Gyrosine Function).** *Let ABC be a right gyroangled gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with acute gyroangles α and β, Fig. 8.12. Then*

$$\begin{aligned} \sin \alpha &= \frac{a_\gamma}{c_\gamma} \\ \sin \beta &= \frac{b_\gamma}{c_\gamma} \end{aligned} \tag{8.101}$$

Following Def. 8.35, identities (8.100) can be written as

$$\begin{aligned} \sin^2 \alpha + \cos^2 \alpha &= 1 \\ \cos^2 \beta + \sin^2 \beta &= 1 \end{aligned} \tag{8.102}$$

for the right gyroangled gyrotriangle in Fig. 8.12, thus uncovering the elementary gyrotrigonometric functions gyrosine and gyrocosine, which share remarkable properties with their trigonometric counterparts. There are also important disanalogies. Thus, for instance, if α and β are the two non-right gyroangles of a right gyroangled gyrotriangle, Fig. 8.12, then in general $\cos\alpha \neq \sin\beta$ and $\sin\alpha \neq \cos\beta$ as implied from formulas shown in Fig. 8.12.

The gyrosine and the gyrocosine functions of gyroangles behave like the sine and the cosine functions of (Euclidean) angles. To see this one can move the vertex of any gyroangle to the origin of its Möbius gyrovector space, where the gyroangle and its gyrosine and gyrocosine becomes an (Euclidean) angle with its sine and cosine, as demonstrated in Figs. 8.2 and 8.3. Hence the gyrotrigonometric functions can be treated in the same way that we commonly treat the trigonometric functions. Thus, for instance, the gyrosine addition formula is the familiar sine addition formula

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{8.103}$$

Accordingly, any trigonometric identity is identical with a corresponding gyrotrigonometric identity.

Theorem 8.36 (The Law of Gyrosines in Möbius Gyrovector Spaces). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{V}_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_s$, and side gyrolengths $a, b, c \in (-s, s)$,*

$$\begin{aligned} \mathbf{a} &= \ominus B \oplus C, & a &= \|\mathbf{a}\| \\ \mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\| \\ \mathbf{c} &= \ominus A \oplus B, & c &= \|\mathbf{c}\| \end{aligned} \tag{8.104}$$

and with gyroangles α, β and γ at the vertices A, B and C, Fig. 8.13. Then

$$\frac{a_\gamma}{\sin\alpha} = \frac{b_\gamma}{\sin\beta} = \frac{c_\gamma}{\sin\gamma} \tag{8.105}$$

Proof. Following Def. 8.35 of the gyrosine function, the proof of the law of gyrosines is fully analogous to its Euclidean counterpart, as shown in Fig. 8.13. □

Employing the identity $\sin^2\gamma = 1 - \cos^2\gamma$ and the condition $\sin\gamma \geq 0$ for any gyroangle γ of a gyrotriangle, we obtain from (8.70) an expression for $\sin\gamma$ in terms of the gyrotriangle sides, Fig. 8.13,

$$\sin\gamma = \frac{\psi(a,b,c;s)\psi(-a,b,c;s)\psi(a,-b,c;s)\psi(a,b,-c;s)}{2a_s b_s}\gamma_c^2 \tag{8.106}$$

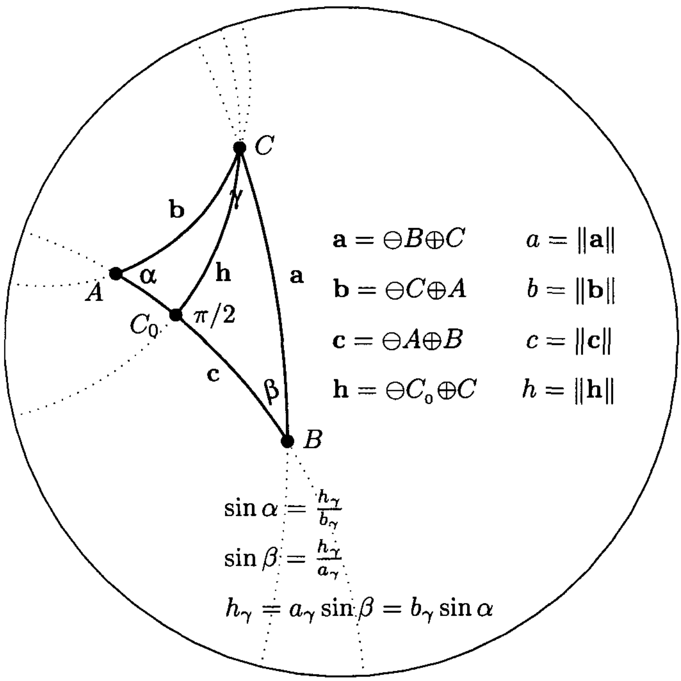


Fig. 8.13 The law of gyrosines in Möbius gyrovector spaces.

where

$$\psi(a, b, c; s) = \sqrt{\frac{1}{s}[(b\oplus c) + a](1 + \frac{bc}{s^2})} = \sqrt{a_s + b_s + c_s + a_s b_s c_s} \tag{8.107}$$

The function $\psi(a, b, c; s)$ is real and symmetric in its variables a, b, c that represent the three sides of a gyrotriangle, Fig. 8.13. It follows from the gyrotriangle inequality $b\oplus c \geq a$ that the function

$$\psi(-a, b, c; s) = \sqrt{\frac{1}{s}[(b\oplus c) - a](1 + \frac{bc}{s^2})} = \sqrt{-a_s + b_s + c_s - a_s b_s c_s} \tag{8.108}$$

is real. A similar remark applies to the functions $\psi(a, -b, c; s)$ and $\psi(a, b, -c; s)$ as well.

Theorem 8.37 (gyroAngle – gyroAngle – Side (AAS)). *If, in two gyrotriangles, two gyroangles and a non-included side of one, are congruent respectively to two gyroangles and a non-included side of the other, then the two gyrotriangles are congruent.*

Proof. Let us consider the gyrotriangle ABC in Fig. 8.13, where gyroangles α and β and side b are given. Then side a can be calculated by means of the law of gyrosines, Theorem 8.36. By the gyrotriangle equality, side c equals the gyrosum

$$c = \|\ominus A \oplus C_0\| \oplus \|\ominus C_0 \oplus B\| \tag{8.109}$$

Part $\|\ominus A \oplus C_0\|$ of the gyrosum can be calculated by means of $\cos\alpha$ and side b, and Part $\ominus C_0 \oplus B$ of the gyrosum can be calculated by means of $\cos\beta$ and side a, as explained in Fig. 8.12. Having sides b and c and gyroangle α, the proof follows from SAS congruency (Theorem 8.31). □

8.5 Gyrotriangle Gyroangles and Side Gyrolengths

Noting the symmetry of the function $\psi(a,b,c;s)$ in a,b,c, it follows from (8.106) by cyclic permutations of the gyrotriangle sides, Fig. 8.11, that the gyroangles α,β,γ of a gyrotriangle ABC with corresponding sides a,b,c are given by the equations

$$\begin{aligned}
\sin\alpha &= \frac{\psi(a,b,c;s)\psi(-a,b,c;s)\psi(a,-b,c;s)\psi(a,b,-c;s)}{2b_s c_s}\gamma_a^2 \\
\sin\beta &= \frac{\psi(a,b,c;s)\psi(-a,b,c;s)\psi(a,-b,c;s)\psi(a,b,-c;s)}{2a_s c_s}\gamma_b^2 \\
\sin\gamma &= \frac{\psi(a,b,c;s)\psi(-a,b,c;s)\psi(a,-b,c;s)\psi(a,b,-c;s)}{2a_s b_s}\gamma_c^2
\end{aligned} \tag{8.110}$$

so that

$$\frac{b_s c_s}{\gamma_a^2}\sin\alpha = \frac{a_s c_s}{\gamma_b^2}\sin\beta = \frac{a_s b_s}{\gamma_c^2}\sin\gamma \tag{8.111}$$

The law of gyrosines (8.105) can be recovered by dividing all sides of (8.111) by abc.

Identity (8.70) with cyclic permutations of the gyrotriangle gyroangles and sides, Fig. 8.11, gives the following

Theorem 8.38 **(The SSS to AAA Conversion Theorem).** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s,\oplus,\otimes)$, Fig. 8.11, p. 258, with vertices A,B,C, corresponding gyroangles α,β,γ, $0<\alpha+\beta+\gamma<\pi$, and side gyrolengths (or, simply, sides) a,b,c.*

The gyroangles of the gyrotriangle ABC are determined by its sides according to the SSS-to-AAA equations

$$\begin{aligned}\cos\alpha &= \frac{-a_s^2+b_s^2+c_s^2-a_s^2b_s^2c_s^2}{2b_sc_s}\gamma_a^2\\ \cos\beta &= \frac{a_s^2-b_s^2+c_s^2-a_s^2b_s^2c_s^2}{2a_sc_s}\gamma_b^2\\ \cos\gamma &= \frac{a_s^2+b_s^2-c_s^2-a_s^2b_s^2c_s^2}{2a_sb_s}\gamma_c^2\end{aligned} \tag{8.112}$$

Solving the first equation in (8.112) for c_s we have

$$c_s = \frac{Q \pm \sqrt{(a_s^2-b_s^2)(1-a_s^2b_s^2)+Q^2}}{1-a_s^2b_s^2} \tag{8.113}$$

where

$$Q = (1-a_s^2)b_s\cos\alpha \tag{8.114}$$

If $a_s = b_s$ then (8.113) reduces to

$$c_s = \frac{2a_s}{1+a_s^2}\cos\alpha \tag{8.115}$$

determining the side of an isosceles gyrotriangle (that is, two of its sides are congruent) in terms of one of the two congruent sides and one of the two congruent gyroangles of the isosceles gyrotriangle.

If $a_s > b_s$ then the ambiguous sign in (8.113) must be replaced by the positive sign to insure that $c_s > 0$.

Theorem 8.39 (Side−side−gyroAngle (SsA)). *If, in two gyrotriangles, two sides and the gyroangle opposite the longer of the two sides in one are congruent respectively to two sides and the gyroangle opposite the longer of the two sides in the other, then the two gyrotriangles are congruent.*

Proof. Let a and b and α be two given sides and the gyroangle opposite the longer side, a, of a gyrotriangle. The third side, c, is determined by (8.113) in which the ambiguous sign specializes to the positive one,

$$c_s = \frac{Q + \sqrt{(a_s^2-b_s^2)(1-a_s^2b_s^2)+Q^2}}{1-a_s^2b_s^2} \tag{8.116}$$

Hence, by SSS congruency, Theorem 8.30, the two gyrotriangles in the theorem are congruent. □

Theorem 8.40 *Let α and β be two gyroangles of a gyrotriangle, and let c be the gyrolength of the included side. Then the third gyroangle, γ, of the gyrotriangle is given by the equation*

$$\cos\gamma = -\cos\alpha\cos\beta + (\gamma_c^2/\beta_c^2)\sin\alpha\sin\beta \tag{8.117}$$

Proof. The proof follows straightforwardly from (8.110) and (8.112). □

In the limit of large s, $s \to \infty$, the gyrotriangle gyroangle identity (8.117) approaches the triangle angle identity

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta = \cos(\pi-\alpha-\beta) \tag{8.118}$$

or, equivalently,

$$\gamma = \pi - \alpha - \beta \tag{8.119}$$

thus recovering an identity for the triangle angles in Euclidean geometry.

Theorem 8.41 *Let a, b, c be the three sides of a gyrotriangle ABC with corresponding heights h_a, h_b, h_c in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.14. Then*

$$a_\gamma (h_a)_\gamma = b_\gamma (h_b)_\gamma = c_\gamma (h_c)_\gamma \tag{8.120}$$

Proof. By Def. 8.35, the three heights of gyrotriangle ABC, Fig. 8.14, are given by the equations

$$\begin{aligned}
(h_a)_\gamma &= b_\gamma \sin\gamma = c_\gamma \sin\beta \\
(h_b)_\gamma &= a_\gamma \sin\gamma = c_\gamma \sin\alpha \\
(h_c)_\gamma &= a_\gamma \sin\beta = b_\gamma \sin\alpha
\end{aligned} \tag{8.121}$$

Hence,

$$\begin{aligned}
a_\gamma (h_a)_\gamma &= a_\gamma b_\gamma \sin\gamma = a_\gamma c_\gamma \sin\beta \\
b_\gamma (h_b)_\gamma &= a_\gamma b_\gamma \sin\gamma = b_\gamma c_\gamma \sin\alpha \\
c_\gamma (h_c)_\gamma &= a_\gamma c_\gamma \sin\beta = b_\gamma c_\gamma \sin\alpha
\end{aligned} \tag{8.122}$$

thus obtaining the desired gyrotriangle heights identity (8.120). □

Definition 8.42 **(The Gyrotriangle Constant).** *Let a, b, c be the three sides of a gyrotriangle ABC with corresponding heights h_a, h_b, h_c in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.14. The number S_{ABC},*

$$S_{ABC} = a_\gamma (h_a)_\gamma = b_\gamma (h_b)_\gamma = c_\gamma (h_c)_\gamma \tag{8.123}$$

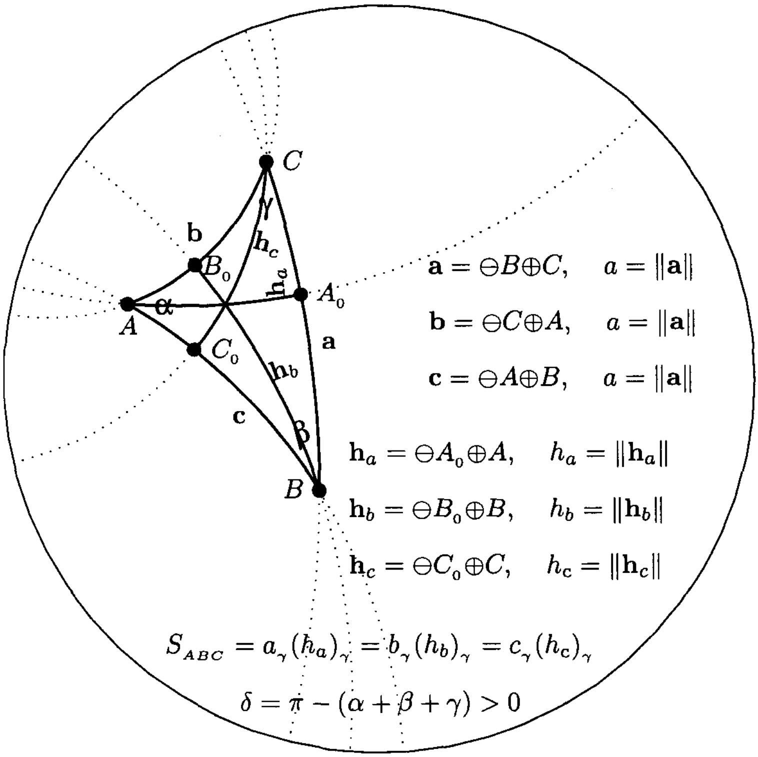


Fig. 8.14 A Möbius gyrotriangle ABC in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ and its heights are shown. The gyrotriangle heights are concurrent, and their lengths can be calculated by gyrotrigonometric techniques.

is called the gyrotriangle constant of gyrotriangle ABC.

8.6 The Gyroangular Defect of Right Gyroangles Gyrotriangles

Definition 8.43 (The Gyrotriangular Defect). *Gyroangle gyrotriangle sum is always smaller than π. The difference between this sum and π is called the defect of the gyrotriangle, Fig. 8.14.*

The sum of the gyroangles α and β of the right gyroangled gyrotriangle ABC in Fig. 8.12 is smaller than $\frac{\pi}{2}$ so that it possesses a positive *gyroangular defect* $\delta = \frac{\pi}{2} - (\alpha + \beta)$. The gyrocosine and the gyrosine of the gyroangular defect δ of gyrotriangle ABC are

$$\begin{aligned}\cos\delta &= \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right) \\ &= \sin(\alpha+\beta) \\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{a_\gamma}{c_\gamma}\frac{a_\beta}{c_\beta} + \frac{b_\beta}{c_\beta}\frac{b_\gamma}{c_\gamma}\end{aligned} \tag{8.124}$$

and

$$\begin{aligned}\sin\delta &= \sin\left(\frac{\pi}{2} - (\alpha+\beta)\right) \\ &= \cos(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ &= \frac{a_\beta b_\beta}{c_\beta^2} - \frac{a_\gamma b_\gamma}{c_\gamma^2}\end{aligned} \tag{8.125}$$

Interestingly, the tangent $\tan(\delta/2)$ of the half gyroangular defect $\delta/2$ is particularly simple and elegant. It follows from (8.124) and (8.125),

$$\tan\delta = \frac{2a_s b_s}{1 - a_s^2 b_s^2} \tag{8.126}$$

so that the γ and β corrections disappear, and

$$\tan\frac{\delta}{2} = a_s b_s \tag{8.127}$$

thus recovering a known result; see [Ungar (2001), (8.127)] and [Hartshorne (2003), Fig. 5]. This result that corresponds to the right-gyroangled gyrotriangle is extended in Theorem 8.44 to any gyrotriangle.

8.7 Gyroangular Defect of the Gyrotriangle

Theorem 8.44 **(The Gyrotriangle Defect Identity, I).** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.11, with vertices A, B, C, corresponding gyroangles α, β, γ, and side gyrolengths a, b, c. The gyroangular defect δ,*

$$\delta = \pi - (\alpha+\beta+\gamma) \tag{8.128}$$

of the gyrotriangle ABC is related to the gyrotriangle side gyrolengths and gyroangles by the identities

$$\tan\frac{\delta}{2} = \frac{a_s b_s \sin\gamma}{1 - a_s b_s \cos\gamma} = \frac{a_s c_s \sin\beta}{1 - a_s c_s \cos\beta} = \frac{b_s c_s \sin\alpha}{1 - b_s c_s \cos\alpha} \tag{8.129}$$

Proof. Let α, β and γ be the gyroangles of the hyperbolic gyrotriangle ABC in Fig. 8.11. The gyrocosines of these gyroangles are calculated by means of the law of gyrocosines, and the gyrosines of these gyroangles are related to each other by the law of gyrosines, enabling us to calculate $\cos\delta$ where $\delta = \pi - (\alpha + \beta + \gamma)$ is the gyrotriangle defect. We thus obtain

$$\begin{aligned}
1 + \cos\delta &= \frac{1}{2}\gamma_a^2\gamma_b^2\gamma_c^2\ (2 + a_s^2 b_s^2 c_s^2 - a_s^2 - b_s^2 - c_s^2)^2 \\
1 - \cos\delta &= \frac{1}{2}\gamma_a^2\gamma_b^2\gamma_c^2\ (a_s + b_s + c_s + a_s b_s c_s)(-a_s + b_s + c_s - a_s b_s c_s)\times \\
&\qquad (a_s - b_s + c_s - a_s b_s c_s)(a_s + b_s - c_s - a_s b_s c_s)
\end{aligned} \tag{8.130}$$

or, equivalently,

$$\begin{aligned}
1 + \cos\delta &= \frac{(\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1)^2}{2\gamma_a^2\gamma_b^2\gamma_c^2} \\
1 - \cos\delta &= \frac{4\gamma_a^2\gamma_b^2\gamma_c^2 - (\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1)^2}{2\gamma_a^2\gamma_b^2\gamma_c^2}
\end{aligned} \tag{8.131}$$

implying

$$\sin\delta = \frac{(\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1)\sqrt{4\gamma_a^2\gamma_b^2\gamma_c^2 - (\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1)^2}}{2\gamma_a^2\gamma_b^2\gamma_c^2} > 0 \tag{8.132}$$

The identities in (8.130) are employed to calculate $\tan(\delta/2)$ by means of the gyrotrigonometric identity $\tan^2(\phi/2) = (1-\cos\phi)/(1+\cos\phi)$, obtaining

$$\tan\frac{\delta}{2} = \frac{\sqrt{4\gamma_a^2\gamma_b^2\gamma_c^2 - (\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1)^2}}{\gamma_a^2 + \gamma_b^2 + \gamma_c^2 - 1} \tag{8.133}$$

or, equivalently [Ungar (2004b)],

$$
\begin{aligned}
\tan\frac{\delta}{2} &= \sqrt{a_s+b_s+c_s+a_sb_sc_s}\sqrt{-a_s+b_s+c_s-a_sb_sc_s}\ \times \\
&\quad \sqrt{a_s-b_s+c_s-a_sb_sc_s}\sqrt{a_s+b_s-c_s-a_sb_sc_s}\ \times \\
&\quad \frac{1}{2+a_s^2b_s^2c_s^2-a_s^2-b_s^2-c_s^2} \\
&= \frac{\psi(a,b,c;s)\psi(-a,b,c;s)\psi(a,-b,c;s)\psi(a,b,-c;s)}{2+a_s^2b_s^2c_s^2-a_s^2-b_s^2-c_s^2}
\end{aligned}
\tag{8.134}
$$

Finally, it follows from (8.134), (8.70) and (8.106) that

$$
\tan\frac{\delta}{2} = \frac{a_sb_s\sin\gamma}{1-a_sb_s\cos\gamma} \tag{8.135}
$$

Identity (8.135), along with permutations of its sides and their corresponding gyroangles, completes the proof. □

Clearly, (8.135) reduces to (8.127) when $\gamma=\pi/2$. Moreover, it follows from (8.135) that

$$
-\frac{2}{K}\tan\frac{\delta}{2} = \frac{1}{2}\frac{ab\sin\gamma}{1-a_sb_s\cos\gamma} \xrightarrow[s\to\infty]{} \frac{1}{2}a_eb_e\sin\gamma \tag{8.136}
$$

$K=-4/s^2$ being the Gaussian curvature, (7.28), where a_e and b_e are corresponding Euclidean side lengths; see Remark 8.45 below.

Remark 8.45 *We should note that in the limit* (8.136) *of large s the gyrotriangle side gyrolengths a and b implicitly depend on s, Fig. 8.11 since they are given by the equations*

$$
\begin{aligned}
a &= \|\ominus B\oplus C\| \\
b &= \|\ominus C\oplus A\|
\end{aligned}
\tag{8.137}
$$

where Möbius addition $\oplus$ depends on s. In the limit of large s Möbius addition $\oplus$ in a Möbius gyrovector space $(\mathbb{V}_s,\oplus,\otimes)$ reduces to vector addition in the vector space $\mathbb{V}$ so that

$$
\begin{aligned}
a_e &= \lim_{s\to\infty} a = \lim_{s\to\infty}\|\ominus B\oplus C\| = \|-B+C\| \\
b_e &= \lim_{s\to\infty} b = \lim_{s\to\infty}\|\ominus C\oplus A\| = \|-C+A\|
\end{aligned}
\tag{8.138}
$$

Accordingly, a_e and b_e are Euclidean side lengths of triangle ABC.

The gyrotriangle gyroangular defect δ is additive. Hence, it is a convenient choice for the measure of the area of gyrotriangles up to a multiplicative constant, so that the area of two disjoint gyrotriangles equals the sum of their areas. However, motivated by analogies with classical results, Identity (8.135) and its associated limit in (8.136) suggest the following

Definition 8.46 *The gyroarea $|ABC|$ of gyrotriangle ABC with corresponding sides a, b, c and gyroangles α, β, γ in a Mobius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.11, is given by the equation*

$$|ABC| = -\frac{2}{K}\tan\frac{\delta}{2} \tag{8.139}$$

where δ is the defect of gyrotriangle ABC, and where $K = -4/s^2$ is the Möbius gyrovector space Gaussian curvature, (7.28).

If we use the notation

$$p = \frac{a_e + b_e + c_e}{2} \tag{8.140}$$

then the first equality in (8.134) gives rise to the limit

$$\begin{aligned}
&\lim_{s\to\infty}\left(-\frac{2}{K}\tan\frac{\delta}{2}\right) = \lim_{s\to\infty}\left(\frac{s^2}{2}\tan\frac{\delta}{2}\right) \\
&= \frac{1}{4}\sqrt{a_e + b_e + c_e}\sqrt{-a_e + b_e + c_e}\sqrt{a_e - b_e + c_e}\sqrt{a_e + b_e - c_e} \\
&= \sqrt{p(p-a_e)(p-b_e)(p-c_e)}
\end{aligned} \tag{8.141}$$

thus recovering Heron's formula for the Euclidean triangle area. Hence, the gyrotriangle gyroarea reduces to the triangle area in the standard limit $s \to \infty$, as one would expect.

Clearly, Remark 8.45 about the limit in (8.136) applies to the limit in (8.141) as well.

We see from (8.135) – (8.141) that gyrotriangle gyroarea measure

$$|ABC| = \tfrac{1}{2}\frac{ab\sin\gamma}{1 - \frac{ab}{s^2}\cos\gamma} \tag{8.142}$$

of gyrotriangle ABC, Fig. 8.14, is a smooth extension of its corresponding triangle area measure,

$$\tfrac{1}{2}a_e b_e \sin\gamma \tag{8.143}$$

to which it reduces in the limit $s \to \infty$. The assumption in the title of [Ruoff (2005)], *"Why Euclidean area measure fails in the noneuclidean plane"*, is therefore wrong.

8.8 Gyroangular Defect of the Gyrotriangle – a Synthetic Proof

In this section we prove the result in (8.135) with $s = 1$ by employing synthetic studies of hyperbolic geometry rather than analytic studies of gyrovector spaces. Accordingly, we use in this section the language of hyperbolic geometry rather than gyrolanguage.

Theorem 8.47 (The Gyrotriangle Defect Identity, II). *Let $\mathbb{D} = \mathbb{R}^2_{s=1}$, Fig. 8.15, be the open unit disc of the Euclidean plane $\mathbb{R}^2$, and let ABC be a hyperbolic triangle in the Poincaré disc model $\mathbb{D}$. The triangle ABC has vertices A, B and C, corresponding hyperbolic angles α, β and γ, and sides BC, AC, and AB with corresponding hyperbolic lengths $a = |BC|$, $b = |AC|$ and $c = |AB|$. The gyroangular defect δ,*

$$\delta = \pi - (\alpha + \beta + \gamma) \tag{8.144}$$

of the hyperbolic triangle ABC is related to the triangle hyperbolic side lengths (that is, gyrolengths) and hyperbolic angles (that is, gyroangles) by the identities

$$\tan\frac{\delta}{2} = \frac{ab\sin\gamma}{1 - ab\cos\gamma} = \frac{ac\sin\beta}{1 - ac\cos\beta} = \frac{bc\sin\alpha}{1 - bc\cos\alpha} \tag{8.145}$$

Proof. Keeping hyperbolic angles and hyperbolic triangle side lengths invariant, we can move triangle ABC in the disc $\mathbb{D}$ by the motions of the disc. By appropriate motions of the disc we place the triangle in the position shown in Fig. 8.15. Vertex C of the triangle is placed at the center of the disc, and vertex A is placed at a point on the positive ray of the horizontal Cartesian coordinate of the disc. The sides AC and BC are now Euclidean straight line segments, and the side AB is a geodesic segment lying on a circle L, as shown in Fig. 8.15. Unlike the hyperbolic length $c = |AB|$ of side AB, the hyperbolic lengths $a = |BC|$ and $b = |AC|$ of sides BC and AC, respectively, coincide with their Euclidean lengths. Furthermore, the included angle γ is both hyperbolic and Euclidean, allowing us to employ in hyperbolic geometry tools from Euclidean geometry.

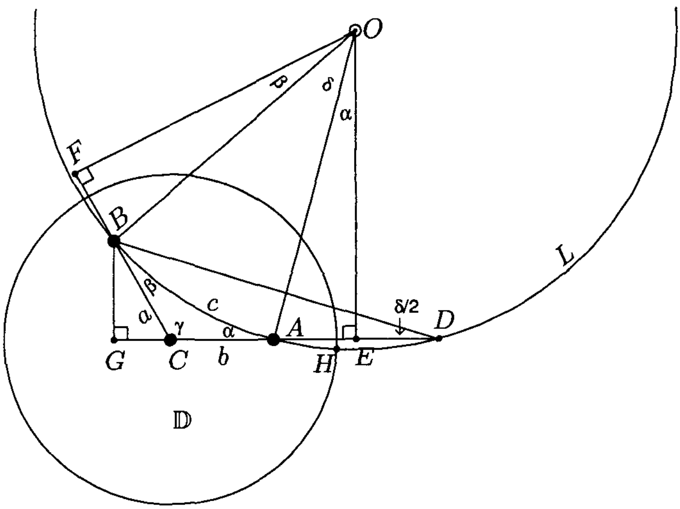


Fig. 8.15 The synthetic approach to the defect δ of gyrotriangle ABC, (8.157).

The circle L intersects the boundary $\partial\mathbb{D}$ of the disc $\mathbb{D}$ orthogonally, and its center, O, is known as the pole of the hyperbolic segment AB. We extend the straight line CA on its right to the point D, where it intersects circle L, and on its left to the point G that forms the Euclidean right-angled triangle DBG. The midpoint E of AD forms the Euclidean right-angled triangle AEO. Similarly, we extend the straight line CB to the point F that forms the Euclidean right-angled triangle BOF.

Let H be a point of intersection of the circles $\partial\mathbb{D}$ and L, Fig. 8.15. Since the two circles are orthogonal, the radius CH of $\partial\mathbb{D}$ is tangent to circle L. Being a secant and a tangent of circle L drawn from the same point C, the Euclidean lengths $|CA|$, $|CD|$ and $|CH|$ of sides CA, CD and CH satisfy the identity [Hartshorne (2000)]

$$|CA|\cdot|CD| = |CH|^2 \tag{8.146}$$

But, $|CA| = b$ and the radius-length of the disc boundary $\partial\mathbb{D}$ is $|CH| = 1$, so that (8.146) gives

$$|CD| = \frac{1}{b} \tag{8.147}$$

The lengths (both Euclidean and hyperbolic) of the two orthogonal sides of triangle GBC are

$$\begin{aligned} |GB| &= a\sin(\pi-\gamma) = a\sin\gamma \\ |GC| &= a\cos(\pi-\gamma) = -a\cos\gamma \end{aligned} \tag{8.148}$$

Hence, by (8.147) – (8.148) we have,

$$|GD| = |GC| + |CD| = \frac{1}{b} - a\cos\gamma \tag{8.149}$$

For the hyperbolic angle α in triangle ABC we clearly have

$$\begin{aligned} \alpha + \frac{\pi}{2} + \angle OAE &= 2\pi \\ \angle AOE + \frac{\pi}{2} + \angle OAE &= 2\pi \end{aligned} \tag{8.150}$$

so that

$$\alpha = \angle AOE \tag{8.151}$$

as shown in Fig. 8.15.

Similarly, for the hyperbolic angle β in triangle ABC we clearly have

$$\begin{aligned} \beta + \frac{\pi}{2} + \angle OBF &= 2\pi \\ \angle BOF + \frac{\pi}{2} + \angle OBF &= 2\pi \end{aligned} \tag{8.152}$$

so that

$$\beta = \angle BOF \tag{8.153}$$

as shown in Fig. 8.15.

Finally, we have

$$\angle FOE + \frac{\pi}{2} + \frac{\pi}{2} + \gamma = 2\pi \tag{8.154}$$

implying $\angle FOE = \pi - \gamma$, so that the angle

$$\delta = \angle FOE - \alpha - \beta = \pi - (\alpha+\beta+\gamma) = \angle AOB \tag{8.155}$$

shown in Fig. 8.15, turns out to be the defect of triangle ABC.

Since the arc AB of circle L subtends the angle δ at O, it subtends the angle $\delta/2$ at D [Hartshorne (2000)], as shown in Fig. 8.15. Thus,

$$\angle GDB = \frac{\delta}{2} \tag{8.156}$$

Hence, by (8.148) – (8.149) and (8.156) we have,

$$\tan\frac{\delta}{2} = \frac{|GB|}{|GD|} = \frac{ab\sin\gamma}{1-ab\cos\gamma} \tag{8.157}$$

Similarly, by circular permutations on the triangle parameters we obtain the triangle angular defect identities (8.145). □

8.9 The Gyrotriangle Side Gyrolengths in Terms of its Gyroangles

The following Theorem 8.48 presents a most important disanalogy with Euclidean triangle similarity. This theorem and Theorem 8.38 are the converse of each other while, in contrast, the Euclidean counterpart of Theorem 8.38 has no converse.

Theorem 8.48 **(The AAA to SSS Conversion Theorem).** *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.11, p. 258, with vertices A, B, C, corresponding gyroangles α, β, γ, $0 < \alpha+\beta+\gamma < \pi$, and side gyrolengths (or, simply, sides) a, b, c.*

The sides of the gyrotriangle ABC are determined by its gyroangles according to the AAA-to-SSS equations

$$\begin{aligned} a_s^2 &= \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)} \\ b_s^2 &= \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)} \\ c_s^2 &= \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)} \end{aligned} \tag{8.158}$$

Conversely, the angles of the gyrotriangle ABC are determined by its sides according to the system of equations, (8.112),

$$\begin{aligned} \cos\alpha &= \frac{-a_s^2 + b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2b_s c_s}\gamma_a^2 \\ \cos\beta &= \frac{a_s^2 - b_s^2 + c_s^2 - a_s^2 b_s^2 c_s^2}{2a_s c_s}\gamma_b^2 \\ \cos\gamma &= \frac{a_s^2 + b_s^2 - c_s^2 - a_s^2 b_s^2 c_s^2}{2a_s b_s}\gamma_c^2 \end{aligned} \tag{8.159}$$

Proof. Solving the three identities of Theorem 8.47,

$$\begin{aligned} \frac{a_s b_s \sin\gamma}{1 - a_s b_s \cos\gamma} &= \tan\frac{\delta}{2} \\ \frac{a_s c_s \sin\beta}{1 - a_s c_s \cos\beta} &= \tan\frac{\delta}{2} \\ \frac{b_s c_s \sin\alpha}{1 - b_s c_s \cos\alpha} &= \tan\frac{\delta}{2} \end{aligned} \tag{8.160}$$

for a_s, b_s, c_s, we have

$$\begin{aligned} a_s^2 &= \frac{\sin\alpha + \cos\alpha\tan\frac{\delta}{2}}{(\sin\beta + \cos\beta\tan\frac{\delta}{2})(\sin\gamma + \cos\gamma\tan\frac{\delta}{2})} \tan\frac{\delta}{2} \\ b_s^2 &= \frac{\sin\beta + \cos\beta\tan\frac{\delta}{2}}{(\sin\alpha + \cos\alpha\tan\frac{\delta}{2})(\sin\gamma + \cos\gamma\tan\frac{\delta}{2})} \tan\frac{\delta}{2} \\ c_s^2 &= \frac{\sin\gamma + \cos\gamma\tan\frac{\delta}{2}}{(\sin\alpha + \cos\alpha\tan\frac{\delta}{2})(\sin\beta + \cos\beta\tan\frac{\delta}{2})} \tan\frac{\delta}{2} \end{aligned} \tag{8.161}$$

Substituting

$$\tan\frac{\delta}{2} = \tan(\frac{\pi}{2} - \frac{\alpha+\beta+\gamma}{2}) = \cot\frac{\alpha+\beta+\gamma}{2} \tag{8.162}$$

in (8.161) and simplifying, we obtain the system of equations (8.158).

Finally, the system of equations (8.159) has already been established in (8.112). $\square$

It follows from Theorem 8.48 that any three gyroangles α, β, γ that satisfy the condition $0 < \alpha+\beta+\gamma < \pi$ give rise to three real numbers a_s, b_s, c_s according to (8.158) and, hence, can be realized as the three gyroangles of a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with sides a, b, c.

Example 8.49 (An Isosceles Gyrotriangle). As an elegant example of the use of Theorem 8.48 let us calculate the side gyrolengths a, b and c of a gyrotriangle in a Möbius gyrovector space $(\mathbb{V}_{s=1}, \oplus, \otimes)$ with gyroangles α, $\alpha/2$ and $\alpha/2$, shown in Fig. 8.16. By Theorem 8.48 these are

$$\begin{aligned} a^2 &= \frac{2\cos\alpha}{1+\cos\alpha} \\ b^2 = c^2 &= \frac{\cos\frac{\alpha}{2} + \cos\frac{3\alpha}{2}}{2\cos\frac{\alpha}{2}} = \cos\alpha \end{aligned} \tag{8.163}$$

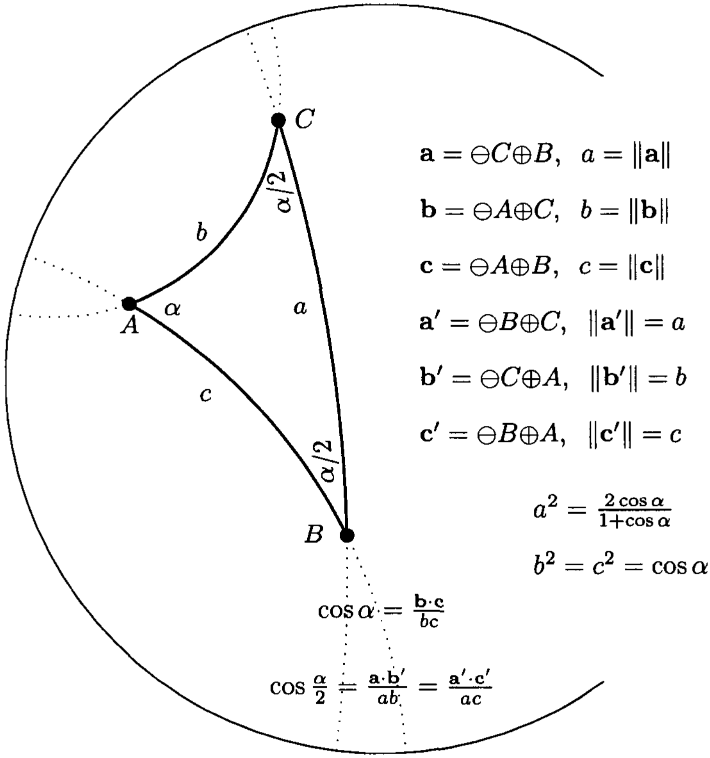


Fig. 8.16 A special isosceles gyrotriangle in a Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$, illustrating Example 8.49. Note that owing to the presence of a gyration, in general $\mathbf{a}' \neq \ominus\mathbf{a}$, etc.

We see from (8.158) that if the gyrotriangle gyroangle sum is π, $\alpha + \beta + \gamma = \pi$, than the gyrotriangle sides vanish. If a gyroangle gyrotriangle vanishes than its two generating sides have gyrolengths s, as we see from (8.158), so that its corresponding gyrotriangle vertex, called an asymptotic vertex, lies on the boundary of the ball $\mathbb{V}_s$ in the space $\mathbb{V}$. Accordingly, a gyrotriangle with a single vanishing gyroangle is called an asymptotic gyrotriangle. Similarly, a gyrotriangle with two (three) vanishing gyroangles is called a doubly (triply) asymptotic gyrotriangle [Ryan (1986)].

Solving the third identity in (8.158) for $\cos\gamma$ we obtain the identity

$$\cos\gamma = 2\gamma_c^2 \sin\alpha \sin\beta - \cos(\alpha - \beta) \tag{8.164}$$

that determines a gyroangle γ of a gyrotriangle in terms of the other two gyrotriangle gyroangles, α and β, and the gyrolength c of their included side.

In the limit of large s, $s \to \infty$, we have $\gamma_c \to 1$ so that (8.164) reduces to the familiar trigonometric result for Euclidean triangles,

$$\begin{aligned} \cos\gamma &= 2\sin\alpha\sin\beta - \cos(\alpha - \beta) \\ &= \cos(\pi - (\alpha + \beta)) \end{aligned} \tag{8.165}$$

which is equivalent to the condition

$$\alpha + \beta + \gamma = \pi \tag{8.166}$$

Theorem 8.50 (gyroAngle – gyroAngle – gyroAngle (AAA)). *If, in two gyrotriangles, three gyroangles of one are congruent to three gyroangles of the other, then the two gyrotriangles are congruent.*

Proof. The three gyroangles of a gyrotriangle determine the three side gyrolengths of the gyrotriangle by Theorem 8.48. Hence, by SSS congruency (Theorem 8.30), the two gyrotriangles are congruent. □

Theorem 8.51 (gyroAngle – Side – gyroAngle (ASA)). *If, in two gyrotriangles, two gyroangles and the included side of one, are congruent to two gyroangles and the included side of the other, then the two gyrotriangles are congruent.*

Proof. The two gyroangles of a gyrotriangle and the gyrolength of the included side determine its third gyroangle by (8.117) or, equivalently, by (8.164), or by the $\cos\gamma$ equation in Fig. 8.11. Hence, by AAA congruency, Theorem 8.50, the two gyrotriangles are congruent. □

In the special case when gyrotriangle ABC is right-gyroangled, with $\gamma = \pi/2$, Fig. 8.12, p. 263, the identities in (8.158) reduce to

$$\begin{aligned} a_s^2 &= \frac{\cos\alpha - \sin\beta}{\cos\alpha + \sin\beta} \\ b_s^2 &= \frac{\cos\beta - \sin\alpha}{\cos\beta + \sin\alpha} \\ c_s^2 &= \frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} \\ &= \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta} \end{aligned} \tag{8.167}$$

satisfying the hyperbolic Pythagorean identity

$$c_s^2 = \frac{a_s^2 + b_s^2}{1 + a_s^2 b_s^2} \tag{8.168}$$

in Möbius gyrovector spaces, Fig. 8.12.

Formally, replacing

$$(\cos\alpha, \sin\beta, \cos\beta, \sin\alpha) \quad \rightarrow \quad (x, y, x', y') \tag{8.169}$$

Identities (8.167) – (8.168) suggest the following elegant one-dimensional Möbius addition formula

$$\frac{x-y}{x+y} \oplus \frac{x'-y'}{x'+y'} = \frac{xx'-yy'}{xx'+yy'} \tag{8.170}$$

that holds for any real (or complex) numbers x, x', y, y' as long as the denominator does not vanish. The one-dimensional Möbius addition $\oplus$ in (8.170), shown for instance in (8.168) or (3.133), is obviously both commutative and associative, as expected. Furthermore, it coincides with the one-dimensional Einstein addition as we see from (3.133) and (3.151). It is owing to (8.170) that Einstein addition emerges in some linear boundary value problems [Loewenthal and Robinson (2000); Vigoureux (1993); Vigoureux (1994)]. In the special case when $x = x' = 1$, Identity (8.170) reduces to a "scalar Q function" identity of Lindell and Sihova [Lindell and Sihova (1998)].

8.10 The Semi-Gyrocircle Gyrotriangle

Theorem 8.52 **(The Semi-Gyrocircle Theorem).** *The point C lies on a gyrocircle with gyrodiameter AB in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, Fig. 8.17, if and only if*

$$\angle ABC + \angle BAC = \angle ACB \tag{8.171}$$

Proof. Let O be the center of the gyrocircle with gyrodiameter AB that contains the point C, Fig. 8.17. Then, the gyrotriangles BOC and AOC are isosceles so that $\alpha = \gamma_1$ and $\beta = \gamma_2$, and $\gamma_1 + \gamma_2 = \gamma$, where $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$, $\gamma_1 = \angle OCA$, and $\gamma_2 = \angle OCB$, thus verifying $\alpha + \beta = \gamma$, (8.171).

Conversely, if $\gamma = \alpha + \beta$, let O be the point on the gyrodiameter AB such that $\angle ACO = \alpha$ and $\angle BCO = \beta$. Then gyrotriangles OAC and OBC are

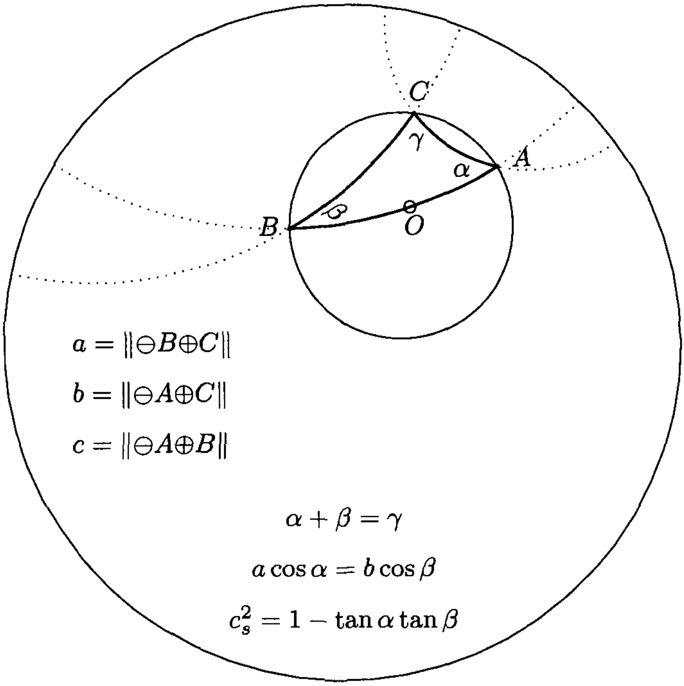


Fig. 8.17 The Semi-Gyrocircle Theorem. The point C lies on the gyrocircle with gyrodiameter AB in a Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ if and only if $\alpha + \beta = \gamma$. The two equations $\alpha + \beta = \gamma$ and $a \cos \alpha = b \cos \beta$ are also valid in the corresponding Euclidean semi-circle theorem. In contrast, owing to the condition $\alpha+\beta+\gamma = \pi$ for Euclidean angle sum and $\alpha + \beta + \gamma < \pi$ for hyperbolic angle sum, the condition $\gamma = \pi/2$ is valid in the Euclidean semi-circle theorem but not in its hyperbolic counterpart. The gyrodiameter gyrolength $c = \|\ominus A \oplus B\|$ satisfies the gyrodiameter identity $c_s^2 = 1 - \tan \alpha \tan \beta$, where $c_s = c/s$. As expected, the Euclidean counterpart of the gyrodiameter identity is trivial, $0 = 1 - \tan \alpha \tan \beta$.

isosceles, satisfying $|OB| = |OC| = |OA|$. Hence $r = |OB| = |OC| = |OA|$ is the radius gyrolength and C lies on the gyrocircle. □

Let C be a point lying on a gyrocircle with gyrodiameter AB in a Möbius gyrovector space $(\mathbb{V}_{s=1}, \oplus, \otimes)$, and let the sides of gyrotriangle ABC be $a = \|\ominus B \oplus C\|$, $b = \|\ominus A \oplus C\|$, $c = \|\ominus A \oplus B\|$. Furthermore, let the gyroangles of gyrotriangle ABC be $\alpha = \angle BAC$, $\beta = \angle ABC$, $\gamma = \angle ACB$, Fig. 8.17. Then, by Theorem 8.52,

$$\gamma = \alpha + \beta \tag{8.172}$$

Hence, by Theorem 8.48,

$$a^2 = \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)}$$

$$= \frac{\cos(\beta-\gamma) + \cos(\beta+\gamma)}{2\cos\alpha} \tag{8.173a}$$

$$= \frac{\cos\beta\cos\gamma}{\cos\alpha}$$

and, similarly,

$$b^2 = \frac{\cos\alpha\cos\gamma}{\cos\beta} \tag{8.173b}$$

and

$$c^2 = \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma - \cos(\alpha-\beta)}$$

$$= \frac{\cos(\alpha+\beta)}{\cos\alpha\cos\beta} \tag{8.173c}$$

$$= 1 - \tan\alpha\tan\beta$$

It follows from (8.173a) (8.173b) that the gyrolengths of the two non-diametric sides of gyrotriangle ABC in Fig. 8.17 are related by the elegant equation

$$a\cos\alpha = b\cos\beta \tag{8.174}$$

Remarkably, the relation (8.174) is valid in the corresponding Euclidean semi-circle theorem as well.

8.11 Gyrotriangular Gyration and Defect

In this section we relate the gyrotriangular defect δ, studied in Sec. 8.8, to gyrations.

Definition 8.53 **(The Gyrotriangular Gyration).** *Let ABC be a gyrotriangle in a gyrovector space $(G, \oplus, \otimes)$ with vertices A, B, C, sides* $\mathbf{a}, \mathbf{b}, \mathbf{c}$, *and defect δ, Fig. 8.11. The three successive gyrations*

$$\mathrm{gyr}[A, \ominus B]\mathrm{gyr}[B, \ominus C]\mathrm{gyr}[C, \ominus A] \tag{8.175}$$

generated by the gyrotriangle vertices can be written as a single gyration, Theorem 3.14,

$$\mathrm{gyr}[A, \ominus B]\mathrm{gyr}[B, \ominus C]\mathrm{gyr}[C, \ominus A] = \mathrm{gyr}[\ominus A \oplus B, \ominus(\ominus A \oplus C)] \qquad (8.176)$$

called a gyrotriangle gyration of the gyrotriangle ABC.

If the three points A, B and C are gyrocollinear, then, by the gyrotransitive gyration law, Theorem 6.29, the gyrotriangle gyration (8.175) is trivial. Hence, a gyrotriangle gyration is trivial when the area of its "gyrotriangle" vanishes (Of course, a "gyrotriangle" with vanishing area is not considered a gyrotriangle in the usual sense). Indeed, the rotation gyroangle that the gyrotriangle gyration (8.176) generates in the gyrotriangle gyroplane equals the gyrotriangle gyroangular defect δ as shown in Identity (8.177) below. The gyrotriangle gyroangular defect δ, in turn, measures the gyrotriangle area.

Remark 8.54 **(The Gyroangle of the Gyrotriangular Gyration Equals the Gyrotriangular Defect).** *Let $\mathbf{x}$ be any gyrovector in the gyroplane generated by the gyrotriangle ABC as, for instance, $\mathbf{x} = \ominus C \oplus B$, or $\mathbf{x} = \ominus C \oplus A$, or $\mathbf{x} = \ominus B \oplus A$. We may interject here that as in Euclidean geometry, a gyroplane generated by a gyrotriangle is the set of all points lying on gyrolines that intersect the gyrotriangle sides at two points. Then, a gyrotriangle gyration $\mathrm{gyr}[\ominus A \oplus B, \ominus(\ominus A \oplus C)]$ of the gyrotriangle ABC rotates $\mathbf{x}$ by the gyrotriangle gyroangular defect δ, (8.129), that is,*

$$\mathrm{gyr}[\ominus A \oplus B, \ominus(\ominus A \oplus C)]\frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \frac{\mathbf{x}}{\|\mathbf{x}\|} = \cos\delta \qquad (8.177)$$

8.12 The Equilateral Gyrotriangle

Theorem 8.55 **(The Equilateral Gyrotriangle).** *Let ABC be an equilateral gyrotriangle (that is, all its sides are congruent) in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, the side gyrolengths of which are a and the gyroangles of which are $\alpha, \alpha \geq 0$, Fig. 8.18. Then*

$$a_s = \sqrt{2\cos\alpha - 1} \qquad (8.178)$$

$0 < a < s$, *and*

$$0 \leq \alpha < \frac{\pi}{3} \qquad (8.179)$$

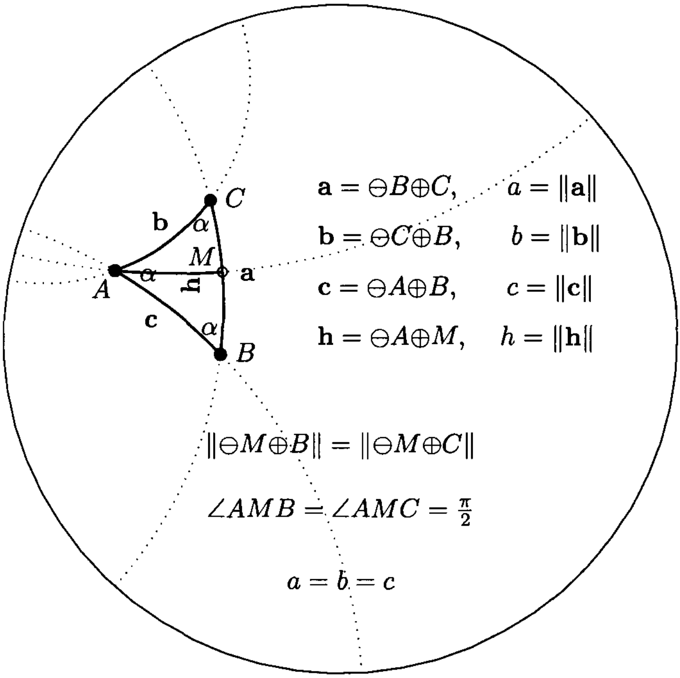


Fig. 8.18 The Möbius Equilateral Gyrotriangle. A Möbius equilateral gyrotriangle ABC in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$ is shown. Its sides have equal gyrolengths, $a = b = c$, its interior gyroangles have equal measures, α, and its altitude MA bisects both the base BC and the gyroangle α at the vertex A. The gyrovectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{h}$, rooted respectively at the points B, C, A, and again A, form the sides of the equilateral gyrotriangle ABC and one of its heights.

Conversely, let $a \in \mathbb{R}_s^{>0} = (0, s)$. Then there exists an equilateral gyrotriangle the gyrolength of each side of which is a, and the measure α of each of its gyroangles is

$$\alpha = \cos^{-1} \frac{1 + a_s^2}{2} \tag{8.180}$$

satisfying the condition

$$\tfrac{1}{2} < \cos\alpha < 1 \tag{8.181}$$

Furthermore, the gyrolength h of each height of the gyrotriangle is given *by the equation*

$$h_s = \sqrt{\frac{2\gamma_a^2 - \gamma_a - 1}{2\gamma_a^2 + \gamma_a - 1}} \tag{8.182}$$

where $h_s = \frac{h}{s}$.

Proof. By Theorem 8.48 for $\alpha = \beta = \gamma$ we have

$$a_s^2 = \frac{\cos\alpha + \cos 2\alpha}{1 + \cos\alpha} = 2\cos\alpha - 1 \tag{8.183}$$

where $\alpha \geq 0$ must satisfy the condition (8.179) to insure the reality of a_s and to exclude $a_s = 0$, thus verifying (8.178). Solving (8.183) for $\cos\alpha$ we obtain (8.180). In the proof of (8.182) we use the notation shown in Fig. 8.18.

The midpoint $\mathbf{M}$ of points B and C is given by the equation, (6.85),

$$\mathbf{M} = B\oplus(\ominus B\oplus C)\otimes\tfrac{1}{2} = B\oplus\tfrac{1}{2}\otimes\mathbf{a} \tag{8.184}$$

so that, by (6.235),

$$\ominus B\oplus\mathbf{M} = \tfrac{1}{2}\otimes\mathbf{a} = \frac{\gamma_a}{1+\gamma_a}\mathbf{a} \tag{8.185}$$

By the Möbius hyperbolic Pythagorean theorem 8.32, Fig. 8.18, we have for the right gyroangled gyrotriangle $A\mathbf{M}B$

$$\frac{1}{s}\|\ominus A\oplus\mathbf{M}\|^2\oplus\frac{1}{s}\|\ominus B\oplus\mathbf{M}\|^2 = \frac{1}{s}\|\mathbf{c}\|^2 \tag{8.186}$$

or equivalently, noting that $\|\mathbf{c}\|^2 = c^2 = a^2$,

$$\frac{h^2}{s}\oplus\frac{1}{s}\left(\frac{\gamma_a}{1+\gamma_a}\right)^2 a^2 = \frac{a^2}{s} \tag{8.187}$$

Hence

$$\frac{h^2}{s} = \frac{a^2}{s}\ominus\frac{1}{s}\left(\frac{\gamma_a}{1+\gamma_a}\right)^2 a^2 \tag{8.188}$$

noting that $\oplus$ in (8.187) – (8.188) is a vector space operation (commutative and associative).

Substituting

$$a^2 = s^2\frac{\gamma_a^2 - 1}{\gamma_a^2} \tag{8.189}$$

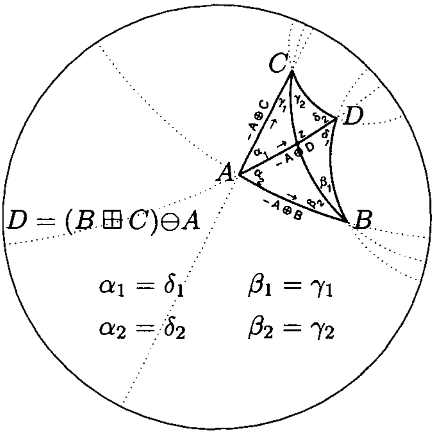


Fig. 8.19 A Möbius gyroparallelogram.

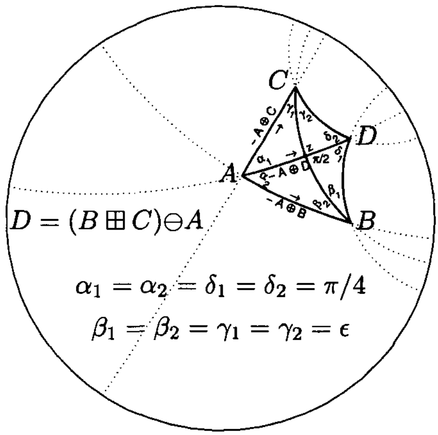


Fig. 8.20 A Möbius gyrosquare.

in (8.188), and expressing the operation $\ominus$ in (8.188) according to (3.133) we finally obtain (8.182). □

In Euclidean geometry equilateral triangles come with a unique angle $\alpha = \pi/3$, but with an arbitrary side length. In contrast, in hyperbolic geometry equilateral gyrotriangles come with arbitrary gyroangles α, $0 < \alpha < \pi/3$, but each admissible gyroangle α allows a unique equilateral gyrotriangle with side gyrolength given by (8.178). Clearly, equilateral triangles lack the richness of equilateral gyrotriangles that we see in gyrovector spaces.

8.13 The Möbius Gyroparallelogram

A gyroparallelogram $ABDC$, Def. 6.40, is shown in Figs. 8.19 and 8.20 in the Möbius gyrovector plane. By Theorem 6.45 opposite sides of a gyroparallelogram are equal modulo gyrations, so that they have equal gyrolengths. Furthermore, by applying Theorem 6.45 to the gyroparallelogram $ABDC$ in each of Figs. 8.19 and 8.20 we have

$$\begin{aligned} \ominus A \oplus B &= \mathrm{gyr}[A, \ominus B]\mathrm{gyr}[B, \ominus C](\ominus C \oplus D) \\ \ominus A \oplus C &= \mathrm{gyr}[A, \ominus C]\mathrm{gyr}[C, \ominus B](\ominus B \oplus D) \end{aligned} \tag{8.190}$$

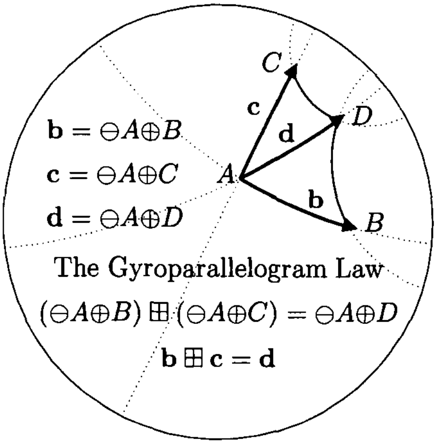


Fig. 8.21 The gyroparallelogram law and the Möbius gyroparallelogram $ABDC$ in Fig. 8.19 give rise to the commutative gyroparallelogram addition law of gyrovectors, shown here as a first example.

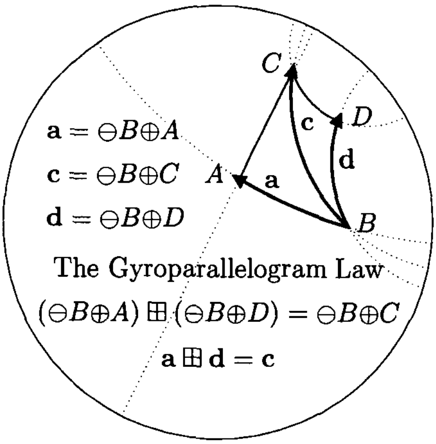


Fig. 8.22 The gyroparallelogram law and the Möbius gyroparallelogram $ABDC$ in Fig. 8.19 give rise to the commutative gyroparallelogram addition of gyrovectors, shown here as a second example.

so that,

$$\begin{aligned} \|\ominus A\oplus B\| &= \|\ominus C\oplus D\| \\ \|\ominus A\oplus C\| &= \|\ominus B\oplus D\| \end{aligned} \tag{8.191}$$

Moreover, by Theorem 6.39 the gyroparallelogram diagonals intersect at their midpoints. The midpoints of the diagonals AD and BC are, respectively,

$$\begin{aligned} P^m_{A,D} &= \tfrac{1}{2}\otimes(A \boxplus D) \\ P^m_{B,C} &= \tfrac{1}{2}\otimes(B \boxplus C) \end{aligned} \tag{8.192}$$

and their equality follows from Def. 6.40 of the gyroparallelogram according to which $D = (B \boxplus C)\ominus A$, implying $A \boxplus D = B \boxplus C$.

Each diagonal of the gyroparallelogram forms two congruent gyrotriangles, from which several congruent gyroangles can be recognized. If two adjacent sides of the gyroparallelogram are congruent, then the diagonals are perpendicular, as shown in Fig. 8.20. If two adjacent sides of the gyroparallelogram are congruent and perpendicular, the gyroparallelogram is called a gyrosquare. A gyrosquare in the Möbius gyrovector plane is shown in Fig. 8.20.

By the gyroparallelogram law, Theorem 6.42, we have for the Möbius gyroparallelogram $ABDC$, Figs. 8.21 and 8.22,

$$
\begin{aligned}
(\ominus A \oplus B) \boxplus (\ominus A \oplus C) &= \ominus A \oplus D \\
(\ominus B \oplus A) \boxplus (\ominus B \oplus D) &= \ominus B \oplus C \\
(\ominus C \oplus A) \boxplus (\ominus C \oplus D) &= \ominus C \oplus B \\
(\ominus D \oplus B) \boxplus (\ominus D \oplus C) &= \ominus D \oplus A
\end{aligned}
\tag{8.193}
$$

The first gyroparallelogram addition in (8.193) can be written as

$$
(\ominus A \oplus B) \oplus \text{gyr}[\ominus A \oplus B, \ominus(\ominus A \oplus C)](\ominus A \oplus C) = \ominus A \oplus D \tag{8.194}
$$

as we see from the definition of the gyrogroup cooperation $\boxplus$, (2.2).

Remark 8.56 *Interestingly, the gyration* $\text{gyr}[\ominus A \oplus B, \ominus(\ominus A \oplus C)]$ *in* (8.194) *is the gyrotriangle gyration of the gyrotriangle ABC, Fig. 8.21, as we see from Def. 8.53. This gyration, in turn, gyrates (rotates) the side $\ominus A \oplus C$ of the gyroparallelogram $ABDC$ in Fig. 8.21 by the gyroangular defect,* (8.176), *of the gyrotriangle ABC so as to "close" the gyroparallelogram. It is also interesting to realize that the gyration in* (8.194) *that "closes" the gyroparallelogram $ABDC$ is the defect of the gyrotriangle ABC, as remarked in Remark 8.54.*

The gyroparallelogram is a gyrovector space object, Theorem 6.38, so that it can be moved in its gyrovector space by the gyrovector space motions without distorting its internal structure. The gyroparallelogram internal structure, in turn, gives rise to the gyroparallelogram law of gyrovector addition. Figure 8.21 shows the application of the gyroparallelogram law to the addition of the gyrovectors $\mathbf{b} = \ominus A \oplus B$ and $\mathbf{c} = \ominus A \oplus C$, (8.193), the latter being equivalence classes by Def. 5.4.

Hence, gyrovectors are equivalence classes of directed gyrosegments that add according to the gyroparallelogram law just like vectors, which are equivalence classes of directed segments that add according to the common parallelogram law.

Some gyroparallelogram properties are presented in Sec. 6.7, p. 160. Following the gyroangle definition, Def. 8.2, we can now explore relationships between gyroparallelogram gyroangles.

Theorem 8.57 **(Gyroparallelogram Opposite Gyroangles).** *Opposite gyroangles of a gyroparallelogram are congruent.*

Proof. Let $ABA'B'$ be a gyroparallelogram in a gyrovector space, and let $\alpha = \angle BAB'$ and $\alpha' = \angle BA'B'$ be opposite gyroangles, Fig. 8.23. Then, by (6.129) of Theorem 6.45 we have (identifying the gyroparallelogram **abdc** of Theorem 6.45 with the gyroparallelogram $ABA'B'$ of the present theorem)

$$\begin{aligned} \ominus A' \oplus B &= \ominus \mathrm{gyr}[A', \ominus A](\ominus A \oplus B') \\ \ominus A' \oplus B' &= \ominus \mathrm{gyr}[A', \ominus A](\ominus A \oplus B) \end{aligned} \tag{8.195}$$

so that

$$\begin{aligned} \|\ominus A' \oplus B\| &= \|\ominus A \oplus B'\| \\ \|\ominus A' \oplus B'\| &= \|\ominus A \oplus B\| \end{aligned} \tag{8.196}$$

and

$$\begin{aligned} (\ominus A' \oplus B)\cdot(\ominus A' \oplus B') &= \mathrm{gyr}[A', \ominus A](\ominus A \oplus B')\cdot \mathrm{gyr}[A', \ominus A](\ominus A \oplus B) \\ &= (\ominus A \oplus B')\cdot(\ominus A' \oplus B) \end{aligned} \tag{8.197}$$

since gyrations preserve the inner product. Hence,

$$\cos\alpha' = \frac{(\ominus A' \oplus A)\cdot(\ominus A' \oplus B)}{\|\ominus A' \oplus A\| \|(\ominus A' \oplus B\|} = \frac{(\ominus A \oplus B')\cdot(\ominus A \oplus B)}{\|\ominus A \oplus B'\| \|(\ominus A \oplus B\|} = \cos\alpha \tag{8.198}$$

The proof of $\beta' = \beta$ for the opposite gyroangles $\beta = \angle ABA'$ and $\beta' = \angle AB'A'$ is similar. □

8.14 Gyrotriangle Defect in the Möbius Gyroparallelogram

Definition 8.58 **(The Gyroparallelogram Defect).** *Gyroangle gyroparallelogram sum is always smaller than 2π. The difference between this sum and 2π is called the defect of the gyroparallelogram*

Instructively, a numerical example is found useful to illustrate gyrotriangle defects in the Möbius gyroparallelogram, Fig. 8.23. For our numerical demonstration we arbitrarily select the three points

$$\begin{aligned} A &= (0.10000000000000,\ 0.20000000000000) \\ B &= (0.60882647943831,\ -0.02106318956871) \\ B' &= (0.34782608695652,\ 0.69565217391304) \end{aligned} \tag{8.199}$$

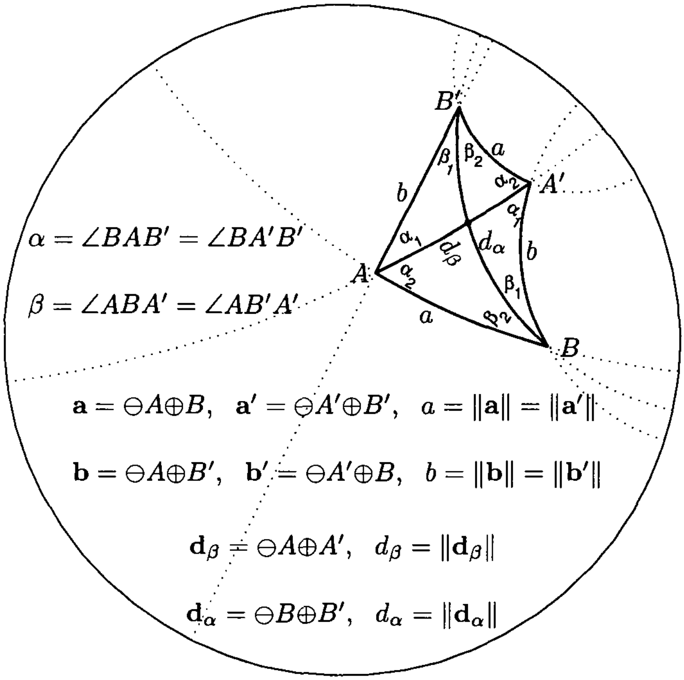


Fig. 8.23 Gyrotriangle defects in the Möbius gyroparallelogram in the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$. The gyroparallelogram $ABA'B'$ has (i) two side gyrolengths, a and b; opposite sides are congruent; (ii) two gyroangles, α and β; opposite gyroangles are congruent; and (iii) two diagonals, d_α and d_β. The diagonals concurrent point coincides with their gyromidpoints. Alternate gyroangles are congruent.

in a Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$, Fig. 8.23, and complete them to a gyroparallelogram $ABA'B'$ by adding the fourth point A' which is determined by the gyroparallelogram condition in Def. 6.40,

$$A' = (B \boxplus B') \ominus A \tag{8.200}$$

that is,

$$A' = (0.34782608695652,\ 0.69565217391304) \tag{8.201}$$

The resulting four gyrovectors that form the gyroparallelogram sides are therefore

$$\begin{aligned}
\mathbf{a} &= \ominus A \oplus B = (0.50000000000000, \ -0.30000000000000) \\
\mathbf{b} &= \ominus A \oplus B' = (0.30000000000000, \ 0.60000000000000) \\
\mathbf{a}' &= \ominus A' \oplus B' = (-0.53960557633458, \ 0.22096565793948) \\
\mathbf{b}' &= \ominus A' \oplus B = (-0.20584835090105, \ -0.63845630737844)
\end{aligned} \tag{8.202}$$

As expected from Theorem 6.45, opposite sides of the gyroparallelogram are congruent,

$$\begin{aligned}
\|\mathbf{a}\| &= 0.58309518948453 \\
\|\mathbf{b}\| &= 0.67082039324994 \\
\|\mathbf{a}'\| &= 0.58309518948453 \\
\|\mathbf{b}'\| &= 0.67082039324994
\end{aligned} \tag{8.203}$$

The four gyroangles of the gyroparallelogram are

$$\begin{aligned}
\alpha = \angle BAB' &= \cos^{-1} \frac{(\ominus A \oplus B) \cdot (\ominus A \oplus B')}{\|\ominus A \oplus B)\| \|\ominus A \oplus B'\|} = 1.64756821806467 \\
\beta = \angle ABA' &= \cos^{-1} \frac{(\ominus B \oplus A) \cdot (\ominus B \oplus A')}{\|\ominus B \oplus A\| \|\ominus B \oplus A'\|} = 0.77010692104966 \\
\alpha' = \angle B'A'B &= \cos^{-1} \frac{(\ominus A' \oplus B') \cdot (\ominus A' \oplus B)}{\|\ominus A' \oplus B' \| \ominus A' \oplus B\|\|} = 1.64756821806468 \\
\beta' = \angle A'B'A &= \cos^{-1} \frac{(\ominus B' \oplus A') \cdot (\ominus B' \oplus A)}{\|\ominus B' \oplus A'\| \|\ominus B' \oplus A\|} = 0.77010692104966
\end{aligned} \tag{8.204}$$

so that opposite gyroangles of the gyroparallelogram are congruent, $\alpha = \alpha'$ and $\beta = \beta'$, as expected from Theorem 8.57.

According to Theorem 8.44, the defects $\delta(ABA')$, $\delta(AB'A')$, $\delta(BAB')$, and $\delta(BA'B')$ of the gyrotriangles ABA', $AB'A'$, $AB'A'$, and $BA'B'$, that comprise the gyroparallelogram $ABA'B'$ in Fig. 8.23, assume the values

$$\begin{aligned}
\delta(ABA') &= 0.72391751447546 \\
\delta(AB'A') &= 0.72391751447546 \\
\delta(BAB') &= 0.72391751447546 \\
\delta(BA'B') &= 0.72391751447546
\end{aligned} \tag{8.205}$$

The equalities $\delta(BAB') = \delta(BA'B')$ and $\delta(ABA') = \delta(AB'A')$, observed numerically in (8.205), are not surprising since, according to (8.129), they are generated by two equal side gyrolengths and equal included gyroangles,

$$\begin{aligned}\delta(BAB') &= \delta(BA'B') = 2\tan^{-1}\frac{ab\sin\alpha}{1-ab\cos\alpha}\\ \delta(ABA') &= \delta(AB'A') = 2\tan^{-1}\frac{ab\sin\beta}{1-ab\cos\beta}\end{aligned}\tag{8.206}$$

However, on first glance, the equality $\delta(BAB') = \delta(ABA')$, observed numerically in (8.205), is surprising since, by (8.129), while its two sides are generated by the same side gyrolengths they are generated by different included gyroangles,

$$\begin{aligned}\delta(BAB') &= 2\tan^{-1}\frac{ab\sin\alpha}{1-ab\cos\alpha}\\ \delta(ABA') &= 2\tan^{-1}\frac{ab\sin\beta}{1-ab\cos\beta}\end{aligned}\tag{8.207}$$

The defect $\delta(ABA'B')$ of gyroparallelogram $ABA'B'$ equals the sum of the defects of its comprising gyrotriangles,

$$\begin{aligned}\delta(ABA'B') &= \delta(ABA') + \delta(AB'A')\\ &= \delta(BAB') + \delta(BA'B')\\ &= 1.44783502895092\end{aligned}\tag{8.208}$$

which equals

$$2\pi - 2(\alpha+\beta) = 1.44783502895092 \tag{8.209}$$

It follows from the defect equalities in (8.208) and (8.207) that the two consecutive gyroparallelogram gyroangles α and β, Fig. 8.23, are related by the equation

$$\frac{\sin\alpha}{1-ab\cos\alpha} = \frac{\sin\beta}{1-ab\cos\beta} \tag{8.210}$$

or, equivalently, by the equation

$$\frac{1-\cos^2\alpha}{(1-ab\cos\alpha)^2} = \frac{1-\cos^2\beta}{(1-ab\cos\beta)^2} \tag{8.211}$$

The resulting quadratic equation (8.211) admits two solutions for $\cos\beta$,

$$\cos\beta = \cos\alpha$$
$$\cos\beta = \frac{2ab - (1 + a^2b^2)\cos\alpha}{(1 + a^2b^2) - 2ab\cos\alpha} \tag{8.212}$$

giving rise to a necessary condition for β to be the second gyroangle of a gyroparallelogram with sides gyrolengths a and b and with an included gyroangle α. On first glance, it seems from the two equations in (8.212) that a gyroparallelogram with sides gyrolengths a and b with an included gyroangle α, Fig. 8.23, admits two distinct gyroparallelograms corresponding to a gyroangle β, $\beta = \alpha$, that comes from the first condition in (8.212) and to a different gyroangle β that comes from the second condition in (8.212). This is, however, not the case since the first condition in (8.212) is included in the second one as a special case, as we show below.

When $\beta = \alpha$ in the gyroparallelogram $ABA'B'$, Fig. 8.23, its included gyrotriangle $B'AB$ has two unspecified sides a and b with an included gyroangle α and two remaining gyroangles $\alpha/2$ and $\alpha/2$. The three gyroangles α, $\alpha/2$ and $\alpha/2$ of gyrotriangle $B'AB$ determine its side gyrolengths a and b according to Theorem 8.48, obtaining

$$a^2 = b^2 = \frac{\cos\frac{\alpha}{2} + \cos\frac{3\alpha}{2}}{2\cos\frac{\alpha}{2}} = \cos\alpha \tag{8.213}$$

as in Fig. 8.16 (where the notation for sides is slightly different).

The substitution of the gyrolengths a and b from (8.213) into the second equation in (8.212) gives $\cos\beta = \cos\alpha$, thus recovering the first equation in (8.212). This demonstrates that the second condition in (8.212) includes the first one as a special case. Hence, the first condition in (8.212) can be deleted with no loss of generality, as we formalize in the following

Theorem 8.59 *Let P be a gyroparallelogram in a Möbius gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ with two distinct side gyrolengths a and b and a gyroangle α. Then, the remaining distinct gyroangle β of P, Fig. 8.23, is given by the equation*

$$\cos\beta = \frac{2a_sb_s - (1 + a_s^2b_s^2)\cos\alpha}{1 + a_s^2b_s^2 - 2a_sb_s\cos\alpha} \tag{8.214}$$

In the limit of large s, $s \to \infty$, (8.214) reduces to the identity

$$\cos\beta = -\cos\alpha \tag{8.215}$$

that recovers a known result about the Euclidean parallelogram, according to which $\alpha + \beta = \pi$.

Theorem 8.60 *With the notation of Fig. 8.23, let P be a gyroparallelogram in a Möbius gyrovector space $(\mathbb{V}_{s=1}, \oplus, \otimes)$ with two distinct side gyrolengths a and b, two distinct gyroangles α and β, and two diagonal gyrolengths d_α and d_β. Then, the two diagonal gyrolengths are related to the gyroparallelogram side gyrolengths by the identity*

$$d_\alpha^2 d_\beta^2 - (d_\alpha^2 + d_\beta^2) - \frac{(a^2 + b^2 - 2a^2b^2)(a^2 + b^2 - 2)}{(1 - a^2b^2)^2} = 0 \tag{8.216}$$

or, equivalently, by the identity

$$\gamma_{d_\alpha}\gamma_{d_\beta} = \gamma_a^2 + \gamma_b^2 - 1 \tag{8.217}$$

Proof. Applying the law of gyrocosines, Theorem 8.25, to each of the gyroparallelogram comprising gyrotriangles ABB' and $A'B'B$, Fig. 8.23, we respectively obtain the following two identities,

$$\begin{aligned} d_\alpha^2 &= a^2 \oplus b^2 \ominus \frac{2\beta_a^2 a \beta_b^2 b \cos\alpha}{1 - 2\beta_a^2 a \beta_b^2 b \cos\alpha} \\ d_\beta^2 &= a^2 \oplus b^2 \ominus \frac{2\beta_a^2 a \beta_b^2 b \cos\beta}{1 - 2\beta_a^2 a \beta_b^2 b \cos\beta} \end{aligned} \tag{8.218}$$

Here one should note that the Möbius addition $\oplus$ in (8.218) is a commutative group operation, as remarked and presented in Remark 8.27.

Eliminating $\cos\alpha$ and $\cos\beta$ between the three equations in (8.214) and (8.218) one recovers (8.216), which turns out to be equivalent to (8.217). $\square$

For the sake of simplicity it is assumed $s = 1$ in Theorem 8.60. However, its extension to $s > 0$ is instructive and simple. Thus, (i) Identity (8.217) remains valid for all $s > 0$, the parameter s being implicit in γ factors; and (ii) Identity (8.216) takes the form

$$\frac{1}{s^4} d_\alpha^2 d_\beta^2 - \frac{1}{s^2}(d_\alpha^2 + d_\beta^2) - \frac{(a_s^2 + b_s^2 - 2a_s^2 b_s^2)(a_s^2 + b_s^2 - 2)}{(1 - a_s^2 b_s^2)^2} = 0 \tag{8.219}$$

or, equivalently,

$$\frac{1}{s^2} d_\alpha^2 d_\beta^2 - (d_\alpha^2 + d_\beta^2) - \frac{(a^2 + b^2 - \frac{2}{s^2} a^2 b^2)(a_s^2 + b_s^2 - 2)}{(1 - \frac{1}{s^4} a^2 b^2)^2} = 0 \tag{8.220}$$

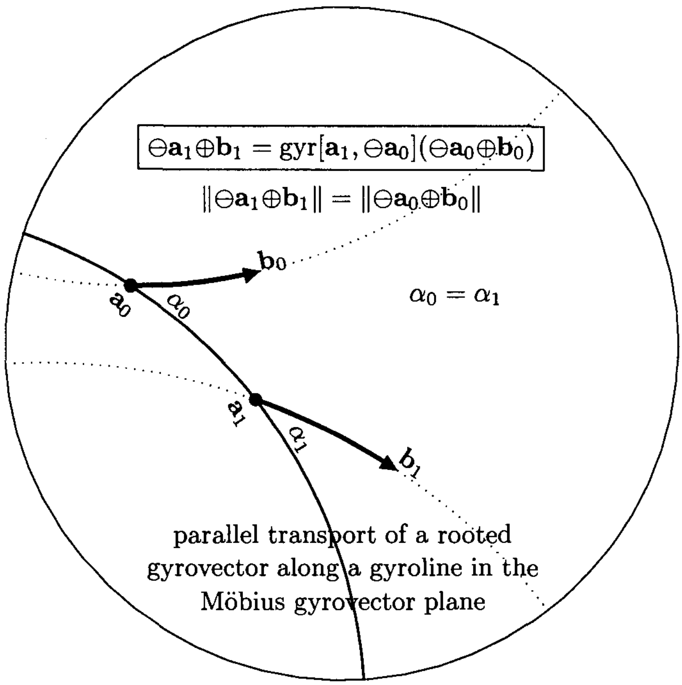


Fig. 8.24 Parallel Transport. The parallel transport of a rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ rooted at $\mathbf{a}_0$ to the rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ rooted at $\mathbf{a}_1$ along the gyroline that links the points $\mathbf{a}_0$ and $\mathbf{a}_1$ in the Möbius gyrovector plane $(\mathbb{R}_s^2, \oplus, \otimes)$.

In the limit of large s, $s \to \infty$, (8.220) reduces to the identity

$$d_\alpha^2 + d_\beta^2 = 2(a^2 + b^2) \tag{8.221}$$

that recovers the known result of Euclidean geometry, according to which the sum of the squares of lengths of the four sides of a parallelogram equals the sum of the squares of the lengths of the two diagonals.

The reduction of the hyperbolic identities (8.214) and (8.220) to corresponding familiar Euclidean identities (8.215) and (8.221) that the free parameter s allows demonstrates the usefulness of the parameter.

8.15 Parallel Transport

Definition 8.61 (Parallel Transport). *A rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ is a parallel transport (or translation) of a rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, $\mathbf{a}_0 \neq$*

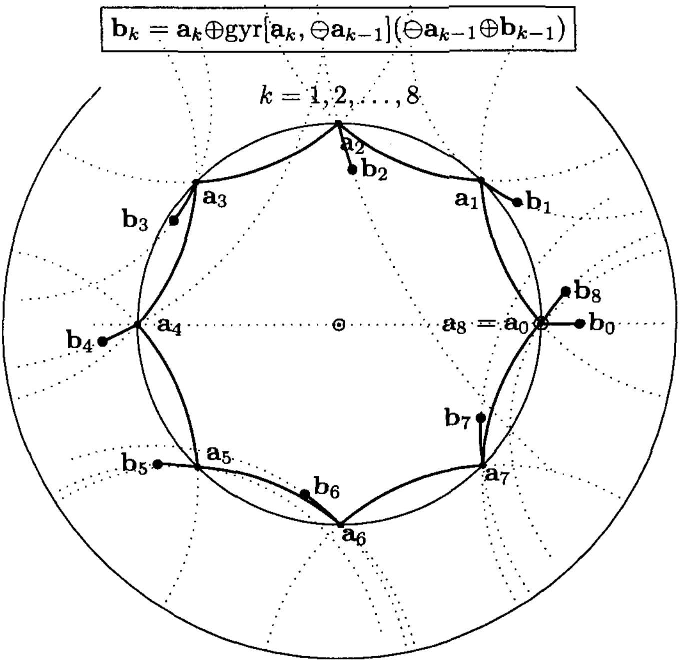


Fig. 8.25 The accrued circular gyrophase shift is approximated by the accrued polygonal gyrophase shift $\theta_8 = \angle\mathbf{b}_0\mathbf{a}_0\mathbf{b}_8 = \angle\mathbf{b}_0\mathbf{a}_8\mathbf{b}_8$ generated by the parallel transport of a geodesic segment along a hyperbolic regular polygonal path. Shown is a hyperbolic regular polygon (gyropolygon) with 8 sides approximating a hyperbolic circle centered at the center of the Möbius disc $(\mathbb{R}^2_s, \oplus, \otimes)$. An initial gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ rooted at vertex $\mathbf{a}_0$ of the gyropolygon is parallel transported counterclockwise along the gyropolygonal path back to its initial root $\mathbf{a}_0 = \mathbf{a}_8$, resulting in the final vector $\ominus\mathbf{a}_8\oplus\mathbf{b}_8$. The angular defect of the resulting accrued gyrophase shift is the angle θ_8 formed by the initial gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ and the final gyrovector $\ominus\mathbf{a}_8\oplus\mathbf{b}_8$, both rooted at the point $\mathbf{a}_8 = \mathbf{a}_0$.

$\mathbf{a}_1$, *(along the gyroline L that joins the points $\mathbf{a}_0$ and $\mathbf{a}_1$, Fig. 8.24) in a gyrovector space $(G, \oplus, \otimes)$ if*

$$\ominus\mathbf{a}_1\oplus\mathbf{b}_1 = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{8.222}$$

Successive parallel transports along a closed regular gyropolygonal path that approximates a gyrocircular path is shown in Fig. 8.25. By increasing the number of vertices of the regular gyropolygonal path one can improve the approximation, as shown in [Ungar (2001), Figs. 7.19–7.21]

In full analogy with vector spaces, a parallel transport in a gyrovector space is a way to transport gyrovectors from a point, $\mathbf{a}_0$, to another point, $\mathbf{a}_1$, along the gyroline, $\mathbf{a}_0\mathbf{a}_1$, connecting the points, Fig. 8.24, such that they stay "parallel" in the following sense. The gyrolength of a parallel transported gyrovector and its gyroangle with the connecting gyroline must remain invariant. The following theorem shows that this is indeed the case.

Theorem 8.62 **(The Parallel Transport Theorem I).** *Let the rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ be a parallel transport of the rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ $\mathbf{a}_0 \neq \mathbf{a}_1$, in a gyrovector space $(G,\oplus,\otimes)$, let L be the gyroline passing through the points $\mathbf{a}_0$ and $\mathbf{a}_1$, and let $\mathbf{a}$ be a point on L such that $\mathbf{a}_1$ lies between $\mathbf{a}_0$ and $\mathbf{a}$, Fig. 8.24. Then, the two rooted gyrovectors $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ and $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ have equal gyrolengths,*

$$\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\| = \|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\| \tag{8.223}$$

and equal gyroangles with the gyroline L,

$$\angle\mathbf{b}_0\mathbf{a}_0\mathbf{a}_1 = \angle\mathbf{b}_1\mathbf{a}_1\mathbf{a} \tag{8.224}$$

Proof. Identity (8.223) follows immediately from (8.222) since gyrations keep the norm invariant.

The representation of the gyroline L in terms of the points $\mathbf{a}_0$ and $\mathbf{a}_1$ that it contains is

$$L = \mathbf{a}_0\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t \tag{8.225}$$

so that a representation of a point $\mathbf{a}$ of L that is not on the gyrosegment $\mathbf{a}_0\mathbf{a}_1$ is

$$\mathbf{a} = \mathbf{a}_0\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t' \tag{8.226}$$

where $t' > 1$.

The following chain of equations, in which equalities are numbered for subsequent explanation, verifies (8.224).

$$
\begin{aligned}
\cos\angle\mathbf{b}_1\mathbf{a}_1\mathbf{a} &= \frac{\ominus\mathbf{a}_1\oplus\mathbf{b}_1}{\|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\|}\cdot\frac{\ominus\mathbf{a}_1\oplus\mathbf{a}}{\|\ominus\mathbf{a}_1\oplus\mathbf{a}\|} \\
&\overset{(1)}{=} \frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\|}\cdot\frac{\ominus\mathbf{a}_1\oplus\{\mathbf{a}_0\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\}}{\|\ominus\mathbf{a}_1\oplus\{\mathbf{a}_0\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\}\|} \\
&\overset{(2)}{=} \frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\|} \\
&\qquad\cdot\frac{(\ominus\mathbf{a}_1\oplus\mathbf{a}_0)\oplus\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'}{\|(\ominus\mathbf{a}_1\oplus\mathbf{a}_0)\oplus\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\|} \\
&\overset{(3)}{=} \frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\|} \\
&\qquad\cdot\frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0]\{\ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\}}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0]\{\ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\}\|} \\
&\overset{(4)}{=} \frac{\ominus\mathbf{a}_0\oplus\mathbf{b}_0}{\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\|}\cdot\frac{\ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'}{\|\ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\oplus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes t'\|} \\
&\overset{(5)}{=} \frac{\ominus\mathbf{a}_0\oplus\mathbf{b}_0}{\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\|}\cdot\frac{(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes(-1+t')}{\|(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)\otimes(-1+t')\|} \\
&\overset{(6)}{=} \frac{\ominus\mathbf{a}_0\oplus\mathbf{b}_0}{\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\|}\cdot\frac{\ominus\mathbf{a}_0\oplus\mathbf{a}_1}{\|\ominus\mathbf{a}_0\oplus\mathbf{a}_1\|} \\
&= \cos\angle\mathbf{b}_0\mathbf{a}_0\mathbf{a}_1
\end{aligned}
\tag{8.227}
$$

The derivation of the numbered equalities in (8.227) follows:

(1) Follows from (8.226).
(2) Follows from the left gyroassociative law.
(3) Follows from the gyrocommutative law and the gyroautomorphic inverse property.
(4) Follows from the invariance of the inner product under gyrations.
(5) Follows from the scalar distributive law.
(6) Follows from the scaling property, noting that $-1+t'>0$. □

Theorem 8.63 (Gyrovector Parallel Transport Head). *Let P, Q, P' be any three points of a gyrovector space $(G, \oplus, \otimes)$. The paral-*

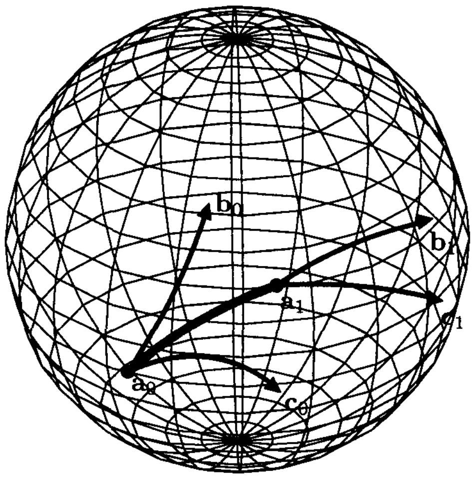


Fig. 8.26 Three dimensional parallel transports in the Möbius gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$, illustrating Theorem 8.64. Parallel transport keeps relative orientations invariant. Shown are two gyrovectors, $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ and $\ominus\mathbf{a}_0\oplus\mathbf{c}_0$, in a Möbius gyrovector space $(\mathbb{R}^3_s, \oplus, \otimes)$ parallel transported, respectively, to the two gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_1\oplus\mathbf{c}_1$ along the gyrosegment $\mathbf{a}_0\mathbf{a}_1$. The parallel transported included gyroangle remains invariant, that is, $\angle\mathbf{b}_0\mathbf{a}_0\mathbf{c}_0 = \angle\mathbf{b}_1\mathbf{a}_1\mathbf{c}_1$.

lel transport of the rooted gyrovector $PQ = \ominus P\oplus Q$ to the rooted gyrovector $P'X = \ominus P'\oplus X$ with tail P' determines its head X,

$$X = P'\oplus\text{gyr}[P', \ominus P](\ominus P\oplus Q) \tag{8.228}$$

Proof. The gyrovector $P'Q'$ is a parallel transport of the gyrovector PQ along the gyroline containing points P and P'. Hence, by Def. 8.61,

$$\ominus P'\oplus X = \text{gyr}[P', \ominus P](\ominus P\oplus Q) \tag{8.229}$$

from which (8.228) follows by a left cancellation. □

Theorem 8.64 (The Parallel Transport Theorem II). *Let $\mathbf{a}_0, \mathbf{b}_0, \mathbf{c}_0, \mathbf{a}_1$ be four given points of a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$. Let the gyrovectors $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ and $\ominus\mathbf{a}_0\oplus\mathbf{c}_0$ be parallel transported to the gyrovectors $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ and $\ominus\mathbf{a}_1\oplus\mathbf{c}_1$, respectively, along the geodesic segment that joins the points $\mathbf{a}_0$ and $\mathbf{a}_1$, Fig. 8.26. The gyroangle between the two gyrovectors remains invariant under their parallel transport, that is,*

$$\angle\mathbf{b}_0\mathbf{a}_0\mathbf{c}_0 = \angle\mathbf{b}_1\mathbf{a}_1\mathbf{c}_1 \tag{8.230}$$

Proof. By the parallel transport equation (8.222), and since gyrations preserve the inner product we have

$$\begin{aligned}\cos\angle\mathbf{b}_1\mathbf{a}_1\mathbf{c}_1 &= \frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\|}\cdot\frac{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{c}_0)}{\|\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{c}_0)\|}\\ &= \frac{\ominus\mathbf{a}_0\oplus\mathbf{b}_0}{\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\|}\cdot\frac{\ominus\mathbf{a}_0\oplus\mathbf{c}_0}{\|\ominus\mathbf{a}_0\oplus\mathbf{c}_0\|}\\ &= \cos\angle\mathbf{b}_0\mathbf{a}_0\mathbf{c}_0\end{aligned} \tag{8.231}$$

□

It follows from Theorem 8.64 that any configuration of several gyrovectors that are parallel transported along a gyrosegment turns as a "gyrorigid" whole. Thus, for instance, it follows from Theorems 8.62 and 8.64 and from SAS congruency that the gyrotriangle $\mathbf{a}_0\mathbf{b}_0\mathbf{c}_0$ and its parallel transported gyrotriangle $\mathbf{a}_1\mathbf{b}_1\mathbf{c}_1$ in Fig. 8.26 are congruent.

Theorem 8.65 *Two successive parallel transports along the same gyroline are equivalent to a single parallel transport along the gyroline.*

Proof. Let $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ be three gyrocollinear points, and let L be the gyroline containing these points in a gyrovector space $(G, \oplus, \otimes)$. Furthermore, let the rooted gyrovector $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ be the parallel transport of a rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ along L, where the latter, in turn, is the parallel transport of a rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ along L, Fig. 8.27. Then, by Def. 8.61,

$$\begin{aligned}\ominus\mathbf{a}_1\oplus\mathbf{b}_1 &= \mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\\ \ominus\mathbf{a}_2\oplus\mathbf{b}_2 &= \mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{b}_1)\end{aligned} \tag{8.232}$$

Eliminating the rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$ from the two equations in (8.232) we obtain the single equation

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{8.233}$$

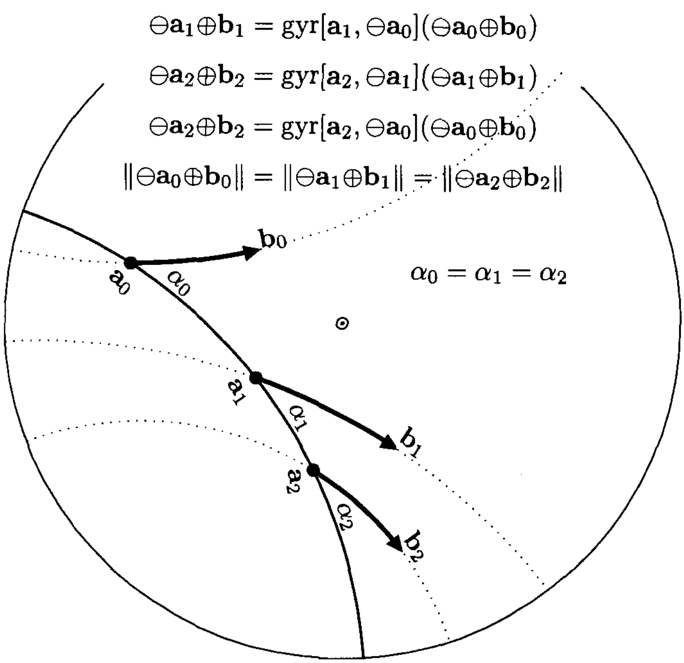


Fig. 8.27 Two successive parallel transports along a gyroline are equivalent to a single parallel transport along the gyroline.

Noting that the three points $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2$ are gyrocollinear, (8.233) can be simplified by means of the gyration transitive law, (6.80), obtaining

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{8.234}$$

so that, by Def. 8.61, the rooted gyrovector $\ominus\mathbf{a}_2\oplus\mathbf{b}_2$ is the parallel transport of the rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ along the gyroline L. □

Parallel translation in Euclidean geometry splits into two kinds of hyperbolic translations. These are (i) the parallel transport and (ii) the gyrovector translation. The former is well-known in hyperbolic geometry and, more generally, in differential geometry, while the latter is unheard of in the nongyro-literature. When translated along a closed gyropolygonal path, the latter regains its original orientation while the former loses it, as explained in Sec. 8.16 and illustrated in Fig. 8.29.

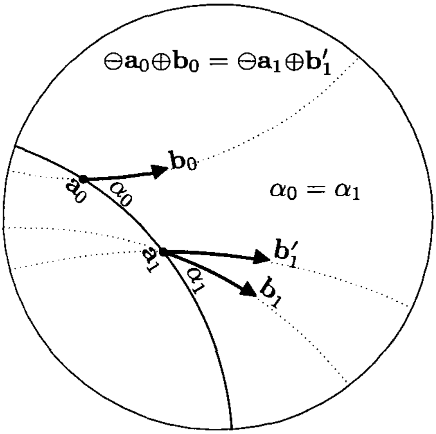


Fig. 8.28 The parallel transport in Fig. 8.24 is contrasted here with a corresponding gyrovector translation in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. The gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, rooted at $\mathbf{a}_0$, is (i) parallel transported to the gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$, rooted at $\mathbf{a}_1$, and (ii) gyrovector translated to the gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}'_1$, also rooted at $\mathbf{a}_1$. The two gyrovectors $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ and $\ominus\mathbf{a}_1\oplus\mathbf{b}'_1$ are equal since they belong to the same equivalence class, Def. 5.4. The rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}'_1$ is called the gyrovector translated companion of the parallel transported rooted gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$. The latter is rotated relative to the former by the gyration $\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0]$, as we see from (8.235).

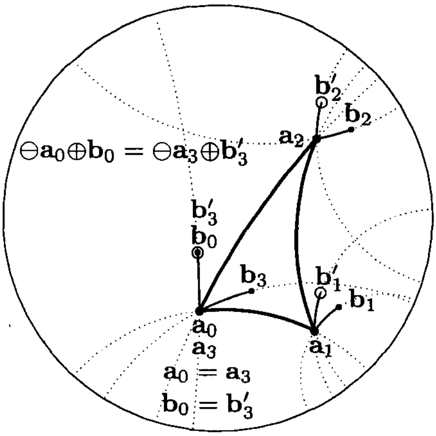


Fig. 8.29 Contrasting successive parallel translations $\ominus\mathbf{a}_k\oplus\mathbf{b}_k$ with successive gyrovector translations $\ominus\mathbf{a}_k\oplus\mathbf{b}'_k$, $k = 1, 2, 3$, of a rooted gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ along a closed gyropolygonal path (a gyrotriangular path in this figure) in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$. Owing to the presence of gyrations, following successive parallel transports along a closed, gyrotriangular path the original gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ loses its original orientation, that is,
$\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\| = \|\ominus\mathbf{a}_3\oplus\mathbf{b}_3\|$ but
$\ominus\mathbf{a}_0\oplus\mathbf{b}_0 \neq \ominus\mathbf{a}_3\oplus\mathbf{b}_3$.
The gyrovector translated companion, in contrast, regains its original orientation, that is, $\ominus\mathbf{a}_0\oplus\mathbf{b}_0 = \ominus\mathbf{a}_3\oplus\mathbf{b}'_3$.

8.16 Parallel Transport vs. Gyrovector Translation

In Fig. 8.28 a gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ is

(1) parallel transported into the gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}_1$; and
(2) gyrovector translated into the gyrovector $\ominus\mathbf{a}_1\oplus\mathbf{b}'_1$.

Hence, by (8.222) of Def. 8.61 and (5.17) of Theorem 5.7, we have

$$\begin{aligned} \ominus\mathbf{a}_1\oplus\mathbf{b}_1 &= \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \\ \ominus\mathbf{a}_1\oplus\mathbf{b}'_1 &= \ominus\mathbf{a}_0\oplus\mathbf{b}_0 \end{aligned} \tag{8.235}$$

The two translations of the gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$ in (8.235) differ by the presence of a gyration in the parallel transport and its absence in the gyrovector translation. The first translation in (8.235), the parallel transport, is well-known in differential geometry in the context of manifolds. Here, in the context of gyrovector spaces, it involves a gyration. The second translation in (8.235), the gyrovector translation, is gyration free. The gyrovector translation was suggested in Def. 5.6 by the introduction of gyrovectors as equivalence classes in Def. 5.4.

A vector space is a special case of a gyrovector space in which all gyrations are trivial. In this special case, the concepts of parallel transport and gyrovector translation, therefore, coincide as we see from (8.235). It is interesting to visualize geometrically, in Fig. 8.29, the significant difference between these two kinds of translations, which emerges in the passage from vector to gyrovector spaces.

As Fig. 8.29 indicates,

(1) following successive gyrovector translations along a closed gyropolygonal path, a rooted gyrovector returns to its original position. In contrast,
(2) following successive parallel transports along a closed gyropolygonal path, a rooted gyrovector does not return to its original position.

In Fig. 8.29 we see three successive gyrovector translations and parallel transports along a closed, gyrotriangular path $\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, where $\mathbf{a}_3 = \mathbf{a}_0$.

The three successive gyrovector translations in Fig. 8.29 are

$$\begin{aligned}\ominus\mathbf{a}_1\oplus\mathbf{b}_1' &= \ominus\mathbf{a}_0\oplus\mathbf{b}_0\\ \ominus\mathbf{a}_2\oplus\mathbf{b}_2' &= \ominus\mathbf{a}_1\oplus\mathbf{b}_1'\\ \ominus\mathbf{a}_3\oplus\mathbf{b}_3' &= \ominus\mathbf{a}_2\oplus\mathbf{b}_2'\end{aligned} \tag{8.236}$$

implying

$$\ominus\mathbf{a}_3\oplus\mathbf{b}_3' = \ominus\mathbf{a}_0\oplus\mathbf{b}_0 \tag{8.237}$$

But the path is closed, $\mathbf{a}_3 = \mathbf{a}_0$, so that

$$\mathbf{b}_3' = \mathbf{b}_0 \tag{8.238}$$

as shown in Fig. 8.29. Hence, following the three successive gyrovector translations along a closed path of the original gyrovector $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, the final gyrovector coincides with the original one,

$$\ominus\mathbf{a}_3\oplus\mathbf{b}_3 = \ominus\mathbf{a}_0\oplus\mathbf{b}_0 \tag{8.239}$$

The three successive parallel transports in Fig. 8.29 are

$$\begin{aligned}\ominus\mathbf{a}_1\oplus\mathbf{b}_1 &= \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\\ \ominus\mathbf{a}_2\oplus\mathbf{b}_2 &= \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{b}_1')\\ \ominus\mathbf{a}_3\oplus\mathbf{b}_3 &= \mathrm{gyr}[\mathbf{a}_3, \ominus\mathbf{a}_2](\ominus\mathbf{a}_2\oplus\mathbf{b}_2')\end{aligned} \tag{8.240}$$

so that

$$\ominus\mathbf{a}_3\oplus\mathbf{b}_3 = \mathrm{gyr}[\mathbf{a}_3, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0) \tag{8.241}$$

Hence, by employing the equality $\mathbf{a}_3 = \mathbf{a}_0$ and the gyrocommutative gyrogroup identity (3.34), we have

$$\begin{aligned}\ominus\mathbf{a}_3\oplus\mathbf{b}_3 &= \mathrm{gyr}[\mathbf{a}_3, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\\ &= \mathrm{gyr}[\mathbf{a}_0, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\\ &= \mathrm{gyr}[\ominus\mathbf{a}_0\oplus\mathbf{a}_2, \ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)\end{aligned} \tag{8.242}$$

It follows from (8.242) that the final gyrovector, $\ominus\mathbf{a}_3\oplus\mathbf{b}_3$, equals the original gyrovector, $\ominus\mathbf{a}_0\oplus\mathbf{b}_0$, gyrated by the gyration

$$\mathrm{gyr}[\ominus\mathbf{a}_0\oplus\mathbf{a}_2, \ominus(\ominus\mathbf{a}_0\oplus\mathbf{a}_1)] \tag{8.243}$$

The gyration (8.243) of the Möbius gyrovector plane turns out to be the gyrotriangular gyration of Def. 8.53, studied in Sec. 8.11. Furthermore, this gyration appears in the Gyroparallelogram Law in (8.194), illustrated in Fig. 8.22. It represents a rotation of the gyroplane about its origin by a gyroangle that equals the gyrotriangular defect of the gyrotriangle $\mathbf{a}_0\mathbf{a}_1\mathbf{a}_2$; see Remarks 8.54 and 8.56.

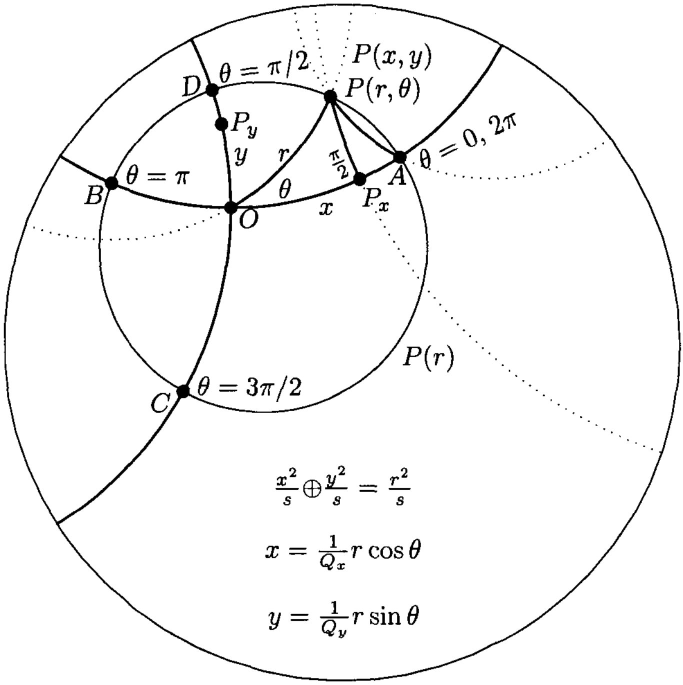


Fig. 8.30 Gyrocircle Gyrotrigonometry. Gyro-Cartesian coordinates (x, y) and gyropolar coordinates (r, θ) of any point $P = P(x, y) = P(r, \theta)$ of the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are shown. Here $-s < x, y < s$, $0 < r < s$, and $0 \le \theta < 2\pi$. The transformation from gyropolar coordinates (r, θ) of a point to its gyro-Cartesian coordinates (x, y) is given by (8.249); and its inverse transformation, the transformation from gyro-Cartesian coordinates (x, y) of a point to its gyropolar coordinates (r, θ) is given by (8.247) - (8.248).

8.17 Gyrocircle Gyrotrigonometry

Gyrotrigonometry, along with parallel transport of gyrosegments, enables us to develop gyrocircle gyrotrigonometry in a way fully analogous to circle trigonometry. This, in turn, gives rise to gyro-Cartesian and gyropolar coordinates for the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$, shown in Fig. 8.30.

Let us arbitrarily select a point O, and construct two mutually orthogonal gyrolines passing through the point. The two gyrolines, called "x-axis" and "y-axis" form a coordinate system with origin O for the Möbius gyrovector plane. In Fig. 8.30 the x-axis is the gyroline containing the points A, O, B and the y-axis is the gyroline containing the points C, O, D.

Let P be any point of the Möbius gyrovector plane other than O. We wish to describe it as the point $P = P(x, y)$ relative to the gyro-Cartesian coordinate system (x, y) or, equivalently, as the point $P = P(r, \theta)$ relative to a corresponding gyropolar coordinate system (r, θ).

Denoting the gyrodistance of P from O by r, we have $r = \|\ominus O \oplus P\|$. Let $P(r)$ be the gyrocircle of radius r centered at the coordinates origin O. The gyrocircle $P(r)$ intersects the x-axis at the points A and B and the y-axis at the points C and D, Fig. 8.30. Let θ be the gyroangle

$$\theta = \angle AOP \tag{8.244}$$

so that the point P is uniquely determined by its gyropolar coordinates, $P = P(r, \theta)$. Furthermore, let P_x be the point on the x-axis such that

$$\angle OP_xP = \frac{\pi}{2} \tag{8.245}$$

and let

$$\begin{aligned} x &= \|\ominus O \oplus P_x\| \\ y &= \|\ominus P_x \oplus P\| \end{aligned} \tag{8.246}$$

be the gyrolengths of sides OP_x and P_xP of the right gyroangled gyrotriangle OP_xP. Then, by the Hyperbolic Pythagorean Theorem 8.32, we have

$$\frac{x^2}{s} \oplus \frac{y^2}{s} = \frac{r^2}{s} \tag{8.247}$$

By (8.89) and Def. 8.35, as shown in Fig. 8.12, we have

$$\begin{aligned} \cos\theta &= \frac{\beta_x^2}{\beta_r^2}\frac{x}{r} = \frac{s^2 + r^2}{s^2 + x^2}\frac{x}{r} \\ \sin\theta &= \frac{\gamma_y^2}{\gamma_r^2}\frac{y}{r} = \frac{s^2 - r^2}{s^2 - y^2}\frac{y}{r} \end{aligned} \tag{8.248}$$

Solving the system (8.248) of two equations for the unknowns x and y we have

$$\begin{aligned} x &= \frac{1}{Q_x} r \cos\theta \\ y &= \frac{1}{Q_y} r \sin\theta \end{aligned} \tag{8.249}$$

where

$$Q_x = \tfrac{1}{2}\{1 + \tfrac{r^2}{s^2} + \sqrt{1 + \tfrac{r^4}{s^4} - 2\tfrac{r^2}{s^2}\cos 2\theta}\}$$
$$Q_y = \tfrac{1}{2}\{1 - \tfrac{r^2}{s^2} + \sqrt{1 + \tfrac{r^4}{s^4} - 2\tfrac{r^2}{s^2}\cos 2\theta}\} \tag{8.250}$$

Let P_y be the point on the y-axis between points O and D, Fig. 8.30, such that

$$y = \|\ominus O \oplus P_y\| \tag{8.251}$$

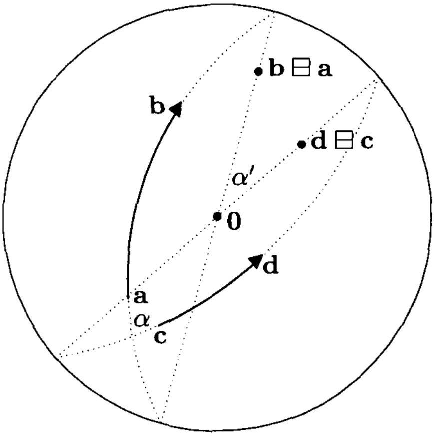
in analogy with the first equation in (8.246). Then, the gyrosegments OP_y and P_xP in Fig. 8.30 are related to one another by a parallel transport. Parallel transporting the gyrovector OP_y to a gyrovector with tail at P_x gives the gyrovector P_xP. We may recall here that the parallel transport of a gyrovector along a gyroline keeps the gyrolength of the transported gyrovector and its gyroangle with the gyroline invariant, Fig. 8.27.

In Euclidean geometry parallel transport and gyrovector translations coincide, as we found in Sec. 8.16. Hence, in the Euclidean counterpart of Fig. 8.30 the vectors OP_y and P_xP are equal. In gyrogeometry, in contrast, parallel transports and gyrovector translations are different notions. Hence, the gyrovectors OP_y and P_xP are different in general.

The pair (x, y) represents the gyro-Cartesian coordinates of the point P, and the pair (r, θ) represents the gyropolar coordinates of the point P relative to the xy-coordinate system in Fig. 8.30.

(1) The conversion from gyro-Cartesian coordinates (x, y) of a point to its gyropolar coordinates (r, θ) is given by (8.247) – (8.248); and
(2) The conversion from gyropolar coordinates (r, θ) of a point to its gyro-Cartesian coordinates (x, y) is given by (8.249).

In the limit of large s, $s \to \infty$, the conversions between gyro-Cartesian and gyropolar coordinates reduce to the familiar conversions between Cartesian and polar coordinates. Thus, in particular, (8.247) reduces to the familiar relation $x^2 + y^2 = r^2$ between polar and Cartesian coordinates of the Euclidean plane, as explained in Remark 8.33.



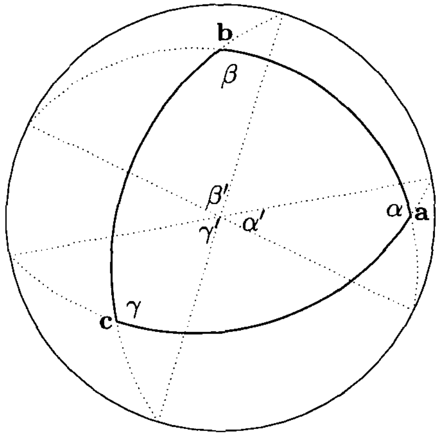


Fig. 8.31 The cogyroangle α generated by the two rooted cogyrovectors $\mathbf{b}\boxminus\mathbf{a}$ and $\mathbf{d}\boxminus\mathbf{c}$ in the Möbius gyrovector plane is shown. Its cosine is given by (8.254), and is numerically equal to the Euclidean angle α' generated by the corresponding supporting diameters.

Fig. 8.32 The cogyroangle between cogeodesics and its associated Euclidean angle between corresponding supporting diameters are equal. Hence, by inspection, the sum of the cogyroangles of a cogyrotriangle is π. $\alpha = \alpha'$, $\beta = \beta'$, $\gamma = \gamma'$, and $\alpha' + \beta' + \gamma' = \pi$.

8.18 Cogyroangles

Definition 8.66 **(Unit Cogyrovectors).** *Let $\ominus\mathbf{a} \boxplus \mathbf{b}$ be a nonzero rooted cogyrovector, Def. 5.9, in a gyrovector space $(G, \oplus, \otimes)$. Its cogyrolength is $\|\ominus\mathbf{a} \boxplus \mathbf{b}\|$ and its associated rooted cogyrovector*

$$\frac{\ominus\mathbf{a} \boxplus \mathbf{b}}{\|\ominus\mathbf{a} \boxplus \mathbf{b}\|} \tag{8.252}$$

is called a unit cogyrovector.

We may note that by (2.8) and Theorem 3.4 we have

$$\ominus\mathbf{a} \boxplus \mathbf{b} = \boxminus\mathbf{a} \boxplus \mathbf{b} = \mathbf{b} \boxminus \mathbf{a} \tag{8.253}$$

Unit cogyrovectors represent "cogyrodirections". A cogyroangle is, accordingly, a relation between two cogyrodirections.

Definition 8.67 **(Cogyroangles).** *Let $\ominus\mathbf{a} \boxplus \mathbf{b}$ and $\ominus\mathbf{c} \boxplus \mathbf{d}$ be two nonzero rooted cogyrovectors in a gyrovector space $(G, \oplus, \otimes)$. The gyrocosine of the measure of the cogyroangle α that the two rooted cogyrovectors generate is given by the equation, Fig. 8.31,*

$$\cos\alpha = \frac{\ominus\mathbf{a} \boxplus \mathbf{b}}{\|\ominus\mathbf{a} \boxplus \mathbf{b}\|} \cdot \frac{\ominus\mathbf{c} \boxplus \mathbf{d}}{\|\ominus\mathbf{c} \boxplus \mathbf{d}\|} \tag{8.254}$$

If $\mathbf{a} = \mathbf{c}$ then cogyroangle α in (8.254) is denoted by $\alpha = \angle\mathbf{bad}$ or, equivalently, $\alpha = \angle\mathbf{dab}$ (it should always be clear from the context whether α is a gyroangle or a cogyroangle). Two cogyroangles are congruent if they have the same measure.

Cogyroangles are invariant under automorphisms of their gyrovector spaces. However, unlike gyroangles, cogyroangles are not invariant under left gyrotranslations.

Theorem 8.68 *The cogyroangle between two cogyrolines equals the cogyroangle (which is also a gyroangle) between the two corresponding origin-intercept cogyrolines (which are also gyrolines).*

Origin-intercept gyrolines, Def. 6.17, are origin-intercept cogyroline by Theorem 6.18. The origin-intercept gyroline that corresponds to a given cogyroline is defined in Def. 6.67. The origin-intercept gyroline-cogyroline in the Möbius gyrovector space (that is, in the Poincaré ball model of hyperbolic geometry) turns out to be the supporting gyrodiameter, as remarked in Remark 6.69 and illustrated in Fig. 8.31.

Proof. Let us consider two cogyrolines that contain, respectively, the cogyrosegments $\mathbf{ab}$ and $\mathbf{cd}$, Fig. 8.31. The gyrocosine of the cogyroangle α between these cogyrolines is given by the equation, Def. 8.67,

$$\cos\alpha = \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{d} \boxminus \mathbf{c}}{\|\mathbf{d} \boxminus \mathbf{c}\|} \tag{8.255}$$

By Theorem 6.68, the cogyrodifference $\mathbf{b} \boxminus \mathbf{a}$ lies on the origin-intercept cogyroline that corresponds to the cogyroline containing the cogyrosegment $\mathbf{ab}$ and, similarly, the cogyrodifference $\mathbf{d} \boxminus \mathbf{c}$ lies on the origin-intercept cogyroline that corresponds to the cogyroline containing the cogyrosegment $\mathbf{cd}$ (as shown in Fig. 8.31, where corresponding origin-intercept cogyrolines are corresponding supporting gyrodiameters). The gyrocosine of the cogyroangle α' (which is also a gyroangle) between these corresponding origin-intercept cogyroline (which are also gyrolines) is given by the equation

$$\begin{aligned}
\cos\alpha' &= \frac{(\mathbf{b}\boxminus\mathbf{a})\boxminus\mathbf{0}}{\|(\mathbf{b}\boxminus\mathbf{a})\boxminus\mathbf{0}\|}\cdot\frac{(\mathbf{d}\boxminus\mathbf{c})\boxminus\mathbf{0}}{\|(\mathbf{d}\boxminus\mathbf{c})\boxminus\mathbf{0}\|} \\
&= \frac{\mathbf{b}\boxminus\mathbf{a}}{\|\mathbf{b}\boxminus\mathbf{a}\|}\cdot\frac{\mathbf{d}\boxminus\mathbf{c}}{\|\mathbf{d}\boxminus\mathbf{c}\|} \\
&= \cos\alpha
\end{aligned} \tag{8.256}$$

□

We may remark that α',

$$\alpha' = \angle(\mathbf{b}\boxminus\mathbf{a})\mathbf{0}(\mathbf{d}\boxminus\mathbf{c}) \tag{8.257}$$

is treated in (8.256) as a cogyroangle. However, it can equivalently be treated as a gyroangle simply by replacing "$\boxminus\mathbf{0}$" by "$\ominus\mathbf{0}$" in (8.256). Hence, α' is both a gyroangle and a cogyroangle. In the models that we study in this book, a gyroline that coincides with a cogyroline turns out to be a Euclidean line and, similarly, a gyroangle that coincides with a cogyroangle turns out to be a Euclidean angle.

Theorem 8.69 *Cogyroangles are invariant under gyrovector space automorphisms.*

Proof. We have to show that the cogyroangle $\alpha = \angle\mathbf{bac}$ for any points $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ of a gyrovector space $(G, \oplus, \otimes)$ is invariant under the gyrovector space automorphisms, Def. 6.5. Equivalently, we have to show that

$$\angle\mathbf{bac} = \angle(\tau\mathbf{b})(\tau\mathbf{a})(\tau\mathbf{c}) \tag{8.258}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{c} \in G$ and all $\tau \in Aut(G)$. Indeed, by (2.52) and the third identity in (6.7) we have

$$
\begin{aligned}
\cos\angle(\tau\mathbf{b})(\tau\mathbf{a})(\tau\mathbf{c}) &= \frac{\ominus\tau\mathbf{a}\boxplus\tau\mathbf{b}}{\|\ominus\tau\mathbf{a}\boxplus\tau\mathbf{b}\|}\cdot\frac{\ominus\tau\mathbf{a}\boxplus\tau\mathbf{c}}{\|\ominus\tau\mathbf{a}\boxplus\tau\mathbf{c}\|} \\
&= \frac{\tau(\ominus\mathbf{a}\boxplus\mathbf{b})}{\|\tau(\ominus\mathbf{a}\boxplus\mathbf{b})\|}\cdot\frac{\tau(\ominus\mathbf{a}\boxplus\mathbf{c})}{\|\tau(\ominus\mathbf{a}\boxplus\mathbf{c})\|} \\
&= \frac{\tau(\ominus\mathbf{a}\boxplus\mathbf{b})}{\|\ominus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\tau(\ominus\mathbf{a}\boxplus\mathbf{c})}{\|\ominus\mathbf{a}\boxplus\mathbf{c}\|} \\
&= \frac{\ominus\mathbf{a}\boxplus\mathbf{b}}{\|\ominus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\boxplus\mathbf{c}}{\|\ominus\mathbf{a}\boxplus\mathbf{c}\|} \\
&= \cos\angle\mathbf{bac}
\end{aligned}
\tag{8.259}
$$

□

Theorem 8.70 *The measure of a cogyroangle is model independent.*

Proof. Following Def. 6.85 let $(G_1, \oplus_1, \otimes_1)$ and $(G_2, \oplus_2, \otimes_2)$ be two isomorphic gyrovector spaces with isomorphism $\phi: G_1 \to G_2$. Furthermore, let α_1 be a cogyroangle in G_1 given by

$$
\cos\alpha_1 = \frac{\ominus_1\mathbf{a}_1\boxplus_1\mathbf{b}_1}{\|\ominus_1\mathbf{a}_1\boxplus_1\mathbf{b}_1\|}\cdot\frac{\ominus_1\mathbf{c}_1\boxplus_1\mathbf{d}_1}{\|\ominus_1\mathbf{c}_1\boxplus_1\mathbf{d}_1\|}
\tag{8.260}
$$

and let α_2 be the corresponding gyroangle in G_2,

$$
\cos\alpha_2 = \frac{\ominus_2\mathbf{a}_2\boxplus_2\mathbf{b}_2}{\|\ominus_2\mathbf{a}_2\boxplus_2\mathbf{b}_2\|}\cdot\frac{\ominus_2\mathbf{c}_2\boxplus_2\mathbf{d}_2}{\|\ominus_2\mathbf{c}_2\boxplus_2\mathbf{d}_2\|}
\tag{8.261}
$$

where $\mathbf{a}_2 = \phi(\mathbf{a}_1)$, $\mathbf{b}_2 = \phi(\mathbf{b}_1)$, $\mathbf{c}_2 = \phi(\mathbf{c}_1)$, and $\mathbf{d}_2 = \phi(\mathbf{d}_1)$.

Then, by (6.295) and (6.292) we have

$$\begin{aligned}
\cos\alpha_2 &= \frac{\ominus_2\mathbf{a}_2 \boxplus_2 \mathbf{b}_2}{\|\ominus_2\mathbf{a}_2 \boxplus_2 \mathbf{b}_2\|}\cdot\frac{\ominus_2\mathbf{c}_2 \boxplus_2 \mathbf{d}_2}{\|\ominus_2\mathbf{c}_2 \boxplus_2 \mathbf{d}_2\|} \\
&= \frac{\ominus_2\phi(\mathbf{a}_1) \boxplus_2 \phi(\mathbf{b}_1)}{\|\ominus_2\phi(\mathbf{a}_1) \boxplus_2 \phi(\mathbf{b}_1)\|}\cdot\frac{\ominus_2\phi(\mathbf{c}_1) \boxplus_2 \phi(\mathbf{d}_1)}{\|\ominus_2\phi(\mathbf{c}_1) \boxplus_2 \phi(\mathbf{d}_1)\|} \\
&= \frac{\phi(\ominus_1\mathbf{a}_1 \boxplus_1 \mathbf{b}_1)}{\|\phi(\ominus_1\mathbf{a}_1 \boxplus_1 \mathbf{b}_1)\|}\cdot\frac{\phi(\ominus_1\mathbf{c}_1 \boxplus_1 \mathbf{d}_1)}{\|\phi(\ominus_1\mathbf{c}_1 \boxplus_1 \mathbf{d}_1)\|} \\
&= \frac{\ominus_1\mathbf{a}_1 \boxplus_1 \mathbf{b}_1}{\|\ominus_1\mathbf{a}_1 \boxplus_1 \mathbf{b}_1\|}\cdot\frac{\ominus_1\mathbf{c}_1 \boxplus_1 \mathbf{d}_1}{\|\ominus_1\mathbf{c}_1 \boxplus_1 \mathbf{d}_1\|} \\
&= \cos\alpha_1
\end{aligned} \tag{8.262}$$

so that α_1 and α_2 have the same measure, $\alpha_1 = \alpha_2$. □

To calculate the gyrocosine of a cogyroangle α generated by two directed cogyrolines we place on the two cogyrolines two nonzero rooted cogyrovectors $\ominus\mathbf{a} \boxplus \mathbf{b}$ and $\ominus\mathbf{c} \boxplus \mathbf{d}$ or, equivalently, $\ominus\mathbf{a}' \boxplus \mathbf{b}'$ and $\ominus\mathbf{c}' \boxplus \mathbf{d}'$, Fig. 8.31. The measure of the cogyroangle α is given by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a} \boxplus \mathbf{b}}{\|\ominus\mathbf{a} \boxplus \mathbf{b}\|}\cdot\frac{\ominus\mathbf{c} \boxplus \mathbf{d}}{\|\ominus\mathbf{c} \boxplus \mathbf{d}\|} \tag{8.263}$$

where it is represented by the points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ or, equivalently, by the equation

$$\cos\alpha = \frac{\ominus\mathbf{a}' \boxplus \mathbf{b}'}{\|\ominus\mathbf{a}' \boxplus \mathbf{b}'\|}\cdot\frac{\ominus\mathbf{c}' \boxplus \mathbf{d}'}{\|\ominus\mathbf{c}' \boxplus \mathbf{d}'\|} \tag{8.264}$$

where it is represented by the points $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ and $\mathbf{d}'$.

In the following theorem we show that, as anticipated in the cogyroangle calculation, (8.263) and (8.264) give the same cogyroangle measure for the cogyroangle α.

Theorem 8.71 *The cogyroangle generated by two directed cogyrolines is representation independent.*

Proof. Let

$$\begin{aligned}
L_{\mathbf{ab}} &= (\mathbf{b} \boxminus \mathbf{a})\otimes t \oplus \mathbf{a} \\
L_{\mathbf{cd}} &= (\mathbf{d} \boxminus \mathbf{c})\otimes t \oplus \mathbf{c}
\end{aligned} \tag{8.265}$$

be two cogyrolines in a gyrovector space $(G, \oplus, \otimes)$, Fig. 8.31. Let $\mathbf{a}'$ and $\mathbf{b}'$ be two distinct points on $L_{\mathbf{ab}}$ and, similarly, let $\mathbf{c}'$ and $\mathbf{d}'$ be two distinct points on $L_{\mathbf{cd}}$. Then

$$\begin{aligned}
\mathbf{a}' &= (\mathbf{b} \boxminus \mathbf{a})\otimes t_1 \oplus \mathbf{a} \\
\mathbf{b}' &= (\mathbf{b} \boxminus \mathbf{a})\otimes t_2 \oplus \mathbf{a} \\
\mathbf{c}' &= (\mathbf{d} \boxminus \mathbf{c})\otimes t_3 \oplus \mathbf{c} \\
\mathbf{d}' &= (\mathbf{d} \boxminus \mathbf{c})\otimes t_4 \oplus \mathbf{c}
\end{aligned} \tag{8.266}$$

for some $t_k \in \mathbb{R}$, $k = 1, \ldots, 4$. Furthermore, we assume that the cogyrolines are directed such that $t_2 > t_1$ and $t_4 > t_3$.

The cogyroangle α generated by the two directed cogyrolines (directed in the direction of increasing their parameter t) $L_{\mathbf{ab}}$ and $L_{\mathbf{cd}}$ is given by

$$\cos\alpha = \frac{\ominus\mathbf{a}' \boxplus \mathbf{b}'}{\|\ominus\mathbf{a}' \boxplus \mathbf{b}'\|} \cdot \frac{\ominus\mathbf{c}' \boxplus \mathbf{d}'}{\|\ominus\mathbf{c}' \boxplus \mathbf{d}'\|} \tag{8.267}$$

where it is represented by the points $\mathbf{a}'$ and $\mathbf{b}'$ on $L_{\mathbf{ab}}$ and the points $\mathbf{c}'$ and $\mathbf{d}'$ on $L_{\mathbf{cd}}$.

Let us express $\mathbf{a}'$ and $\mathbf{b}'$ ($\mathbf{c}'$ and $\mathbf{d}'$) in terms of $\mathbf{a}$ and $\mathbf{b}$ ($\mathbf{c}$ and $\mathbf{d}$). As in the derivation of (6.163), by Identity (2.44) of the Cogyrotranslation Theorem 2.16, and the scalar distributive law we have

$$\begin{aligned}
\mathbf{b}' \boxminus \mathbf{a}' &= \{(\mathbf{b} \boxminus \mathbf{a})\otimes t_2 \oplus \mathbf{a}\} \boxminus \{(\mathbf{b} \boxminus \mathbf{a})\otimes t_1 \oplus \mathbf{a}\} \\
&= (\mathbf{b} \boxminus \mathbf{a})\otimes t_2 \boxminus (\mathbf{b} \boxminus \mathbf{a})\otimes t_1 \\
&= (\mathbf{b} \boxminus \mathbf{a})\otimes t_2 \ominus (\mathbf{b} \boxminus \mathbf{a})\otimes t_1 \\
&= (\mathbf{b} \boxminus \mathbf{a})\otimes (t_2 - t_1)
\end{aligned} \tag{8.268}$$

and, similarly,

$$\mathbf{d}' \boxminus \mathbf{c}' = (\mathbf{d} \boxminus \mathbf{c})\otimes (t_4 - t_3) \tag{8.269}$$

Hence, by the scaling property $(V4)$ of gyrovector spaces, noting that $t_2 - t_1 > 0$ and $t_4 - t_3 > 0$, we manipulate the representation of $\cos\alpha$ by the points $\mathbf{a}'$ and $\mathbf{b}'$ on $L_{\mathbf{ab}}$ and $\mathbf{c}'$ and $\mathbf{d}'$ on $L_{\mathbf{cd}}$,

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}'\boxminus\mathbf{a}'}{\|\mathbf{b}'\boxminus\mathbf{a}'\|}\cdot\frac{\mathbf{d}'\boxminus\mathbf{c}'}{\|\mathbf{d}'\boxminus\mathbf{c}'\|} \\ &= \frac{(\mathbf{b}\boxminus\mathbf{a})\otimes(t_2-t_1)}{\|(\mathbf{b}\boxminus\mathbf{a})\otimes(t_2-t_1)\|}\cdot\frac{(\mathbf{d}\boxminus\mathbf{c})\otimes(t_4-t_3)}{\|(\mathbf{d}\boxminus\mathbf{c})\otimes(t_4-t_3)\|} \\ &= \frac{\mathbf{b}\boxminus\mathbf{a}}{\|\mathbf{b}\boxminus\mathbf{a}\|}\cdot\frac{\mathbf{d}\boxminus\mathbf{c}}{\|\mathbf{d}\boxminus\mathbf{c}\|}\end{aligned} \tag{8.270}$$

obtaining a representation of $\cos\alpha$ by the points $\mathbf{a}$ and $\mathbf{b}$ on $L_{\mathbf{ab}}$ and the points $\mathbf{c}$ and $\mathbf{d}$ on $L_{\mathbf{cd}}$. Hence, the value of $\cos\alpha$ is representation independent. □

8.19 The Cogyroangle in the Three Models

It follows from (3.130) that the cogyroangle α generated by the cogyrolines $L_{\mathbf{ab}}$ and $L_{\mathbf{cd}}$ in a Möbius gyrovector space, Fig. 8.31, is given by the equation

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\boxminus_M\mathbf{a}}{\|\mathbf{b}\boxminus_M\mathbf{a}\|}\cdot\frac{\mathbf{d}\boxminus_M\mathbf{c}}{\|\mathbf{d}\boxminus_M\mathbf{c}\|} \\ &= \frac{\frac{\gamma_{\mathbf{b}}^2\mathbf{b}-\gamma_{\mathbf{a}}^2\mathbf{a}}{\gamma_{\mathbf{a}}^2+\gamma_{\mathbf{b}}^2-1}}{\left\|\frac{\gamma_{\mathbf{b}}^2\mathbf{b}-\gamma_{\mathbf{a}}^2\mathbf{a}}{\gamma_{\mathbf{a}}^2+\gamma_{\mathbf{b}}^2-1}\right\|}\cdot\frac{\frac{\gamma_{\mathbf{d}}^2\mathbf{d}-\gamma_{\mathbf{c}}^2\mathbf{c}}{\gamma_{\mathbf{c}}^2+\gamma_{\mathbf{d}}^2-1}}{\left\|\frac{\gamma_{\mathbf{d}}^2\mathbf{d}-\gamma_{\mathbf{c}}^2\mathbf{c}}{\gamma_{\mathbf{c}}^2+\gamma_{\mathbf{d}}^2-1}\right\|} \\ &= \frac{\gamma_{\mathbf{b}}^2\mathbf{b}-\gamma_{\mathbf{a}}^2\mathbf{a}}{\|\gamma_{\mathbf{b}}^2\mathbf{b}-\gamma_{\mathbf{a}}^2\mathbf{a}\|}\cdot\frac{\gamma_{\mathbf{d}}^2\mathbf{d}-\gamma_{\mathbf{c}}^2\mathbf{c}}{\|\gamma_{\mathbf{d}}^2\mathbf{b}-\gamma_{\mathbf{c}}^2\mathbf{a}\|}\end{aligned} \tag{8.271}$$

Similarly, it follows from (3.156) that the cogyroangle α generated by the cogyrolines $L_{\mathbf{ab}}$ and $L_{\mathbf{cd}}$ in an Einstein gyrovector space is given by the equation

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\boxminus_E\mathbf{a}}{\|\mathbf{b}\boxminus_E\mathbf{a}\|}\cdot\frac{\mathbf{d}\boxminus_E\mathbf{c}}{\|\mathbf{d}\boxminus_E\mathbf{c}\|} \\ &= \frac{\gamma_{\mathbf{b}}\mathbf{b}-\gamma_{\mathbf{a}}\mathbf{a}}{\|\gamma_{\mathbf{b}}\mathbf{b}-\gamma_{\mathbf{a}}\mathbf{a}\|}\cdot\frac{\gamma_{\mathbf{d}}\mathbf{d}-\gamma_{\mathbf{c}}\mathbf{c}}{\|\gamma_{\mathbf{d}}\mathbf{b}-\gamma_{\mathbf{c}}\mathbf{a}\|}\end{aligned} \tag{8.272}$$

Finally, it follows from (3.162) that the cogyroangle α generated by the cogyrolines $L_{\mathbf{ab}}$ and $L_{\mathbf{cd}}$ in a PV gyrovector space is given by the equation

$$\begin{aligned}\cos\alpha &= \frac{\mathbf{b}\boxminus_{\mathrm{U}}\mathbf{a}}{\|\mathbf{b}\boxminus_{\mathrm{U}}\mathbf{a}\|}\cdot\frac{\mathbf{d}\boxminus_{\mathrm{U}}\mathbf{c}}{\|\mathbf{d}\boxminus_{\mathrm{U}}\mathbf{c}\|}\\ &= \frac{\mathbf{b}-\mathbf{a}}{\|\mathbf{b}-\mathbf{a}\|}\cdot\frac{\mathbf{d}-\mathbf{c}}{\|\mathbf{b}-\mathbf{a}\|}\end{aligned} \tag{8.273}$$

Interestingly, cogyrolines in PV gyrovector spaces are straight lines, as we see from Figs. 6.13, 6.14 and 6.15, and cogyroangles that cogyrolines generate have the same measure as their Euclidean counterparts, as we see from (8.273). As a result, the cogyrotriangle cogyroangle sum is π in all gyrovector space models that are isomorphic to a PV gyrovector space. We will see in Theorem 8.73 that the resulting "π-Theorem" is, in fact, valid in any gyrovector space.

8.20 Parallelism in Gyrovector Spaces

Theorem 8.72 (Hyperbolic Alternate Interior Cogyroangles Theorem). *Let L_1 and L_2 be two parallel cogyrolines in gyrovector space $(G,\oplus,\otimes)$ that are intersected by a cogyroline at the points P_1 and P_4 of L_1 and L_2 respectively. Furthermore, let P_2 and P_3 be points on L_1 and L_2 respectively, located on opposite sides of the intersecting cogyroline, Fig. 8.33. Then, the two alternate interior cogyroangles are equal,*

$$\angle P_1P_4P_3 = \angle P_4P_1P_2 \tag{8.274}$$

Proof. Let

$$\begin{aligned} &L_1 : \mathbf{a}\otimes t\oplus\mathbf{v}_1, \qquad && P_1 = \mathbf{a}\otimes t_1\oplus\mathbf{v}_1, \qquad && P_2 = \mathbf{a}\otimes t_2\oplus\mathbf{v}_1\\ &L_2 : \mathbf{a}\otimes t\oplus\mathbf{v}_2, && P_3 = \mathbf{a}\otimes t_3\oplus\mathbf{v}_2, && P_4 = \mathbf{a}\otimes t_4\oplus\mathbf{v}_2\end{aligned} \tag{8.275}$$

$t \in \mathbb{R}$, be two parallel cogyrolines where P_1 and P_2 are two points on L_1, and P_3 and P_4 are two points on L_2 in a gyrovector space $(G,\oplus,\otimes)$, as shown in Fig. 8.33 for the Möbius gyrovector plane. The points P_1 and P_2 on L_1 and the points P_3 and P_4 on L_2 correspond to selected values t_1, t_2, t_3 and t_4 of the parameter t in (8.275). These values are arbitrarily selected such that $t_1 - t_2$ and $t_4 - t_3$ have opposite signs so that the points P_2 and P_3 are located on opposite sides of the cogyroline passing through P_1 and P_4, as shown in Fig. 8.33.

The alternate cogyroangles α and α', Fig. 8.33, are given by the equations

$$\begin{aligned}\cos\alpha &= \frac{\boxminus P_4 \boxplus P_1}{\|\boxminus P_4 \boxplus P_1\|}\cdot\frac{\boxminus P_4 \boxplus P_3}{\|\boxminus P_4 \boxplus P_3\|}\\ \cos\alpha' &= \frac{\boxminus P_1 \boxplus P_4}{\|\boxminus P_1 \boxplus P_4\|}\cdot\frac{\boxminus P_1 \boxplus P_2}{\|\boxminus P_1 \boxplus P_2\|}\\ &= \frac{\boxminus P_4 \boxplus P_1}{\|\boxminus P_4 \boxplus P_1\|}\cdot\frac{\boxminus P_2 \boxplus P_1}{\|\boxminus P_2 \boxplus P_1\|}\end{aligned} \tag{8.276}$$

By (8.275), and by Identity (2.44) of the Cogyrotranslation Theorem 2.16, we have

$$\begin{aligned}\boxminus P_4 \boxplus P_3 &= \boxminus(\mathbf{a}\otimes t_4 \oplus \mathbf{v}_2) \boxplus (\mathbf{a}\otimes t_3 \oplus \mathbf{v}_2)\\ &= \boxminus \mathbf{a}\otimes t_4 \boxplus \mathbf{a}\otimes t_3\\ &= \mathbf{a}\otimes(-t_4 + t_3)\end{aligned} \tag{8.277}$$

and

$$\begin{aligned}\boxminus P_2 \boxplus P_1 &= \boxminus(\mathbf{a}\otimes t_2 \oplus \mathbf{v}_1) \boxplus (\mathbf{a}\otimes t_1 \oplus \mathbf{v}_1)\\ &= \boxminus \mathbf{a}\otimes t_2 \boxplus \mathbf{a}\otimes t_1\\ &= \mathbf{a}\otimes(-t_2 + t_1)\end{aligned} \tag{8.278}$$

By assumption, $t_1 - t_2$ and $t_3 - t_4$ have equal signs so that, by the scaling property $(V4)$ of gyrovector spaces, we have

$$\begin{aligned}\frac{\boxminus P_4 \boxplus P_3}{\|\boxminus P_4 \boxplus P_3\|} &= \frac{\mathbf{a}\otimes(t_3 - t_4)}{\|\mathbf{a}\otimes(t_3 - t_4\|} = \pm\frac{\mathbf{a}}{\|\mathbf{a}\|}\\ \frac{\boxminus P_2 \boxplus P_1}{\|\boxminus P_2 \boxplus P_1\|} &= \frac{\mathbf{a}\otimes(t_1 - t_2)}{\|\mathbf{a}\otimes(t_1 - t_2\|} = \pm\frac{\mathbf{a}}{\|\mathbf{a}\|}\end{aligned} \tag{8.279}$$

where the ambiguous signs go together, so that

$$\frac{\boxminus P_4 \boxplus P_3}{\|\boxminus P_4 \boxplus P_3\|} = \frac{\boxminus P_2 \boxplus P_1}{\|\boxminus P_2 \boxplus P_1\|} \tag{8.280}$$

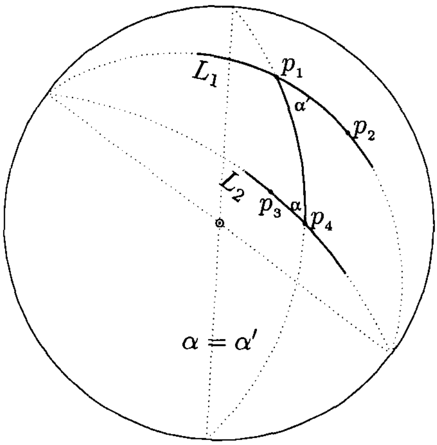


Fig. 8.33 Alternate interior cogyroangles, α and α', generated by a cogyroline intersecting two parallel cogyrolines, are equal.

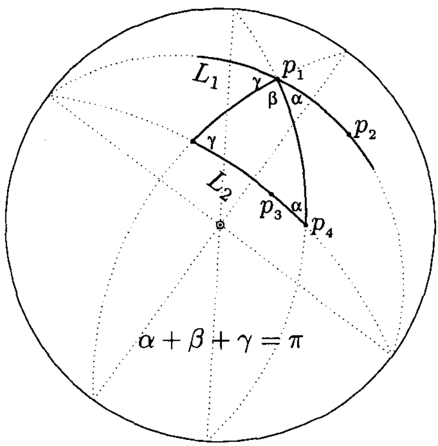


Fig. 8.34 The cogyrotriangle cogyroangle sum, π, is shown in the Möbius gyrovector plane.

Hence, finally, by (8.276) and (8.280) we have

$$\begin{aligned}\cos\alpha &= \frac{\boxminus P_4 \boxplus P_1}{\|\boxminus P_4 \boxplus P_1\|} \cdot \frac{\boxminus P_4 \boxplus P_3}{\|\boxminus P_4 \boxplus P_3\|} \\ &= \frac{\boxminus P_4 \boxplus P_1}{\|\boxminus P_4 \boxplus P_1\|} \cdot \frac{\boxminus P_2 \boxplus P_1}{\|\boxminus P_2 \boxplus P_1\|} \\ &= \cos\alpha'\end{aligned} \tag{8.281}$$

□

Theorem 8.73 **(The π-Theorem).** *The sum of the three cogyroangles of any cogyrotriangle in a gyrovector space is π, Fig. 8.34.*

Proof. The proof, fully analogous to its Euclidean counterpart, follows from Theorem 8.72 and Fig. 8.34. □

8.21 Reflection, Gyroreflection, and Cogyroreflection

In this section we show graphically (leaving the proof to interested readers) that the notions of (i) Euclidean reflection relative to a circle, (ii) gyroreflection, and (iii) cogyroreflection relative to a gyroline in a Möbius gyrovector plane are coincident.

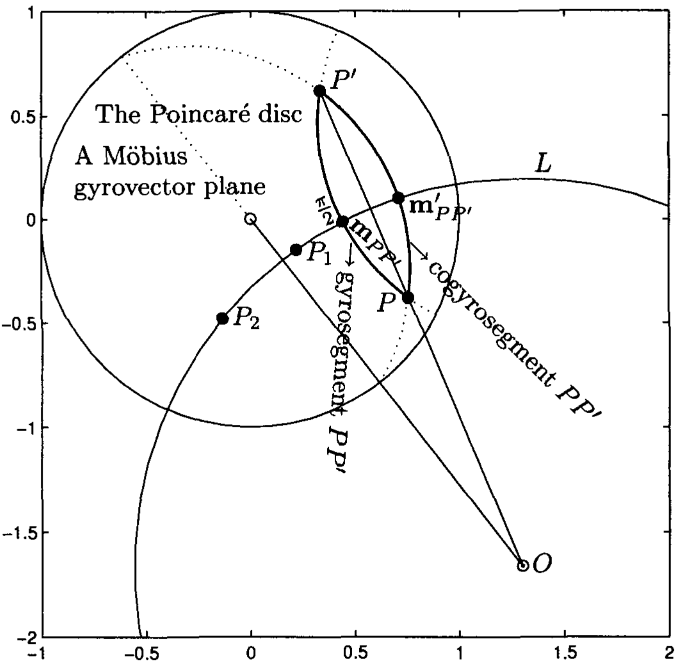


Fig. 8.35 Reflection, Gyroreflection, Cogyroreflection. The point P' is the Euclidean reflection of the point P relative to the circle L in the Poincaré disc $\mathbb{R}^2_{s=1}$ or, equivalently, in the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$. The circle L, in turn, is the circle that carries the unique gyroline that passes through any two given points, P_1 and P_2, of the disc. Coincidentally, the point P' is also the gyroreflection and cogyroreflection of the point P relative to the gyroline L. Interestingly, (i) the gyroline L intersects the gyrosegment PP' at the gyrosegment gyromidpoint $\mathbf{m}_{PP'}$; and also (ii) the gyroline L intersects the cogyrosegment PP' at the cogyrosegment cogyromidpoint $\mathbf{m}'_{PP'}$. Moreover, (i) the gyroline L is perpendicular to the gyrosegment PP'; and also (ii) the gyroline L is "perpendicular" to the cogyrosegment PP' in the sense that the gyroline L is perpendicular to the cogyrosegment supporting diameter. Accordingly, this figure illustrates remarkable duality symmetries that the gyromidpoint and the cogyromidpoint share.

Let P_1 and P_2 be two given points of the Poincaré unit disc, that is, the Möbius gyrovector plane $(\mathbb{R}^2_{s=1}, \oplus, \otimes)$, let L be the gyroline passing through these points, and let P be a point of the unit disc $\mathbb{R}^2_{s=1}$ not on L, Fig. 8.35. Furthermore, let O and r be the Euclidean center and radius of the circle L that carries the gyroline L in the Euclidean plane $\mathbb{R}^2$. The two points P and P' are said to be symmetric relative to the gyroline L if O,

P and P' are Euclidean-collinear and if, in the Euclidean geometry sense,

$$\|P - O\|\|P' - O\| = r^2 \tag{8.282}$$

The points P and P' lie on opposite sides of L in the Poincaré disc, Fig. 8.35, and are said to be *reflections* of each other relative to the circle L.

Coincidentally, (i) the gyroline L intersects the gyrosegment PP' at its gyromidpoint $\mathbf{m}_{PP'}$; and also (ii) the gyroline L is perpendicular to the gyrosegment PP'. Hence, the gyroline L is the perpendicular bisector of the gyrosegment PP', Fig. 8.35, and the points P and P' are said to be *gyroreflections* of each other relative to the gyroline L.

Furthermore, coincidentally (i) the Euclidean line containing the points P and P' intersects the supporting gyrodiameter of the cogyroline containing the same points P and P' at the center O of circle L; (ii) the gyroline L intersects the cogyrosegment PP' at its cogyromidpoint $\mathbf{m}'_{PP'}$; and also (iii) the gyroline L is perpendicular to the cogyrosegment PP' in the following sense: The supporting gyrodiameter of the cogyrosegment PP' is perpendicular to L, as shown in Fig. 8.35. Noting that the orientation of the cogyrosegment PP' is the Euclidean orientation of its supporting gyrodiameter, the cogyrosegment PP' is perpendicular to the gyroline L. Hence, the gyroline L is the perpendicular bisector of the cogyrosegment PP', Fig. 8.35, and the points P and P' are said to be *cogyroreflections* of each other relative to the gyroline L.

Employing the Möbius gyrovector plane and its underlying Euclidean open unit disc we have thus observed the remarkable coincidence of the notions of reflection relative to a circle in Euclidean geometry, gyroreflection in hyperbolic geometry and cogyroreflection in cohyperbolic geometry relative to a gyroline. The bifurcation of non-Euclidean geometry into hyperbolic geometry and cohyperbolic geometry is explained in Fig. 8.38, p. 326.

8.22 Tessellation of the Poincaré Disc

For any given triangle one can reflect each of its vertices relative to its opposite side obtaining three new triangles. Starting with a single triangle and applying successive reflections in its sides, and in the sides of the newly obtained triangles, one may attempt to arrive at a complete triangulation of the Euclidean or hyperbolic plane. Let the angles of the initial triangle be $\pi/k, \pi/l, \pi/m$, where k, l, m are integers. A complete triangulation of the Euclidean plane is obtained by successive triangle reflections

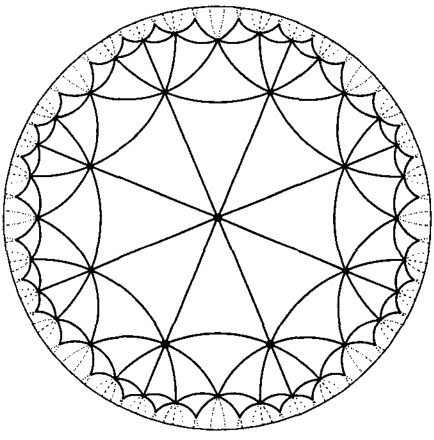


Fig. 8.36 Tessellation of the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes))$ or, equivalently, the Poincaré disc, by equilateral gyrotriangles with gyroangle $\alpha = \pi/4$. The gyrolength of each side of each equilateral gyrotriangle in the tessellation is $\sqrt{\sqrt{2}-1}$ in accordance with Theorem 8.55.

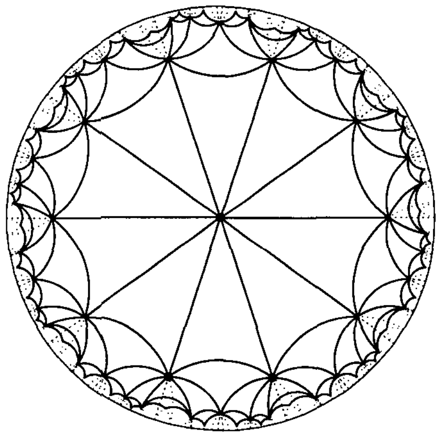


Fig. 8.37 Tessellation of the Möbius gyrovector plane $(\mathbb{R}^2_{c=1}, \oplus, \otimes))$ by equilateral gyrotriangles with gyroangle $\alpha = \pi/5$. The gyrolength of each side of each equilateral gyrotriangle in the tessellation is $\sqrt{2\cos(\pi/5)-1}$ in accordance with Theorem 8.55, p. 287.

if [Carathéodory (1954), p. 174]

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1 \tag{8.283}$$

Following Poincaré's theorem [Poincaré (1882)] a complete triangulation of the hyperbolic plane is obtained by successive hyperbolic triangle reflections (that is, gyrotriangle gyroreflections) if

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} < 1 \tag{8.284}$$

Elementary proof of Poincaré's theorem is found in [Carathéodory (1954), pp. 174–184] and in [Maskit (1971)].

Complete triangulations of the hyperbolic plane by equilateral triangles are special tessellations of the hyperbolic plane [Magnus (1974)], giving elegant demonstrations of hyperbolic reflectional symmetry. Owing to condition (8.283) the Euclidean plane allows only the single case of $k = l = m = 3$. In contrast, owing to condition (8.284), the hyperbolic plane allows the infinitely many cases of $k = l = m \geq 4$. Two of these cases, $k = l = m = 4$ and $k = l = m = 5$, are shown in Figs. 8.36

and 8.37. Evidently, the hyperbolic plane is richer in structure than the Euclidean plane. Various tilings of the Poincaré disc by triangles are available in the literature [Goodman-Strauss (2001)]. For instance, the case of $(k, l, m) = (2, 3, 7)$ is presented in [Mumford, Series and Wright (2002), p. 382], and the cases of $(7, 3, 2)$ and $(6, 4, 2)$ are presented in [Montesinos (1987), p. 178].

Let $\mathbf{abc}$ be an equilateral triangle. The reflection $\mathbf{a}'$ of $\mathbf{a}$ relative to its opposite side $\mathbf{bc}$ is given by the equation

$$\mathbf{a}' = (\mathbf{b} + \mathbf{c}) - \mathbf{a} \tag{8.285}$$

in the Euclidean plane $(\mathbb{R}^2, +)$, and by the equation

$$\mathbf{a}' = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a} \tag{8.286}$$

in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus)$. The latter, (8.286), is obviously related to the gyroparallelogram in Def. 6.40. Indeed, the hyperbolic triangulations in Figs. 8.36 and 8.37 were obtained by the use of (8.286). The analogies that (8.285) and (8.286) share demonstrate that in order to capture analogies between Euclidean and hyperbolic geometry both the gyrogroup operation $\oplus$ and the gyrogroup cooperation $\boxplus$ must be employed.

8.23 The Bifurcation Approach to Non-Euclidean Geometry

Non-Euclidean geometry emerged from the denial of parallelism in Euclidean geometry. However, we have seen in Sec. 8.20 that parallelism reappears in the cogyrolines of cohyperbolic geometry of non-Euclidean geometry, Figs. 6.6, 6.15, 8.33, and 8.34. We thus observe a Euclidean property which is denied in hyperbolic geometry, but reappears in cohyperbolic geometry.

The result that the Euclidean parallelism is retained in a branch of non-Euclidean geometry that has gone unnoticed, is a part of the bifurcation pattern demonstrated in Fig. 8.38. Some Euclidean properties that have seemingly been lost in the passage from Euclidean geometry to non-Euclidean geometry reappear in cohyperbolic geometry. Evidently, Euclidean geometry bifurcates into two non-Euclidean complementary branches, one of which is the hyperbolic geometry of Bolyai and Lobachevsky. Naturally, the branch that has gone unnoticed is called cohyperbolic geometry. The Euclidean geometry bifurcation into the two

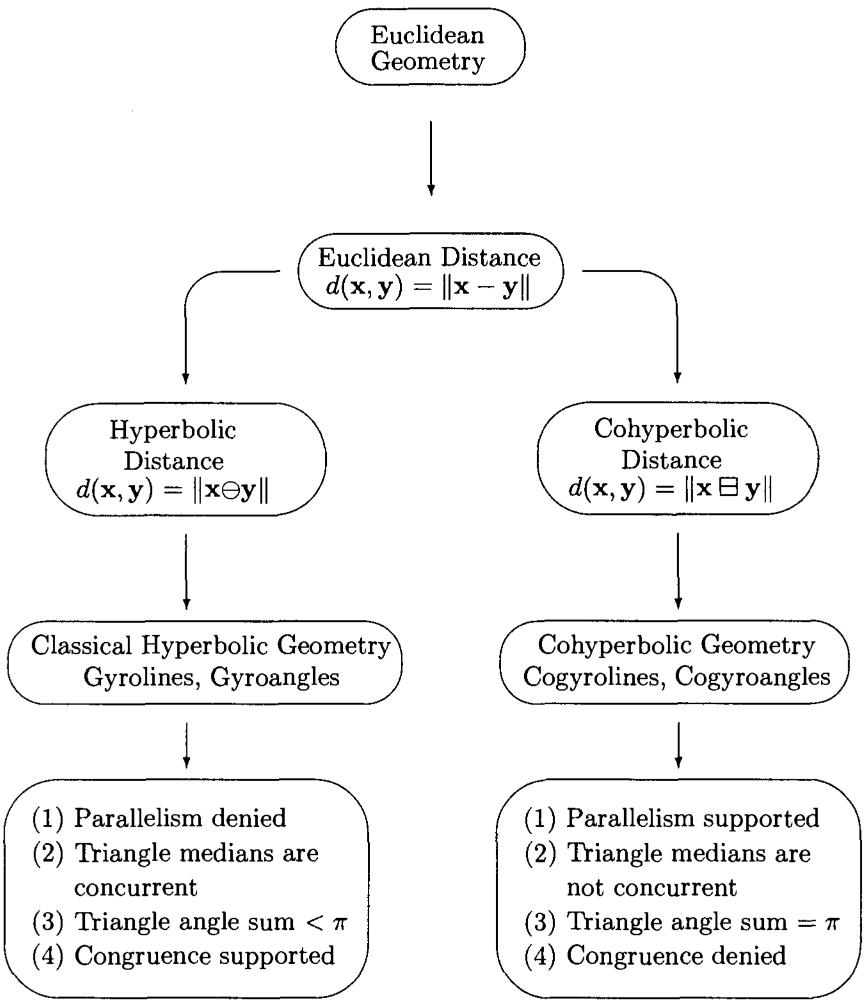


Fig. 8.38 The Hyperbolic Bifurcation Diagram.

complementary branches of non-Euclidean geometry is illustrated schematically in Fig. 8.38.

The two branches, hyperbolic and cohyperbolic geometry, of non-Euclidean geometry complement each other in the sense that they retain all the basic properties of Euclidean geometry. Those Euclidean properties that have seemingly been lost in the passage to hyperbolic geometry reappear in cohyperbolic geometry. Moreover, the two branches of non-Euclidean geometry share duality symmetries in (i) gyrovectors and cogyrovectors, (ii) gyrolines and cogyrolines, (iii) gyroangles and cogyroangles. These duality symmetries include, for instance, Theorem 2.21 and the gyration and cogyration transitive law in Theorems 6.29 and 6.62.

Owing to the bifurcation of Euclidean geometry properties in the transition to non-Euclidean geometry, any set of axioms that determines Euclidean geometry can be classified into the following three classes,

(1) Euclidean axioms that are valid in hyperbolic geometry but invalid in cohyperbolic geometry (like the congruence axioms).
(2) Euclidean axioms that are valid in cohyperbolic geometry but invalid in hyperbolic geometry (like the parallel axiom).
(3) Euclidean axioms that are valid in both hyperbolic and cohyperbolic geometry.

The three classes of Euclidean geometry axioms are disjoint. The *Bifurcation Principle*, that Fig. 8.38 demonstrates, states that the union of these three disjoint classes of Euclidean geometry axioms equals the set of all Euclidean geometry axioms. It follows from the Bifurcation Principle that there is no axiom of Euclidean geometry that is invalid in both hyperbolic and cohyperbolic geometry.

8.24 Exercises

(1) Verify Theorem 8.15 in detail.
(2) Let ABC and $A'B'C'$ be two gyrotriangles with equal defects. Show by numerical examples that their gyrotriangle constants, S_{ABC} and $S_{A'B'C'}$, Def. 8.42, need not be equal.
(3) Show that if two gyrotriangle gyroangles are congruent their opposite sides are congruent.
(4) Show that if two gyrotriangle sides are congruent their opposite gyroangles are congruent. Hint: Use Theorem 8.48.
(5) Verify the relation (8.177) between gyrotriangular gyration and gyroangular defect.
(6) Translate the gyrotriangle defect formula (8.132) for $\sin\delta$ in a Möbius gyrovector space into a corresponding formula for $\sin\delta$ in an Einstein gyrovector space.

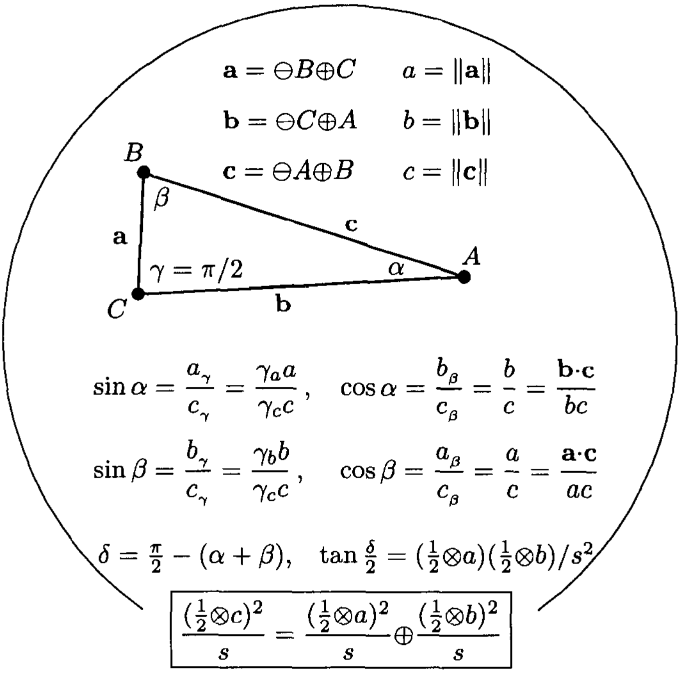


Fig. 8.39 Gyrotrigonometry in Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$.

Hints: (i) Use the translation identities in (6.309); and (ii) show that your result coincides with [Chen and Ge (1998), Eq. (14)].

(7) Verify the $\cos\gamma$ equation in Fig. 8.11.

(8) Figure 8.39 is the translation of Fig. 8.12, p. 263, from a Möbius to an Einstein gyrovector plane. Verify the identities presented in Fig. 8.39. Hint: Employ the translation identities in (6.309), noting that also the notation a_β, a_γ, etc., must be translated.

(9) Verify Equation (8.173b).

(10) Use the law of gyrocosines in Theorem 8.36 to calculate an alternative expression for c in (8.173c). Show that the alternative expression for c and (8.173c) are equivalent.

(11) Show that the Möbius gyroparallelogram identity (8.217) gives rise to the corresponding Einstein gyroparallelogram identity

$$\sqrt{1+\gamma_{d_\alpha}}\sqrt{1+\gamma_{d_\beta}} = \gamma_a + \gamma_b \tag{8.287}$$

Hint: Use the relationship $\gamma^2_{\mathbf{v}_m} = (1+\gamma_{\mathbf{v}_e})/2$, (6.310).

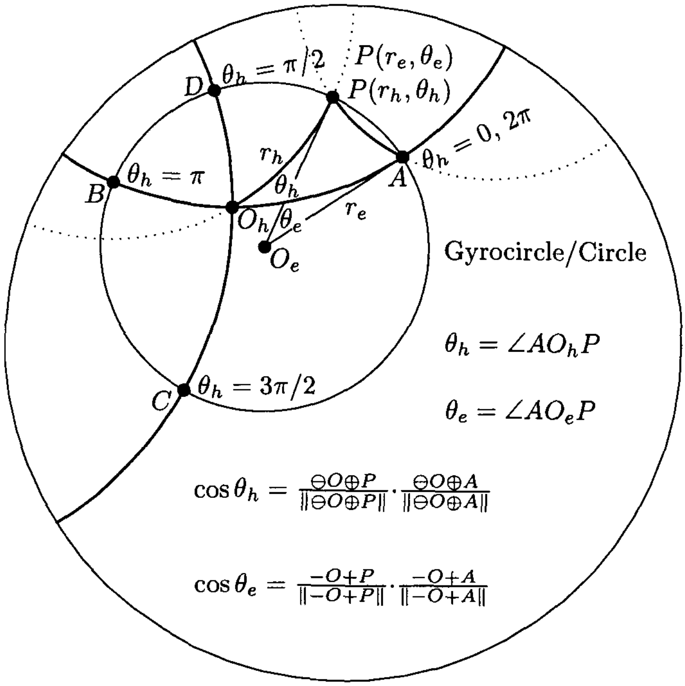


Fig. 8.40 The gyrocircle of Fig. 8.30 is shown here (i) as a gyrocircle with its gyrocenter O_h, gyroradius r_h, and a gyroangle θ_h, $0 \le \theta_h < 2\pi$; and (ii) as a circle with its (Euclidean) center O_e, (Euclidean) radius r_e, and an angle θ_e, $0 \le \theta_e < 2\pi$. The gyroangle θ_h that corresponds to the points A and $P = P(r_h, \theta_h)$ on the circumference of the gyrocircle is the gyroangle between the gyrorays O_hA and O_hP, $\theta_h = \angle AO_hP$. The gyrocosine of the gyroangle θ_h, $\cos\theta_h$, is shown. The angle θ_e that corresponds to the same points A and $P = P(r_e, \theta_e)$ on the circumference of the circle is the angle between the rays O_eA and O_eP, $\theta_e = \angle AO_eP$. The cosine of the angle θ_e, $\cos\theta_e$, is shown.

(12) Figure 8.40 presents analogies between the gyroangle θ_h and the angle θ_e of a gyrocircle/circle about its gyrocenter/center. Show that in general the numerical values of θ_h and θ_e are different. What is the special case when $\theta_h = \theta_e$?

Chapter 9

Bloch Gyrovector of Quantum Computation

The Bloch vector in the open unit ball $\mathbb{B}^3 = \mathbb{R}^3_{s=1}$ of the Euclidean 3-space $\mathbb{R}^3$ is well-known in quantum computation theory. Following a brief introduction, we will find in this chapter that the Bloch vector is, in fact, a gyrovector rather than a vector. Hence, we will discover that the geometry of quantum computation theory is the hyperbolic geometry of Bolyai and Lobachevsky, and its algebra is the algebra of gyrovector spaces.

9.1 The Density Matrix for Mixed State Qubits

A *qubit* is a two state quantum system completely described by the qubit *density matrix* $\rho(\mathbf{v})$,

$$\rho(\mathbf{v}) = \tfrac{1}{2}(\mathbf{1} + \boldsymbol{\sigma}\cdot\mathbf{v}) = \tfrac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{9.1}$$

parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$. Here $\mathbf{1}$ is the unit matrix and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices in vector notation [Chen and Ungar (2001)],

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9.2}$$

Using vector notation we thus have $\boldsymbol{\sigma}\cdot\mathbf{v} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3$ for any $\mathbf{v} \in \mathbb{B}^3$.

The density matrix [Blum (1996)] dates back to the early independent work of Landau and von Neumann, has proved useful in physics [Urbantke (1991); Chen, Ungar and Zhao (2002); Chen, Fu, Ungar and Zhao (2002)]. Researchers have devoted substantial efforts in describing the spaces defined by density matrices [Bloore (1976)], in using them to analyze the separability of quantum systems [Życzkowski (1998); Slater (1999)], in comparing information-theoretic properties of various probability distributions over them [Slater (1998)], as well as studying the question of parallel transport in this context [Uhlmann (1986); Uhlmann (1993); Uhlmann (1987); Urbantke (1991); Lévay (2004a); Lévay (2004b)].

The parameter $\mathbf{v}$ of the density matrix $\rho(\mathbf{v})$ is known as the Bloch vector [Nielsen and Chuang (2000)], and accordingly, the ball $\mathbb{B}^3$ is called the Bloch ball (or, sphere). The determinant of $\rho(\mathbf{v})$ is given by the equation

$$\det(\rho(\mathbf{v})) = \frac{1}{4\gamma_{\mathbf{v}}^2} \tag{9.3}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor (3.129) with $s = 1$.

All the postulates of quantum mechanics can be reformulated in terms of the density matrix language [Nielsen and Chuang (2000), p. 99]. The generic density matrix for a mixed state qubit turns out to be the 2×2 *qubit density matrix* $\rho(\mathbf{v})$ [Nielsen and Chuang (2000), p. 105]. The qubit density matrix (9.1) obeys the constraints of unit trace ($\mathrm{tr}(\rho(\mathbf{v})) = 1$) and positivity ($\det(\rho(\mathbf{v})) > 0$) [Spekkens and Rudolph (2002)].

The qubit is, thus, a two state quantum system completely described by the qubit density matrix $\rho(\mathbf{v})$, parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$, known as the *Bloch vector.*

Remarkable identities linking the qubit density matrices $\rho(\mathbf{v})$ to the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ of their Bloch vector parameter $\mathbf{v} \in \mathbb{B}^3$ will be presented, demonstrating that the Bloch vector deserves the title "gyrovector" rather than "vector".

Let $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ be two qubit density matrices generated by the two Bloch vectors $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$, and let $\rho(\mathbf{u} \oplus \mathbf{v})$ be the qubit density matrix generated by the Möbius sum $\mathbf{u} \oplus \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$, where $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition (3.127), so that $\mathbf{u} \oplus \mathbf{v} \in \mathbb{B}^3$. Furthermore, let $R(\mathbf{u}, \mathbf{v})$ be the 2×2 matrix given by the equation

$$R(\mathbf{u}, \mathbf{v}) = \rho^{-1}(\mathbf{u} \oplus \mathbf{v}) \rho(\mathbf{u}) \rho(\mathbf{v}) \tag{9.4}$$

where we use the notation $\rho^{-1}(\mathbf{v}) = (\rho(\mathbf{v}))^{-1}$ and, more generally, $\rho^r(\mathbf{v}) = (\rho(\mathbf{v}))^r$, $r \in \mathbb{R}$.

Remarkably, the matrix $R(\mathbf{u}, \mathbf{v})$, defined in terms of Möbius addition

$\oplus$, has elegant form,

$$R(\mathbf{u},\mathbf{v}) = \tfrac{1}{2}\begin{pmatrix} 1+\mathbf{u}{\cdot}\mathbf{v}+i(\mathbf{u}\times\mathbf{v})_3 & (\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 \\ -(\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 & 1+\mathbf{u}{\cdot}\mathbf{v}-i(\mathbf{u}\times\mathbf{v})_3 \end{pmatrix} \tag{9.5}$$

and elegant determinant,

$$4\det(R(\mathbf{u},\mathbf{v})) = (1+\mathbf{u}{\cdot}\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2 \tag{9.6}$$

$\mathbf{u},\mathbf{v}\in\mathbb{B}^3$. Hence, $R(\mathbf{u},\mathbf{v})$ has the form

$$R(\mathbf{u},\mathbf{v}) = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \in PSU(2) \tag{9.7}$$

for some complex numbers a and b, with a non zero determinant.

The matrix $R(\mathbf{u},\mathbf{v})$ is a positively scaled unitary unimodular matrix in the sense that it can be written as a positive number, $\sqrt{|a|^2+|b|^2}$, times a unitary unimodular matrix. It is, thus, an element of the group $PSU(2) = (0,\infty)\times SU(2)$ of all 2×2 positively scaled unitary unimodular matrices, a group isomorphic to the multiplicative group of nonzero quaternions [Altmann (1986)]. Moreover, it enjoys the following two remarkable features,

$$\mathrm{tr}(R(\mathbf{u},\mathbf{v})) = 2\mathrm{tr}(\rho(\mathbf{u})\rho(\mathbf{v})) \tag{9.8}$$

and

$$R(\mathbf{u},\mathbf{v}) = \bar{R}^t(\mathbf{v},\mathbf{u}) \tag{9.9}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, $\bar{R}^t$ being the transpose, complex conjugate of R.

The special unitary matrix $R_s(\mathbf{u},\mathbf{v})\in SU(2)$, $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, associated with $R(\mathbf{u},\mathbf{v})$ is

$$\begin{aligned} R_s(\mathbf{u},\mathbf{v}) &= \frac{R(\mathbf{u},\mathbf{v})}{\sqrt{\det(R(\mathbf{u},\mathbf{v}))}} \\ &= \frac{\begin{pmatrix} 1+\mathbf{u}{\cdot}\mathbf{v}+i(\mathbf{u}\times\mathbf{v})_3 & (\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 \\ -(\mathbf{u}\times\mathbf{v})_2+i(\mathbf{u}\times\mathbf{v})_1 & 1+\mathbf{u}{\cdot}\mathbf{v}-i(\mathbf{u}\times\mathbf{v})_3 \end{pmatrix}}{\sqrt{(1+\mathbf{u}{\cdot}\mathbf{v})^2+\|\mathbf{u}\times\mathbf{v}\|^2}} \end{aligned} \tag{9.10}$$

It satisfies the functional equations

$$R_s(\mathbf{u},\mathbf{v}) = R_s(-\mathbf{u},-\mathbf{v}) \tag{9.11}$$

$$R_s^{-1}(\mathbf{u},\mathbf{v}) = R_s(\mathbf{v},\mathbf{u}) \tag{9.12}$$

and

$$R_s(\mathbf{u}\oplus\mathbf{v},\mathbf{v}) = R_s(\mathbf{u},\mathbf{v}) \tag{9.13}$$

$$R_s(\mathbf{u},\mathbf{v}\oplus\mathbf{u}) = R_s(\mathbf{u},\mathbf{v}) \tag{9.14}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$. Furthermore, if $\mathbf{u}$ and $\mathbf{v}$ are parallel, $\mathbf{u}\|\mathbf{v}$, then

$$R_s(\mathbf{u},\mathbf{v}) = I, \qquad \mathbf{u}\|\mathbf{v} \tag{9.15}$$

I being the identity 2×2 matrix. Identities (9.13) – (9.14) clearly indicate the link with Möbius addition $\oplus$ in $\mathbb{B}^3$.

The denominator in (9.10),

$$F(\mathbf{u},\mathbf{v}) = \sqrt{(1+\mathbf{u}\cdot\mathbf{v})^2 + \|\mathbf{u}\times\mathbf{v}\|^2} > 0 \tag{9.16}$$

$\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, the square of which appears in the denominator of Möbius addition law (3.127), can be written as, (3.128),

$$F(\mathbf{u},\mathbf{v}) = \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{9.17}$$

where $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor of special relativity, (3.129), with $s = 1$.

Furthermore, the function $F(\mathbf{u},\mathbf{v})$ satisfies the following cocycle functional equation and normalization conditions [Ungar (2001), p. 289],

$$\begin{aligned} F(\mathbf{u},\mathbf{v}\oplus\mathbf{w})F(\mathbf{v},\mathbf{w}) &= F(\mathbf{v}\oplus\mathbf{u},\mathbf{w})F(\mathbf{u},\mathbf{v}) \\ F(\mathbf{u},\mathbf{0}) = F(\mathbf{0},\mathbf{v}) &= 1 \end{aligned} \tag{9.18}$$

Coincidentally, the cocycle equation (9.18) with an ordinary addition, $+$, rather than Möbius addition, $\oplus$, arises in several branches of mathematics, as pointed out by B. R. Ebanks and C. T. Ng [Ebanks and Ng (1993)]. It is thus interesting to realize that with Möbius addition $\oplus$ replacing the ordinary vector addition, the well-known cocycle equation appears in the study of gyrogroups [Rózga (2000)], in general, and in the present study of the qubit density matrix, in particular.

The matrices $R(\mathbf{u},\mathbf{v})$, $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, play a role in the definition of the gyrator gyr of the gyrocommutative gyrogroup of qubit density matrices shown in (9.19) below.

We define the gyration $\text{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]$ generated by the qubit density matrices $\rho(\mathbf{u})$ and $\rho(\mathbf{v})$ in terms of its effects on a qubit density matrix $\rho(\mathbf{w})$ by the equation

$$\text{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = R(\mathbf{u},\mathbf{v})\rho(\mathbf{w})R^{-1}(\mathbf{u},\mathbf{v}) \tag{9.19}$$

$\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{B}^3$, where we use the notation $R^{-1}(\mathbf{u},\mathbf{v}) = (R(\mathbf{u},\mathbf{v}))^{-1}$.

The right hand side of (9.19) is interesting. It is in a form that describes the evolution of a closed quantum system by the unitary operator $R_s(\mathbf{u},\mathbf{v})$, (9.10), according to [Nielsen and Chuang (2000), p. 99].

The left hand side of (9.19) is interesting too. It satisfies the identity

$$\text{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = \rho(\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}) \tag{9.20}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{B}^3$, where $\text{gyr}[u,v]$ are the gyrations of the Möbius gyrogroup $(\mathbb{B}^3,\oplus)$.

Interested readers may verify that (9.20) follows from (9.19) by lengthy but straightforward algebra that can be done by computer algebra, that is, by a computer software for symbolic manipulation, like MATHEMATICA and MAPLE. Thus, identities (9.19) – (9.20) give a quantum mechanical interpretation for a gyrogroup gyration.

Finally, we define a binary operation $\odot$ on the set

$$D = \{\rho(\mathbf{v}) :\ \mathbf{v}\in\mathbb{B}^3\} \tag{9.21}$$

of all mixed state qubit density matrices by the equation

$$\rho(\mathbf{u})\odot\rho(\mathbf{v}) = \rho(\mathbf{u}\oplus\mathbf{v}) \tag{9.22}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$. Identity (9.20) relates the gyration of $(D,\odot)$, generated by $\rho(\mathbf{u}),\rho(\mathbf{v})\in D$, (9.19), to the gyration of the parameter space $(\mathbb{B}^3,\oplus)$ of D, generated by $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$.

The groupoid $(D,\odot)$ involves the Bloch vector parameter space $\mathbb{B}^3$; and its composition law is given in (9.22) by parameter composition in $\mathbb{B}^3$. The parameter composition in $\mathbb{B}^3 = (\mathbb{B}^3,\oplus)$, in turn, is given by Möbius addition $\oplus$. The groupoid $(D,\odot)$ thus turns out to be a gyrocommutative gyrogroup, which is an isomorphic copy of the Möbius gyrogroup $(\mathbb{B}^3,\oplus)$ with isomorphism given by (9.22). The elements of $\mathbb{B}^3$, the Bloch vectors, now naturally become gyrovectors. The identity element of the gyrocommutative gyrogroup D is $\rho(\mathbf{0})$, where $\mathbf{0}\in\mathbb{B}^3$ is the zero vector of the Euclidean 3-space $\mathbb{R}^3$. The gyrogroup inverse $\rho^{\ominus 1}(\mathbf{v})$ of $\rho(\mathbf{v})\in(D,\odot)$ is

$\rho^{\ominus 1}(\mathbf{v}) = \rho(\ominus\mathbf{v})$ where $\ominus\mathbf{v} = -\mathbf{v}$ in $\mathbb{B}^3$. Indeed, by (9.22) we have,

$$\begin{aligned} \rho^{\ominus 1}(\mathbf{v})\odot\rho(\mathbf{v}) &= \rho(\ominus\mathbf{v})\odot\rho(\mathbf{v}) \\ &= \rho(\ominus\mathbf{v}\oplus\mathbf{v}) \\ &= \rho(\mathbf{0}) \end{aligned} \tag{9.23}$$

Hence, for instance, the gyration effects in (9.19) can also be written, according to Theorem 2.8(10) and (9.23), as

$$\mathrm{gyr}[\rho(\mathbf{u}),\rho(\mathbf{v})]\rho(\mathbf{w}) = \rho(\ominus\mathbf{u}\ominus\mathbf{v})\odot\{\rho(\mathbf{u})\odot(\rho(\mathbf{v})\odot\rho(\mathbf{w}))\} \tag{9.24}$$

as interested readers may verify by computer algebra.

The relationship between the generic qubit density matrix $\rho_{\mathbf{v}} = \rho(\mathbf{v}) \in D$ and its Bloch gyrovector parameter $\mathbf{v} \in \mathbb{B}^3$ is described by the following commutative diagram.

$$\begin{array}{ccccccc} \mathbf{v} & \xrightarrow{\mathbf{u}\oplus} & \mathbf{u}\oplus\mathbf{v} & \xrightarrow{\mathrm{gyr}[\mathbf{v},\mathbf{u}]} & \mathbf{v}\oplus\mathbf{u} & \xrightarrow{\mathbf{w}\oplus} & \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} & & \\ \rho_{\mathbf{v}} & \xrightarrow{\rho_{\mathbf{u}}\odot} & \rho_{\mathbf{u}}\odot\rho_{\mathbf{v}} & \xrightarrow{\mathrm{gyr}[\rho_{\mathbf{v}},\rho_{\mathbf{u}}]} & \rho_{\mathbf{v}}\odot\rho_{\mathbf{u}} & \xrightarrow{\rho_{\mathbf{w}}\odot} & \end{array}$$

$$\begin{array}{ccc} \mathbf{w}\oplus(\mathbf{v}\oplus\mathbf{u}) & \xrightarrow{=} & (\mathbf{w}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{w},\mathbf{v}]\mathbf{u} \\ \downarrow{\scriptstyle\rho} & & \downarrow{\scriptstyle\rho} \\ \rho_{\mathbf{w}}\odot(\rho_{\mathbf{v}}\odot\rho_{\mathbf{u}}) & \xrightarrow{=} & (\rho_{\mathbf{w}}\odot\rho_{\mathbf{v}})\odot\mathrm{gyr}[\rho_{\mathbf{w}},\rho_{\mathbf{v}}]\rho_{\mathbf{u}} \end{array} \tag{9.25}$$

Relationships for qubit density matrices with Einstein addition $\oplus_{\mathrm{E}}$, rather than Möbius addition $\oplus = \oplus_{\mathrm{M}}$, are presented in [Chen and Ungar (2002b)], and will be encountered in (9.68).

Density matrices for pure state qubits are given by (9.1) with a Bloch vector $\mathbf{v} \in \partial\mathbb{B}^3$ on the boundary of the Bloch ball $\mathbb{B}^3$, that is, $\|\mathbf{v}\| = 1$, [Nielsen and Chuang (2000), p. 100]. The qubit composition law (9.22) for $\mathbf{u}, \mathbf{v} \in \mathbb{B}^3$ remains valid on the closure $\bar{\mathbb{B}}^3$ of $\mathbb{B}^3$ with one exception. The composition

$$\rho(\mathbf{v})\odot\rho(-\mathbf{v}) = \rho(\mathbf{v}\ominus\mathbf{v}) \tag{9.26}$$

which is well-defined for all $\mathbf{v}\in\mathbb{B}^3$ as $\rho(\mathbf{0})$, does not exist on $\partial\mathbb{B}^3$ in terms of a limit for any $\mathbf{v}\in\mathbb{B}^3$ approaching $\mathbf{v}\in\partial\mathbb{B}^3$.

In special relativity the seemingly unfortunate failure of the limit of $\mathbf{v}\ominus_{\mathrm{E}}\mathbf{v}$ to exist when $\mathbf{v}$ approaches the boundary, $\partial\mathbb{B}^3$, of the ball $\mathbb{B}^3$ of relativistically admissible velocities turns out to be a blessing. It represents the physical experimental fact that, unlike subluminal particles, the photon has no rest frame. The role of the non-existence of $\mathbf{v}\ominus\mathbf{v}$ for any $\mathbf{v} \in \partial\mathbb{B}^3$ in the quantum mechanical interpretation needs to be explored. Here $\mathbf{v}\ominus_{\rm E}\mathbf{v} = \mathbf{v}\oplus_{\rm E}(-\mathbf{v})$, $\oplus_{\rm E}$ being the Einstein addition of relativistically admissible velocities. The relationship between Mobius and Einstein addition, $\oplus = \oplus_{\rm M}$ and $\oplus_{\rm E}$ is presented in Table 6.1, p. 202, and in (6.296).

9.2 The Bloch Gyrovector

Identity (9.4) can be presented as the polar decomposition,

$$\rho(\mathbf{u})\rho(\mathbf{v}) = \rho(\mathbf{u}\oplus\mathbf{v})R(\mathbf{u},\mathbf{v}) = F(\mathbf{u},\mathbf{v})\rho(\mathbf{u}\oplus\mathbf{v})R_s(\mathbf{u},\mathbf{v}) \tag{9.27}$$

demonstrating that, in general, the matrix product of two qubit density matrices is not equivalent to a qubit density matrix but, rather, to a qubit density matrix preceded, or followed, by a positively scaled unitary unimodular matrix. Indeed, Identity (9.27), in which the positively scaled unitary unimodular matrix $R(\mathbf{u},\mathbf{v}) \in PSU(2)$ precedes the qubit density matrix $\rho(\mathbf{u}\oplus\mathbf{v})$, is associated with the polar decomposition

$$\rho(\mathbf{u})\rho(\mathbf{v}) = R(\mathbf{u},\mathbf{v})\rho(\mathbf{v}\oplus\mathbf{u}) = F(\mathbf{u},\mathbf{v})R_s(\mathbf{u},\mathbf{v})\rho(\mathbf{v}\oplus\mathbf{u}) \tag{9.28}$$

in which the positively scaled unitary unimodular matrix $R(\mathbf{u},\mathbf{v})\in PSU(2)$ follows the qubit density matrix $\rho(\mathbf{v}\oplus\mathbf{u})$.

We may note that the qubit density matrix on the right hand side of (9.27) is parametrized by the Möbius sum $\mathbf{u}\oplus\mathbf{v}$ of Bloch gyrovectors, while the qubit density matrix on the right hand side of (9.28) is parametrized by the Möbius sum $\mathbf{v}\oplus\mathbf{u}$ of Bloch gyrovectors.

In general, the matrix product of several qubit density matrices is not equivalent to a single qubit density matrix, as we see from Identities (9.27) and (9.28). However, there are special cases in which the matrix product of several qubit density matrices is equivalent to a single qubit density matrix with a positive coefficient. Each of these cases corresponds to a *symmetric matrix product of qubit density matrices* that we define below. These cases clearly exhibit the nature of the Bloch vector as a gyrovector in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ rather than a vector in the open unit ball $\mathbb{B}^3$ of the Euclidean 3-space $\mathbb{R}^3$.

Definition 9.1 *Since gyrogroup summation is nonassociative, we define general gyrogroup summations by successive left summations as, for in-*

stance,

$$\sum_{k=n}^{1} \oplus g_k = g_1 \oplus (g_2 \oplus \ldots \oplus (g_{n-2} \oplus (g_{n-1} \oplus g_n)) \ldots) \tag{9.29}$$

in a gyrogroup $(G, \oplus)$. *One should note that the index* k *in* (9.29) *runs backwards from* n *to* 1 *and, accordingly, the summation runs from* g_n *to* g_1.

Let

$$\prod_{k=1}^{n} \rho^{r_k}(\mathbf{v}_k) = \rho^{r_1}(\mathbf{v}_1)\rho^{r_2}(\mathbf{v}_2) \ldots \rho^{r_{n-1}}(\mathbf{v}_{n-1})\rho^{r_n}(\mathbf{v}_n) \tag{9.30}$$

be the matrix product of n *qubit density matrices parametrized by the* n *Bloch gyrovectors* $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{n-1}, \mathbf{v}_n \in \mathbb{B}^3$, *and raised to the respective powers* $r_1, r_2, \ldots, r_n \in \mathbb{R}$.

A reversion of the matrix product (9.30) *is the matrix product*

$$\prod_{k=n}^{1} \rho^{r_k}(\mathbf{v}_k) = \rho^{r_n}(\mathbf{v}_n)\rho^{r_{n-1}}(\mathbf{v}_{n-1}) \ldots \rho^{r_2}(\mathbf{v}_2)\rho^{r_1}(\mathbf{v}_1) \tag{9.31}$$

A matrix product, ρ_s , *of qubit density matrices is symmetric if it is identically equal to its own reversion,*

$$\rho_s = \prod_{k=1}^{n} \rho^{r_k}(\mathbf{v}_k) = \prod_{k=n}^{1} \rho^{r_k}(\mathbf{v}_k) \tag{9.32}$$

for all $\mathbf{v}_k \in \mathbb{B}^3$, $k = 1, \ldots, n$.

Let $\rho_s = \prod_{k=1}^{n} \rho^{r_k}(\mathbf{w}_k)$ *be a symmetric matrix product of qubit density matrices. Its Bloch gyrovector* $\mathbf{w}$ *(or, equivalently, the Bloch gyrovector* $\mathbf{w}$ *that it possesses) is given by the equation*

$$\mathbf{w} = \sum_{k=n}^{1} \oplus (r_k \otimes \mathbf{w}_k) \tag{9.33}$$

in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

Examples illustrating Def. 9.1 follow. The reversion of

$$\rho_1 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{w}) \tag{9.34}$$

is

$$\rho_2 = \rho(\mathbf{w})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{9.35}$$

The matrix product ρ_1 is, in general, different from its reversion ρ_2 so that it is not symmetric. The matrix product

$$\rho_3 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{9.36}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{u}) \tag{9.37}$$

Similarly, the matrix product

$$\rho_3 = \rho(\mathbf{u})\rho(\mathbf{v})\rho(\mathbf{v})\rho(\mathbf{u}) \tag{9.38}$$

is symmetric, possessing the Bloch gyrovector

$$\mathbf{w} = \mathbf{u}\oplus(2\otimes\mathbf{v}\oplus\mathbf{u}) = 2\otimes(\mathbf{u}\oplus\mathbf{v}) \tag{9.39}$$

calculated by employing the Two-Sum Identity in Theorem 6.7.

Theorem 9.2 *Let ρ_s be a symmetric matrix product of qubit density matrices, possessing the Bloch gyrovector $\mathbf{w}\in\mathbb{B}^3$. Then*

$$\rho_s = \mathrm{tr}(\rho_s)\rho(\mathbf{w}) \tag{9.40}$$

Theorem 9.2 states that up to a positive coefficient, $\mathrm{tr}(\rho_s)$, a symmetric matrix product of qubit density matrices, ρ_s, is equivalent to a single qubit density matrix $\rho(\mathbf{w})$ parametrized by the Bloch gyrovector $\mathbf{w}$ that it possesses. The latter, in turn, is a gyrovector in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$ generated by the operations $\oplus$ and $\otimes$ in $\mathbb{B}^3$, as (9.33) indicates.

When convenient, we may use the standard notation

$$\rho_{\mathbf{v}} = \rho(\mathbf{v}) \tag{9.41}$$

for the 2×2 *density matrix* $\rho(\mathbf{v})$ in (9.1).

Illustrative examples of results that follow from Theorem 9.2 are presented in Examples $1-4$.

Example 1. Let

$$\rho_s = \rho_{\mathbf{v}}^n \tag{9.42}$$

where n is a positive integer. By Def. 9.1, ρ_s is symmetric, possessing the Bloch gyrovector, (9.33),

$$\mathbf{w} = \overbrace{\mathbf{v}\oplus\cdots\oplus\mathbf{v}}^{n \text{ terms}} = n\otimes\mathbf{v} \tag{9.43}$$

Hence, by Theorem 9.2,

$$\rho_{\mathbf{v}}^{n} = \operatorname{tr}(\rho_{\mathbf{v}}^{n})\rho_{n\otimes\mathbf{v}} \tag{9.44}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $n\in\mathbb{R}$. Remarkably, Identity (9.44) remains valid for any real n as well, expressing any real power $n\in\mathbb{R}$ of a qubit density matrix as a qubit density matrix with a positive coefficient. We may note that this remarkable result follows readily from the spectral theorem [Korányi (2001)].

Let $\mathbf{u} = n\otimes\mathbf{v}$ so that $\mathbf{v} = (1/n)\otimes\mathbf{u}$ in $\mathbb{B}^3$. Then, by (9.44),

$$\rho_{\mathbf{u}} = \frac{\rho_{(1/n)\otimes\mathbf{u}}^{n}}{\operatorname{tr}(\rho_{(1/n)\otimes\mathbf{u}}^{n})} \tag{9.45}$$

for all $n\in\mathbb{R}$, $n \neq 0$. Hence, for $m = 1/n$ (9.45) can be written as

$$\rho_{\mathbf{u}} = \frac{\rho_{m\otimes\mathbf{u}}^{1/m}}{\operatorname{tr}(\rho_{m\otimes\mathbf{u}}^{1/m})} \tag{9.46}$$

for all $\mathbf{u}\in\mathbb{B}^3$ and $m\in\mathbb{R}$, $m \neq 0$.

Renaming m and $\mathbf{u}$ as n and $\mathbf{v}$, (9.46) can be written as

$$\rho_{\mathbf{v}}^{n} = \frac{\rho_{n\otimes\mathbf{v}}}{(\operatorname{tr}(\rho_{n\otimes\mathbf{v}}^{1/n}))^{n}} \tag{9.47}$$

Finally, comparing (9.44) and (9.47) we have,

$$\operatorname{tr}(\rho_{\mathbf{v}}^{n})(\operatorname{tr}(\rho_{n\otimes\mathbf{v}}^{1/n}))^{n} = 1 \tag{9.48}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $n\in\mathbb{R}$, $n \neq 0$.

Identity (9.44) of Example 1 can be used to determine by inspection the matrix $\rho_{\mathbf{v}}^{r}$ and its trace $\operatorname{tr}(\rho_{\mathbf{v}}^{r})$ for all $r\in\mathbb{R}$. Let

$$\begin{aligned} A &= 1 + \|\mathbf{v}\| \\ B &= 1 - \|\mathbf{v}\| \end{aligned} \tag{9.49}$$

$\mathbf{v}\in\mathbb{B}^3$. Then, it follows from (9.1) and (6.234) that

$$\rho_{r\otimes\mathbf{v}} = \frac{1}{A^r + B^r}\frac{1}{2\|\mathbf{v}\|}M \tag{9.50}$$

where M is the matrix

$$M = \begin{pmatrix} (A^r + B^r)\|\mathbf{v}\| + (A^r - B^r)v_3 & (A^r - B^r)(v_1 - iv_2) \\ (A^r - B^r)(v_1 + iv_2) & (A^r + B^r)\|\mathbf{v}\| - (A^r - B^r)v_3 \end{pmatrix} \tag{9.51}$$

$\mathbf{v} \neq \mathbf{0}$, so that from (9.44) and by inspection,

$$\mathrm{tr}(\rho_{\mathbf{v}}^r) = \frac{A^r + B^r}{2^r} \tag{9.52}$$

and

$$\begin{aligned} \rho_{\mathbf{v}}^r &= \mathrm{tr}(\rho_{\mathbf{v}}^r)\rho_{r\otimes\mathbf{v}} \\ &= \frac{A^r + B^r}{2^r}\frac{1}{A^r + B^r}\frac{1}{2\|\mathbf{v}\|}M \\ &= \frac{1}{2^{r+1}\|\mathbf{v}\|}M \end{aligned} \tag{9.53}$$

Finally, (9.44) can be extended to a power of any real number, obtaining

$$\begin{aligned} \rho_{\mathbf{v}}^r &= \mathrm{tr}(\rho_{\mathbf{v}}^r)\rho_{r\otimes\mathbf{v}} \\ &= \frac{A^r + B^r}{2^r}\rho_{r\otimes\mathbf{v}} \end{aligned} \tag{9.54}$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $r\in\mathbb{R}$.

Example 2. Let

$$\rho_s = \rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} \tag{9.55}$$

By Def. 9.1, ρ_s is symmetric, possessing the Bloch gyrovector, (9.33),

$$\begin{aligned} \mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus(\mathbf{v}\oplus\mathbf{u})) \\ &= \mathbf{u}\oplus(2\otimes\mathbf{v}\oplus\mathbf{u}) \\ &= 2\otimes(\mathbf{u}\oplus\mathbf{v}) \end{aligned} \tag{9.56}$$

In (9.56) we use the left gyroassociative law and the Two-Sum Identity in Theorem 6.7.

Hence, by Theorem 9.2,

$$\begin{aligned} \rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} &= \mathrm{tr}(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})\rho_{\mathbf{w}} \\ &= \mathrm{tr}(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})} \end{aligned} \tag{9.57}$$

But, by (9.44),

$$\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})} = \frac{\rho^2_{\mathbf{u}\oplus\mathbf{v}}}{\mathrm{tr}(\rho^2_{(\mathbf{u}\oplus\mathbf{v})})} \tag{9.58}$$

and by (9.48),

$$\frac{1}{\mathrm{tr}(\rho^2_{(\mathbf{u}\oplus\mathbf{v})})} = (\mathrm{tr}(\rho^{1/2}_{2\otimes(\mathbf{u}\oplus\mathbf{v})}))^2 \tag{9.59}$$

Hence, (9.57) can be written as

$$\rho_{\mathbf{u}}\rho^2_{\mathbf{v}}\rho_{\mathbf{u}} = \mathrm{tr}(\rho_{\mathbf{u}}\rho^2_{\mathbf{v}}\rho_{\mathbf{u}})(\mathrm{tr}(\rho^{1/2}_{2\otimes(\mathbf{u}\oplus\mathbf{v})}))^2\rho^2_{\mathbf{u}\oplus\mathbf{v}} \tag{9.60}$$

Example 3. We wish to calculate $[\mathrm{tr}(\sqrt{\rho_{\mathbf{v}}})]^2$ for later reference. By (9.48) and (6.235) we have

$$\begin{aligned}
[\mathrm{tr}(\sqrt{\rho_{\mathbf{v}}})]^2 &= [\mathrm{tr}(\rho^2_{(1/2)\otimes\mathbf{v}})]^{-1} \\
&= [\mathrm{tr}(\rho^2_{\gamma_{\mathbf{v}}\mathbf{v}/(1+\gamma_{\mathbf{v}})})]^{-1} \\
&= \frac{1+\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}}
\end{aligned} \tag{9.61}$$

where the extreme right hand side of (9.61) is calculated by straightforward algebra. Coincidently, the extreme right hand side of (9.61) appears in (6.235) as well.

Example 4. Let

$$\rho_s = \rho^m_{\mathbf{u}}\rho^{2n}_{\mathbf{v}}\rho^m_{\mathbf{u}} \tag{9.62}$$

By Def. 9.1, ρ_s is symmetric, possessing the Bloch gyrovector, (9.33),

$$\begin{aligned}
\mathbf{w} &= m\otimes\mathbf{u}\oplus((2n)\otimes\mathbf{v}\oplus m\otimes\mathbf{u}) \\
&= m\otimes\mathbf{u}\oplus(2\otimes(n\otimes\mathbf{v})\oplus m\otimes\mathbf{u}) \\
&= 2\otimes(m\otimes\mathbf{u}\oplus n\otimes\mathbf{v})
\end{aligned} \tag{9.63}$$

as, similarly, in (9.55) and (9.56).

Hence, by Theorem 9.2,

$$\begin{aligned}\frac{\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m}{\mathrm{tr}(\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m)} &= \rho(m\otimes\mathbf{u}\oplus((2n)\otimes\mathbf{v}\oplus m\otimes\mathbf{u})) \\ &= \rho(m\otimes\mathbf{u}\oplus(2\otimes(n\otimes\mathbf{v})\oplus m\otimes\mathbf{u})) \\ &= \rho(2\otimes(m\otimes\mathbf{u}\oplus n\otimes\mathbf{v})) \\ &= \frac{\rho^2_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}}}{\mathrm{tr}(\rho^2_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}})}\end{aligned} \tag{9.64}$$

Square rooting both sides of (9.64) we have

$$\sqrt{\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m} = \sqrt{\frac{\mathrm{tr}(\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m)}{\mathrm{tr}(\rho^2_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}})}}\, \rho_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}} \tag{9.65}$$

Tracing both sides of (9.65) we have

$$\mathrm{tr}(\sqrt{\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m}) = \sqrt{\frac{\mathrm{tr}(\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m)}{\mathrm{tr}(\rho^2_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}})}} \tag{9.66}$$

so that, finally, by squaring both sides of (9.66) we obtain the identity

$$\left[\mathrm{tr}(\sqrt{\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m})\right]^2 = \frac{\mathrm{tr}(\rho_{\mathbf{u}}^m \rho_{\mathbf{v}}^{2n} \rho_{\mathbf{u}}^m)}{\mathrm{tr}(\rho^2_{m\otimes\mathbf{u}\oplus n\otimes\mathbf{v}})} \tag{9.67}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$ and $m,n\in\mathbb{R}$.

When $m=n=1/2$, (9.67) gives rise to an identity for the so called *Bures fidelity* $\mathcal{F}(\rho_{\mathbf{u}},\rho_{\mathbf{v}})$ [Nielsen and Chuang (2000); Chen and Ungar (2002a)], also known as Uhlmann's transition probability for mixed qubit states [Scutaru (1998)]. Thus,

$$
\begin{aligned}
\mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{v}}) &\overset{(1)}{=\!=\!=} \left[\operatorname{tr}\left(\sqrt{\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}}\,\right)\right]^2 \\
&\overset{(2)}{=\!=\!=} \frac{\operatorname{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}})}{\operatorname{tr}(\rho^2_{(1/2)\otimes\mathbf{u}\oplus(1/2)\otimes\mathbf{v}})} \\
&\overset{(3)}{=\!=\!=} \left[\operatorname{tr}\left(\sqrt{\rho_{2\otimes((1/2)\otimes\mathbf{u}\oplus(1/2)\otimes\mathbf{v})}}\,\right)\right]^2 \operatorname{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(4)}{=\!=\!=} \left[\operatorname{tr}\left(\sqrt{\rho_{\mathbf{u}\oplus_E\mathbf{v}}}\,\right)\right]^2 \operatorname{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(5)}{=\!=\!=} \frac{1+\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}} \operatorname{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) \\
&\overset{(6)}{=\!=\!=} \frac{1}{2}\frac{1+\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}(1+\mathbf{u}\cdot\mathbf{v}) \\
&\overset{(7)}{=\!=\!=} \frac{1}{2}\frac{1+\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}\frac{\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \\
&=\!=\!= \frac{1}{2}\frac{1+\gamma_{\mathbf{u}\oplus_E\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \\
&\overset{def}{=\!=\!=} \mathcal{F}(\mathbf{u},\mathbf{v})
\end{aligned}
\tag{9.68}
$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$.

The equalities in (9.68) are numbered for the respective explanation below.

(1) Presents the definition of Bures fidelity.
(2) Follows from (9.67) with $m = n = 1/2$.
(3) Follows from tracing both sides of (9.47) with $n = 2$.
(4) Follows from the isomorphism (6.296) between $\oplus = \oplus_M$ and $\oplus_E$.
(5) Follows from (9.61).
(6) Follows from (9.71).
(7) Follows from (3.144) with $s = 1$.

Identity (9.68) expresses Bures fidelity in terms of Einstein addition [Chen and Ungar (2002a)],

$$\mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{v}}) = \mathcal{F}(\mathbf{u}, \mathbf{v}) = \tfrac{1}{2}\,\frac{1 + \gamma_{\mathbf{u}\oplus_E \mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{9.69}$$

From (9.68) and (3.146), with $s = 1$, we obtain the elegant, obvious identity

$$\mathrm{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) = \tfrac{1}{2}\frac{\gamma_{\mathbf{u}\oplus_E \mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} = \tfrac{1}{2}(1 + \mathbf{u}\cdot\mathbf{v}) \tag{9.70}$$

Indeed,

$$\mathrm{tr}(\sqrt{\rho_{\mathbf{u}}}\,\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}) = \mathrm{tr}(\rho_{\mathbf{u}}\rho_{\mathbf{v}}) = \frac{1 + \mathbf{u}\cdot\mathbf{v}}{2} \tag{9.71}$$

Substituting (3.146), with $s = 1$, in (9.69) we have, by (3.129),

$$\begin{aligned} \mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{v}}) &= \tfrac{1}{2}\left\{1 + \mathbf{u}\cdot\mathbf{v} + \frac{1}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}}\right\} \\ &= \tfrac{1}{2}\left\{1 + \mathbf{u}\cdot\mathbf{v} + \sqrt{1 - \|\mathbf{u}\|^2}\sqrt{1 - \|\mathbf{v}\|^2}\right\} \end{aligned} \tag{9.72}$$

thus recovering a well-known identity for Bures fidelity [Jozsa (1994)], and a link with Einstein's special relativity theory.

In Examples 1 – 4 we uncovered the tip of the giant iceberg of the rich gyro-algebra that qubit density matrices possess in terms of their Bloch gyrovector parameter in the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

Exploiting the convex structure of the space of all qubit density matrices, the evolution of a density matrix $\rho(\mathbf{u})$ into another density matrix $\rho(\mathbf{v})$ through continuously successive density matrices is demonstrated by the *convex sum*

$$(1 - t)\rho(\mathbf{u}) + t\rho(\mathbf{v}), \qquad 0 \le t \le 1 \tag{9.73}$$

At "time" $t = 0$ the expression (9.73) gives the density matrix $\rho(\mathbf{u})$ and at "time" $t = 1$ the expression (9.73) gives the density matrix $\rho(\mathbf{v})$ [Preskill (2004), Sec. 2.5.1], thus evolving $\rho(\mathbf{u})$ into $\rho(\mathbf{v})$.

However, rather than exploiting the convex structure of the space of all qubit density matrices, it is more natural from the geometric viewpoint to exploit the rich gyrostructure and its gyrogeometry rather than the convex structure of the space. Among the advantages in employing the gyrostructure of the space is that the associated geometric phase emerges naturally as a result in the gyrogeometry of gyrovector spaces.

Exploiting the gyroline (6.56) of the Poincaré ball model of hyperbolic geometry, Fig. 6.1, we generate the evolution of a density matrix $\rho(\mathbf{u})$ into another density matrix $\rho(\mathbf{v})$ through continuously successive density matrices by the parameter transport of "time" along a gyroline,

$$\rho(\mathbf{u}\oplus(\ominus\mathbf{u}\oplus\mathbf{v})\otimes t), \qquad 0 \leq t \leq 1 \tag{9.74}$$

At "time" $t = 0$ the expression (9.74) gives the density matrix $\rho(\mathbf{u})$ and at "time" $t = 1$ the expression (9.74) gives the density matrix $\rho(\mathbf{v})$. As t evolves from $t = 0$ to $t = 1$, the Bloch gyrovector parameter of the qubit density matrix in (9.74) travels from $\mathbf{u}$ to $\mathbf{v}$ along the geodesic segment (that is, the gyrosegment) joining the points $\mathbf{u}$ and $\mathbf{v}$ of the Möbius gyrovector space $(\mathbb{B}^3, \oplus, \otimes)$.

The parallel transport, Fig. 8.24, p. 299, of a Bloch gyrovector $\mathbf{v}$ along a closed path in $\mathbb{B}^3$ results in a geometric angle defect, studied in Chap. 8 and in [Ungar (2001)]. The study of the resulting geometric phase of the evolution of the associated density matrix $\rho(\mathbf{v})$ could be of interest in geometric quantum computation [Ekert, Ericsson, Hayden, Inamori, Jones, Oi and Vedral (2000)]. It, thus, remains to harness the gyrostructure of the qubit density matrix for work in geometric quantum computation and its associated geometric phase.

9.3 The Bures Fidelity

Two Bloch vectors $\mathbf{u}$ and $\mathbf{v}$ generate the two density matrices $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ that, in turn, generate the *Bures fidelity* $\mathcal{F}(\rho_{\mathbf{u}}, \rho_{\mathbf{v}})$ that we may also write as $\mathcal{F}(\mathbf{u}, \mathbf{v})$. The concept of fidelity is a basic ingredient in quantum communication theory [Jozsa (1994)]. A good quantum communication channel must be capable of transferring output quantum states which are close to the input states. To quantify this idea of closeness, it is often necessary to provide a measure to distinguish different quantum states. To do so, one introduces the idea of fidelity. A fidelity of unity implies identical states whereas a fidelity of zero implies orthogonal states. Indeed this idea of fidelity is not just confined to quantum communication. It is also important in quantum optics, quantum computing and quantum teleportation [Wang (2001)]. Furthermore, the corresponding Bures distance has been used [Vedral, Rippin and Knight (1997)] to define a measure of the entanglement as the minimal Bures distance of an entangled state to the set of disentangled states.

The Bures fidelity $\mathcal{F}(\mathbf{u},\mathbf{v})$ is a most important distance measure between quantum states $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ of the qubit in quantum computation and quantum information [Nielsen and Chuang (2000); Wang, Kwek and Oh (2000)], given by (9.69),

$$\mathcal{F}(\mathbf{u},\mathbf{v}) = \left[\mathrm{tr}\sqrt{\sqrt{\rho_{\mathbf{u}}}\rho_{\mathbf{v}}\sqrt{\rho_{\mathbf{u}}}}\right]^2 = \tfrac{1}{2}\,\frac{1+\gamma_{\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}}}{\gamma_{\mathbf{u}}\gamma_{\mathbf{v}}} \tag{9.75}$$

Its values range over the interval (0, 1], quantifying the extent to which $\rho_{\mathbf{u}}$ and $\rho_{\mathbf{v}}$ can be distinguished from one another. From the physical point of view, the properties that the Bures fidelity $\mathcal{F}(\mathbf{u},\mathbf{v})$ possesses are natural, as indicated in [Chen, Ungar and Zhao (2002)].

Unfortunately, Bures fidelity $\mathcal{F}(\mathbf{u},\mathbf{v})$, $\mathbf{u},\mathbf{v}\in\mathbb{B}^3$, is not invariant under left gyrotranslations of its Möbius gyrogroup $(\mathbb{B}^3,\oplus)$. A different definition of fidelity that retains some properties of Bures fidelity while being invariant under left gyrotranslations in its Möbius gyrogroup $(\mathbb{B}^3,\oplus)$, is the *gyrocovariant* fidelity, given by the equation

$$\mathcal{F}^c(\mathbf{u},\mathbf{v}) = 1 - \|\mathbf{u}\ominus\mathbf{v}\| \tag{9.76}$$

Like Bures fidelity, the values of the gyrocovariant fidelity range over the interval (0, 1] so that $\mathcal{F}(\mathbf{v},\mathbf{v}) = \mathcal{F}^c(\mathbf{v},\mathbf{v}) = 1$. Unlike Bures fidelity, the gyrocovariant fidelity is simple and it possesses geometric significance in the Poincaré three-dimensional ball models of hyperbolic geometry and in its algebraic structure, the Möbius gyrovector spaces $(\mathbb{B}^3,\oplus,\otimes)$.

The realization that the home of Bloch vector is the Poincaré ball model of hyperbolic geometry, where it becomes a gyrovector in a Möbius gyrovector space, motivated Péter Lévay to explore the relationship between the Bures metric that results from Bures fidelity [Lévay (2004b), p. 4597] and the Möbius metric $\|\mathbf{u}\ominus\mathbf{v}\|$, (6.257), in the ball $\mathbb{B}^3$, where $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition (7.11) in the ball. He discovered [Lévay (2004a), p. 1838] [Lévay (2004b), p. 4603] that the Bures metric is conformally equivalent to the standard Poincaré metric which is, in turn, equivalent to Möbius gyrometric, as shown in Sec. 6.15, p. 192. Accordingly, Uhlman's parallel transport, used to determine geometric phases [Lévay (2004b)] is equivalent to the parallel transport, Fig. 8.24, in the Möbius gyrovector space $(\mathbb{B}^3,\oplus,\otimes)$.

The Bures fidelity has particularly wide currency today in quantum computation and quantum information geometry. However, as Nielsen and Chuang complain in [Nielsen and Chuang (2000), p. 410], "Unfortunately, no similarly [alluding to the *trace distance*] clear geometric interpretation is known for the fidelity between two states of a qubit". Following the study of the Bloch gyrovector in this chapter it is now clear that the elusive geometric interpretation for the fidelity between two states of a qubit lies in analytic hyperbolic geometry.

Chapter 10

Special Theory of Relativity: The Analytic Hyperbolic Geometric Viewpoint

This chapter commemorates 2005 as the 100th anniversary of Albert Einstein's (1879 – 1955) miraculous year, and the 50th anniversary of his death in April 18, 1955. In 1905 he published the paper [Einstein (1905); Einstein (1998)] that founded the special theory of relativity, a term that he coined about ten years later. As we will see throughout this chapter, in the framework of analytic hyperbolic geometry (gyrogeometry) special theory of relativity reveals more of its intricate beauty, where Einstein velocity addition becomes a gyrovector addition. Analytic hyperbolic geometry, thus, significantly extends Einstein's unfinished symphony. As such, this chapter prepares non-Euclidean geometers and physicists to practice twenty-first century special relativity.

The majestic scientific achievement of the 20th century in mathematical beauty and experimental verifications has been the special theory of relativity that Einstein introduced a century ago in 1905 [Einstein (1905)] [Einstein (1998), p. 141]. Einstein's special relativity theory is one of the foundation blocks of modern theoretical physics. Yet, in the year of 2005 Einstein's special relativity appears the first classical theory that defies common sense. Dedicated to the centenary of the birth of the special theory of relativity, 1905 – 2005, this book presents the analytic theory of hyperbolic geometry in terms of analogies that it shares with the analytic theory of Euclidean geometry. The resulting unification of analytic Euclidean and hyperbolic geometry allows the extension of our intuitive understanding from Euclidean to hyperbolic geometry, and similarly, from classical mechanics to relativistic mechanics as well.

Coincidentally, it is the mathematical abstraction of the relativistic effect known as "Thomas precession" that allows the unification and, hence, the extension of our intuitive understanding. The brief history of the discovery of Thomas precession is described in [Ungar (2001), Sec. 1, Chap. 1]. For a historical account, including the ensuing discussion, see [Mehra and Rechenberg (1982)].

We add physical appeal to Einstein velocity addition law (3.141) of relativistically admissible velocities, thereby gaining new analogies with classical mechanics and invoking new insights into the special theory of relativity. We place Einstein velocity addition in the foundations of both special relativity and its underlying hyperbolic geometry, enabling us to present special relativity in full three space dimensions rather than the usual one-dimensional space, using three-geometry instead of four-geometry. Doing so we uncover unexpected analogies with classical results, enabling the modern and unfamiliar to be studied in terms of the classical and familiar.

In particular, we show in this chapter that while it is well-known that the relativistic mass does not mesh up with the four-geometry of special relativity, it meshes extraordinarily well with the three-geometry, providing unexpected insights that are not easy to come by, by other means. Our novel approach provides powerful, far reaching insights into the Lorentz transformation of Minkowski's four-vectors that could be comprehended, and its beauty appreciated, by the reader. Hence, readers of this book will find this chapter interesting and useful regardless of whether they are familiar with the special theory of relativity.

10.1 Introduction

In the small arena of terrestrial and planetary measurements, Euclidean geometry is the shoe that fits the foot. Over the vast reaches of intergalactic spacetime, Lorentzian geometry, the geometry of general relativity, appears to be what is wanted. Neglecting gravitation, the geometry needed is the hyperbolic geometry of Bolyai and Lobachevsky, the geometry of special relativity, [Criado and Alamo (2001); Criado and Alamo (2002)].

The mere mention of hyperbolic geometry is enough to strike fear in the heart of the undergraduate physics student. Some undergraduate physics students regard themselves as excluded from the profound insights of hyperbolic geometry so that this enormous portion of human achievement is a closed door to them. But this book opens the door on its mission to make the hyperbolic geometry of Bolyai and Lobachevsky, which underlies the special theory of relativity, accessible to a wider audience in terms of gyrogeometry, the super analytic geometry that unifies analytic Euclidean and hyperbolic geometry.

Special relativity was introduced by Einstein a century ago in order to explain the massive experimental evidence against ether as the medium for propagating electromagnetic waves. However, as studied in all modern physics books, special relativity is not Einsteinian special relativity, the special theory of relativity as was originally formulated by Einstein in 1905 [Einstein (1905)]. Rather, it is Minkowskian special relativity, the special theory of relativity as was subsequently reformulated by Minkowski in 1908 [Lorentz, Einstein, Minkowski and Weyl (1952)]. Einsteinian and Minkowskian special relativity form two different approaches to the same special theory of relativity. In Minkowskian special relativity four-velocities and their Lorentz transformation law appear as a primitive, rather than as a derived, concept. In contrast, in Einsteinian special relativity three-velocities and their Einstein velocity addition law appear as a primitive concept, from which four-velocities and their Lorentz transformations are derived. The reason we have the Minkowski spacetime formalism of four-vectors today is that Minkowski's friend Sommerfeld took it upon himself to rewrite Minkowski's formalism and make it look like ordinary vector analysis.

As a result of the dominant position of Minkowskian special relativity some authors of relativity books omit the concept of Einstein's relativistic mass since it does not mesh up with Minkowskian special relativity [Adler (1987)], as will be indicated in Sec. 10.4.

In contrast, Einstein's relativistic mass meshes extraordinarily well with Einsteinian special relativity. We uncover analogies that the relativistic mass captures when it is studied in the context of Einsteinian special relativity and its underlying gyrogeometry.

Employing gyrovector space theoretic techniques, crucial analogies between the pairs

$$\begin{pmatrix} \textit{Euclidean Geometry} \\ \textit{Newtonian Mechanics} \end{pmatrix} \leftrightarrow \begin{pmatrix} \textit{Gyrogeometry} \\ \textit{Einsteinian Mechanics} \end{pmatrix} \tag{10.1}$$

get discovered in diverse situations one of which, concerning the notion of the *relativistic center of momentum (CM) velocity*, is displayed in Figs. 10.2 and 10.3, pp. 370–371. Remarkably, the novel analogies in these figures stem from the relativistic mass correction, according to which the mass of moving objects is velocity dependent [Anderson (1967), p. 199][Tsai (1986)].

Einstein velocity addition provides powerful insights into the Lorentz transformation as well. Einsteinian velocities and space rotations parameterize the Lorentz transformation group of relativistic mechanics just as Newtonian velocities and space rotations parameterize the Galilei transformation group of classical mechanics. Furthermore, the novel composition law of Lorentz transformations in terms of parameter composition is fully analogous to the well-known composition law of Galilei transformations in terms of parameter composition. This and other related novel analogies the Lorentz transformation group shares with its Galilean counterpart, as seen through the novel insights that Einstein velocity addition law provides, are presented in Sec. 10.15. Following these analogies, readers who intuitively understand the parameterized Galilei transformation group can straightforwardly extend their intuitive understanding to the parameterized Lorentz transformation group. Application of the latter to the gyrocovariant CM velocity is presented in Secs. 10.17 and 10.20, giving rise to the notion of the gyrobarycentric coordinates in Secs. 10.22 and 10.23.

Being guided by analogies with classical results we thus place Einstein velocity addition in the foundations of special relativity, enabling us to present special relativity in full three space dimensions rather than the usual one-dimensional space in four-geometry.

10.2 Einstein Velocity Addition

Attempts to measure the absolute velocity of the earth through the hypothetical ether had failed. The most famous of these experiments is one performed by Michelson and Morley in 1887 [Feynman and Sands (1964)]. It was 18 years later before the null results of these experiments were finally explained by Einstein in terms of a new velocity addition law that bears his name, that he introduced in his 1905 paper that founded the special theory of relativity [Einstein (1905); Einstein (1998)].

Contrasting Newtonian velocities, which are vectors in the Euclidean 3-space $\mathbb{R}^3$, Einsteinian velocities must be relativistically admissible, that is, their magnitude must not exceed the vacuum speed of light c. Let, (3.138),

$$\mathbb{R}^3_c = \{\mathbf{v} \in \mathbb{R}^3 : \|\mathbf{v}\| < c\} \tag{10.2}$$

be the c-ball of all relativistically admissible velocities. It is the ball of radius c, centered at the origin of the Euclidean 3-space $\mathbb{R}^3$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^3$ with magnitude $\|\mathbf{v}\|$ smaller than c. Einstein addition $\oplus$ in the ball is given by the equation [Einstein (1905)] [Einstein (1998),

p. 141], (3.141),

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\} \tag{10.3}$$

satisfying the gamma identity, (3.144),

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{10.4}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, where $\gamma_{\mathbf{u}}$ is the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}} \tag{10.5}$$

Einstein addition gives rise to the Einstein groupoid $(\mathbb{R}_c^3, \oplus)$ of Einsteinian velocities.

It is clear from (10.4) that $\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|$. However, it follows from (10.3) that, in general, $\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}$. Indeed, Einstein's exposé of velocity composition for two inertial systems emphasizes the lack of symmetry in the formula for the direction of the relative velocity vector [Einstein (1905), pp. 905 — 906] [Walter (1999b), p. 117]. Borel's attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described in [Walter (1999b), p. 117].

Remarkably, we will see that there is no need to "repair" the breakdown of commutativity and associativity in Einstein velocity addition law. While counterintuitively Einstein velocity addition law is neither commutative nor associative, it reveals intriguing features of hyperbolic geometry. In particular, it gives rise to a gyroparallelogram addition law which is commutative and fully analogous to the common Euclidean parallelogram addition law, Figs. 10.6 and 10.7, pp. 376 – 377. Furthermore, we will see that Einstein velocity addition law gives rise to a higher dimensional gyroparallelepiped law, which is both commutative and associative, (10.66), Fig. 10.13.

For the sake of simplicity some normalize the vacuum speed of light to $c = 1$. We, however, prefer to leave it as a free positive parameter enabling Einstein addition to be reduced to ordinary vector addition in the Newtonian limit, when $c \to \infty$. Thus, in the Newtonian limit the gyrocommutative gyrogroup of Einsteinian velocities, $(\mathbb{R}_c^3, \oplus)$, reduces to the commutative group of Newtonian velocities, $(\mathbb{R}^3, +)$.

While, in general, Einstein velocity addition is neither commutative nor associative, the special case when Einstein velocity addition is restricted to parallel velocities, (3.150), is both commutative and associative. When it is necessary to contrast Einstein velocity addition (10.3) with its restricted velocity addition (3.150), we use the term general (as opposed to restricted) Einstein velocity addition for the binary operation $\oplus$ in (10.3).

10.3 Status of the General Einstein Addition

Unlike the restricted Einstein velocity addition of parallel velocities, (3.150), the general Einstein velocity, (10.3), addition is unheard of in most modern relativity books. Why?

Harmony is the notion that motivates and justifies our desire to impose mathematical order on natural phenomena. The discovery of Vladimir Varičak in 1908–1910 [Varičak (1908); Varičak (1910a)] that Einstein's addition of relativistically admissible three-velocities has natural interpretation in the hyperbolic geometry of Bolyai and Lobachevsky was therefore a great triumph to Riemann and to the principle of harmony between mathematics and physics. For his chagrin, Varičak had to admit in 1924 that the adaption of vector algebra for use in hyperbolic space was just not possible [Varičak (1924), p. 80], as Scott Walter notes in [Walter (1999b), p. 121]. However, following Chap. 5, the introduction of vectors into hyperbolic geometry, where they are called gyrovectors, is now possible.

Riemann was aware of the possible application of his geometry to physics. In his inaugural address in 1854 on the occasion of joining the University Faculty of Göttingen he said that the value of his non-Euclidean geometry can possibly be to liberate us from preconceived ideas, should ever the time come that in the exploration of the laws of physics the concepts of Euclidean geometry may have to be abandoned. These prophetic words were literally fulfilled fifty years later by the special theory of relativity [Lanczos (1970), p. 91], uncovered by Einstein in 1905 [Einstein (1905); Einstein (1998)].

However, as Scott Walter notes [Walter (1999b), p. 94], in contrast to the amount of publicity they received, applications of the hyperbolic geometry of Bolyai and Lobachevsky to relativity physics produced slim results, the value of which was outstripped by the technical intricacy of the methods developed to obtain them. The seemingly lack of harmony between mathematics and Einstein's original formulation of special relativity led Hermann Minkowski to reformulate special relativity, elaborating during the years 1907–1909 a four-dimensional spacetime geometry, now known as the Minkowski space [Lorentz, Einstein, Minkowski and Weyl (1952)]. The basic notion in Minkowski's reformulation of special relativity is the Lorentz transformation law of four-velocities as opposed to Einstein's formulation in terms of his addition law of three-velocities.

Minkowski characterized his spacetime geometry as evidence that *pre-established harmony* between pure mathematics and applied physics does exist [Pyenson (1982)]. Subsequently, the study of special relativity followed the lines laid down by Minkowski, in which the role of Einstein velocity addition and its interpretation in the hyperbolic geometry of Bolyai and Lobachevsky are ignored [Barrett (1998)].

The tension created by the mathematician Minkowski into the specialized realm of theoretical physics, as well as Minkowski's strategy to overcome disciplinary obstacles to the acceptance of his reformulation of special relativity is discussed by Scott Walter in [Walter (1999a)].

According to Leo Corry [Corry (1998)], Einstein considered Minkowski's reformulation of his theory in terms of four-dimensional spacetime to be no more than "superfluous erudition". More generally, the entry of mathematicians into the field of relativity was described by Einstein as an *invasion*, as Sommerfeld later recalled [Schilpp (1949), p. 102] [Walter (1999a)]. But, the importance of the Minkowskian special relativity was quickly grasped by physicists like Arnold Sommerfeld and Max von Laue. Admittedly, later in life Einstein had to adopt the Minkowskian reformulation of his special theory of relativity [Adler (1987), fn. 27]. However, had Einstein known that his velocity addition law of relativistically admissible velocities is a gyrovector space operation in the same way that the addition of Newtonian velocities is a vector space operation, he could have made a better case for his original formulation of special relativity.

The strong objection to place Einstein velocity addition centrally in special relativity is traced back to Minkowski. Scott Walter thus writes:

> Minkowski neither mentioned the [Einstein] law of velocity addition, nor expressed it in formal terms.
>
> Scott Walter [Walter (1999b)]

Moreover, at the September 1909 meeting of the German Association of Natural Scientists in Salzburg, Arnold Sommerfeld attempted to spark physicists' interest in Minkowskian formalism of special relativity. According to Scott Walter:

> As an example of the advantage of the Minkowskian approach, Sommerfeld selected [in his Salzburg talk] the case of Einstein's "famous addition theorem", according to which *velocity parallelograms do not close* [that is, Einstein velocity addition is noncommutative; Italics added]. This "somewhat strange" result, Sommerfeld suggested, became "completely clear" when viewed from Minkowski's standpoint.
>
> Scott Walter [Walter (1999b), p. 110]

Accordingly, Roger Penrose writes:

> My own point of view would be that ... special relativity was not fully appreciated (either by Poincarè or by Einstein) until Herman Minkowski presented, in 1908, the four-dimensional spacetime picture. He gave a now famous lecture at the University of Göttingen in which he proclaimed, 'Henceforth space by itself, and time by itself are doomed to fade away into mere shadow, and only a kind of union of the two will preserve an independent reality.'
>
> Einstein seems not to have appreciated the significance of Minkowski's contribution initially, and for about two years he did not take it seriously. But subsequently he came to realize the full power of Minkowski's point of view. It formed the essential background for Einstein's extraordinary late development of general relativity, in which Minkowski's four-dimensional spacetime geometry becomes curved.
>
> Roger Penrose [Penrose (2002)]

Contrasting Minkowski's *inseparable* spacetime, the desirability of splitting spacetime into space and time in general relativity is expressed by Charles W. Misner, Kip S. Thorne and John Archibald Wheeler in [Misner, Thorne and Wheeler (1973), p. 505].

Contrasting Sommerfeld's view that Einstein addition is geometrically inferior since *"velocity parallelograms do not close"*, we will find in Sec. 10.9 that in Einstein gyrovector spaces *velocity gyroparallelograms do close*, giving rise to the gyroparallelogram law. Moreover, in Sec. 10.14 the Einsteinian gyroparallelogram law will be extended to higher dimensions, giving rise to the Einsteinian gyroparallelepiped law in n-dimensions, $n \geq 3$.

As a result of the trend laid down by Minkowski, the general Einstein velocity addition of relativistically admissible velocities that need not be parallel is unheard of in most books on relativity physics. Among outstanding exceptions are [Fock (1964)], [Bacry (1977)] and [Sexl and Urbantke (2001)]. Not unexpectedly, therefore, the history of gyrogroup theory and its application in relativity physics is presented in [Sexl and Urbantke (2001), pp. 141–142].

The status of Einstein velocity addition prior to the discovery of its gyrostructure is well described in [Brehme (1968)]:

> The transformation law for the spatial components of the coordinate velocity, known as the Einstein (or relativistic) velocity addition theorem, is awkward and difficult to use in any but the very simplest situations [that is, Einstein velocity addition of parallel velocities].
>
> Robert W. Brehme [Brehme (1968)]

With the advent of the theory of gyrogroups and gyrovector spaces, Einstein velocity addition has acquired a rich structure. It became a gyrocommutative, gyroassociative gyrogroup operation, as well as a gyrovector space operation, placing itself on equal footing with Newton velocity addition. Earlier, however, Einstein velocity addition was merely considered as a noncommutative, nonassociative binary operation – too impoverished a structure to stand centrally in the special theory of relativity.

10.4 Einstein Addition is an Indispensable Relativistic Tool

Relativistic three-velocities and their Einstein velocity addition law mesh extraordinarily well with the notion of the relativistic mass, as illustrated in Figs. 10.2 and 10.3, pp. 370–371. These figures indicate that (i) relativistic three-velocities, (ii) relativistic mass, and (iii) hyperbolic geometry, interplay in Einsteinian mechanics in a way fully analogous to the interplay of (i) Newtonian velocities, (ii) classical mass, and (iii) Euclidean geometry in Newtonian mechanics

The considerable increase of the mass of particles in particle accelerators has to be taken into account in the design of accelerators. Owing to the reality of the relativistic mass [Anderson (1967), p. 199][Tsai (1986)], Einstein addition of three-velocities is an indispensable relativistic tool. Contrasting the harmonious interplay between three-velocities and the relativistic mass, the latter is in conflict with Minkowskian four-vector spacetime approach. Explanation and rationalization of the resulting never-ending debates about the status of mass in special relativity, calling it "the messy mass", are provided by Beisbart and Jung [Beisbart and Jung (2004)]. We will find that within the frame of Einsteinian relativity, as opposed to Minkowskian relativity, the problem that Beisbart and Jung explain and rationalize in [Beisbart and Jung (2004)] does not exist.

Taking three-velocities and their Einstein addition law as a primitive concept in special relativity, one can derive Minkowski's four velocities and their Lorentz transformation law, as we show in Sec. 10.15. In contrast, taking four-velocities and their Lorentz transformation law as a primitive concept in special relativity leaves the theory with seemingly no need for Einstein addition. This is, however, not the case since the physically significant relativistic mass does not mesh up with Minkowskian special relativity while it meshes extraordinarily well with Einsteinian special relativity. Moreover, with the omission of Einstein velocity addition, the opportunity to make a very appealing hyperbolic geometrical point is lost. Hence, clearly, the omission of Einstein velocity addition is both unfortunate and unnecessary.

The Einstein relativistic mass $m\gamma_{\mathbf{v}}$ of a particle with rest mass m, $m \in \mathbb{R}^{>0}$, and relative velocity $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^3_c$, is velocity dependent [Anderson (1967), p. 199]. To understand the role that the relativistic mass plays in the geometry of special relativity we need an approach to special relativity that meshes smoothly with the relativistic mass concept. Hence, we are forced to employ Einsteinian, rather than Minkowskian, special relativity as illustrated by Figs. 10.2 and 10.3.

The fact that the relativistic mass does not mesh up with the Minkowskian four-vector spacetime approach to the study of special relativity led several authors to omit the relativistic mass from new books and new editions of old books on relativity physics regardless of its physical reality. Remarking on "The bane of the relativistic mass" [Brehme (1968)], Brehme writes:

> By assigning mass a relativistic character, we obscure both the simplicity and the essentially kinematic nature of relativity.
>
> Robert W. Brehme [Brehme (1968)]

Attributing to the relativistic mass in special relativity the role of an artifact [Adler (1987), pp. 742–743], Adler admits that the "relativistic mass is a concept in turmoil":

> Any one who has tried to teach special relativity using the four-vector spacetime approach knows that relativistic mass and four-vectors make for an ill-conceived marriage. ... The solution is for physics teachers to understand that relativistic mass is a concept in turmoil. If they choose to use it in their course, they should caution the students to this effect.
>
> Carl G. Adler [Adler (1987)]

> Personal experience shows that with a little familiarity with the TW [Taylor and Wheeler, [Taylor and Wheeler (1966)]] approach, the relativistic mass concept appears rather artificial.
>
> M.A.B. Whitaker [Whitaker (1976)]

Einstein derived the equivalence of rest mass m_0 and energy E, expressible as $E^2 - p^2c^2 = m_0^2c^4$ where p is the relativistic momentum, $p = m_0\gamma_{\mathbf{v}}\mathbf{v}$. When relativistic mass $m = m_0\gamma_{\mathbf{v}}$ is used instead, the mass–energy equivalence equation reduces to the famous formula $E = mc^2$.

> In the modern language of relativity theory there is only one mass, the Newtonian mass m, which does not vary with velocity; hence the famous formula $E = mc^2$ has to be taken with a large grain of salt.
>
> Lev B. Okun [Okun (1989)]

> When relativity is put in four-dimensional form, as in Chapter XIII, the idea of relativistic mass is out of place and clumsy.
>
> T.M. Helliwell [Helliwell (1966), p. 149]

The physical significance of the concept of the relativistic, velocity dependent mass is well-known; see, for instance, [Anderson (1967), p. 199], [Tsai (1986)] and [Gabrielse (1995)]. Hence the rejection of this concept, solely based on the breakdown of harmony with Minkowskian special relativity, is not justified [Rindler (1990)]. Contrasting the opinion that the concept of the relativistic mass is in turmoil since it does not mesh up with Minkowskian special relativity, we will soon find that this peaceful concept is rather welcome. It is an asset rather than a liability since it meshes extraordinarily well with Einsteinian special relativity and its hyperbolic geometry, as will be indicated in Figs. 10.2 and 10.3.

Readers who, following this book, choose to study the Minkowski spacetime as a notion derived from Einsteinian special relativity, as we do in Sec. 10.15, rather than as a primitive notion, will be rewarded by encountering no confusion in the concept of the relativistic mass. Rather, they will encounter the harmonious interplay between the relativistic mass and the hyperbolic geometry of Bolyai and Lobachevsky that regulates Einstein addition. The harmonious interplay is presented in Sec. 10.8 and is visually illustrated by Figs. 10.2 and 10.3. Relativistic mass, the ugly duckling of Minkowski's four-vectors will, thus, turn out to be the beautiful swan of Einstein's three-velocities.

The dignity of special relativity theory requires that every possible means for the harmonious introduction of the relativistic mass into the theory be explored. Indeed, the means is provided by Einsteinian relativity. Both Einstein velocity addition and Einstein relativistic mass capture important analogies:

(1) Einstein velocity addition captures analogies with classical velocity addition, so that the hyperbolic geometry of Einsteinian velocities turns out to be fully analogous to the Euclidean geometry of Newtonian velocities; and
(2) Einstein relativistic mass captures analogies with classical mass, so that the relativistic mass is a geometric scalar in the hyperbolic geometry of Einsteinian velocities in full analogy with the classical mass, which is a geometric scalar in the Euclidean geometry of Newtonian velocities.

Einstein velocity addition and relativistic mass will, thus, turn out to be an indispensable ingredient for special relativity and its 3-dimensional hyperbolic geometry.

Relativistic mass came into common usage in the relativity text books of the early 1920s written by Pauli [Pauli (1958)], Eddington [Eddington (1924)] and Born [Born (1964)]. The invariant mass of particles became more significant in the 1950s, and inevitably physicists started to use the term "mass" to mean invariant mass. Gradually this took over as the normal convention, and the concept of relativistic mass increasing with velocity is presently played down.

Einstein's original mechanical formalism of special relativity is described in terms of inertial reference frames, velocities, forces, length contraction and time dilation. Relativistic mass fits naturally into this mechanical framework. In contrast, invariant mass proves to be more fundamental in Minkowski's geometric approach to special relativity and, accordingly, relativistic mass is of no use in general relativity. The black holes of general relativity, if exist, provide evidence against the relativistic mass.

Owing to length contraction and relativistic mass velocity dependence, an observer who moves at a speed sufficiently close to the speed of light should collapse to form a black hole. If the observer moves fast enough relative to a star then that star must appear to the observer as a black hole because of its increased mass observed by the observer. This would be paradoxical since we would expect things to appear very differently to an observer who is stationary relative to the star. So what has gone wrong? Either mass does not increase with velocity or black holes do not exist! Indeed, in general relativity black holes do exist and, accordingly, relativistic mass in general relativity is of no use at all.

10.5 From Thomas Gyration to Thomas Precession

Thomas gyration is the missing link between Einstein addition and ordinary vector addition.

Einstein addition (10.3) forms the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus)$. Hence, its gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}^3_c \to \mathbb{R}^3_c$ are generated by Einstein addition according to the formula, Theorem 2.8(10),

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus(\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})) \tag{10.6}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w}\in\mathbb{R}^3_c$. The velocity $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w}$ is said to be the velocity $\mathbf{w}$ gyrated by the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ generated by the velocities $\mathbf{u}$ and $\mathbf{v}$. For $\mathbf{u} = \mathbf{v} = \mathbf{0}$ we have

$$\mathrm{gyr}[\mathbf{0}, \mathbf{0}]\mathbf{w} = \mathbf{w} \tag{10.7}$$

so that the gyration $\mathrm{gyr}[\mathbf{0}, \mathbf{0}]$ is trivial, being the identity map of $\mathbb{R}^3_c$.

It is clear from (10.6) that gyrations measure the nonassociativity of Einstein addition. Since Einstein addition of parallel velocities (3.150) is associative, gyrations generated by parallel velocities are trivial. Accordingly, one can show that

$$\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w}, \qquad \mathbf{u}\|\mathbf{v} \tag{10.8}$$

for all $\mathbf{w}\in\mathbb{R}_c^3$, whenever $\mathbf{u}$ and $\mathbf{v}$ are parallel in the ball $\mathbb{R}_c^3$ of relativistic velocities.

Owing to the breakdown of associativity in Einstein velocity addition, the self-map $\text{gyr}[\mathbf{u},\mathbf{v}]$ of the ball $\mathbb{R}_c^3$ is, in general, non-trivial. It turns out to be an element of the group $SO(3)$ of all 3×3 real orthogonal matrices with determinant 1. Indeed, the map $\text{gyr}[\mathbf{u},\mathbf{v}]$ can be written as a 3×3 real orthogonal matrix with determinant 1, as shown in [Ungar (1988a)]. As such, the map $\text{gyr}[\mathbf{u},\mathbf{v}]$ represents a rotation of the Euclidean 3-space $\mathbb{R}^3$ about its origin. It preserves the inner product in the ball,

$$\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b} \tag{10.9}$$

and, hence, it also preserves the norm in the ball,

$$\|\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\| = \|\mathbf{a}\| \tag{10.10}$$

for all $\mathbf{a},\mathbf{b},\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, where $\cdot$ and $\|\ \|$ are the inner product and the norm that the ball $\mathbb{R}_c^3$ inherits from its space $\mathbb{R}^3$.

Furthermore, $\text{gyr}[\mathbf{u},\mathbf{v}]$ turns out to be an *automorphism* of the relativistic *groupoid* $(\mathbb{R}_c^3,\oplus)$ in the following sense.

An automorphism $\text{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, of the groupoid $(\mathbb{R}_c^3,\oplus)$ is a bijective (one-to-one) self-map of $\mathbb{R}_c^3$ that preserves its binary operation $\oplus$. Indeed, we have

$$(\text{gyr}[\mathbf{u},\mathbf{v}])^{-1} = \text{gyr}[\mathbf{v},\mathbf{u}] \tag{10.11}$$

where $(\text{gyr}[\mathbf{u},\mathbf{v}])^{-1}$ is the inverse of $\text{gyr}[\mathbf{u},\mathbf{v}]$, so that $\text{gyr}[\mathbf{u},\mathbf{v}]$ is bijective; and

$$\text{gyr}[\mathbf{u},\mathbf{v}](\mathbf{a}\oplus\mathbf{b}) = \text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\oplus\text{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} \tag{10.12}$$

for all $\mathbf{u},\mathbf{v},\mathbf{a},\mathbf{b}\in\mathbb{R}_c^3$, so that $\text{gyr}[\mathbf{u},\mathbf{v}]$ preserves the Einstein addition $\oplus$ in the ball $\mathbb{R}_c^3$.

The automorphism $\text{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}_c^3$, is called the *Thomas gyration* generated by $\mathbf{u}$ and $\mathbf{v}$. It is the mathematical abstraction of the relativistic effect known as Thomas precession [Jackson (1975)], as we will see soon.

Remarkably, Thomas gyration "repairs" the breakdown of commutativity and associativity in Einstein velocity addition, giving rise to their following gyro-counterparts,

$$\begin{aligned} \mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law} \\ \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\ (\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law} \end{aligned} \tag{10.13}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^3_c$. The gyrocommutative and the gyroassociative laws of Einstein velocity addition share obvious analogies with the common commutative and associative laws of vector addition, allowing the classical picture of velocity addition to be restored. Accordingly, the gyrocommutative and the gyroassociative laws of Einstein velocity addition give rise to the mathematical group-like object, the gyrocommutative gyrogroup in Def. 2.6.

Moreover, Thomas gyration possesses a rich structure, including the left and right loop property

$$\begin{aligned} \mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Left Loop Property} \\ \mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Right Loop Property} \end{aligned} \tag{10.14}$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$. For the proof of the identities in (10.9) – (10.14) see Exercise (1) at the end of the chapter.

The prefix "gyro" that stems from Thomas gyration is extensively used to emphasize analogies with classical terms as, for instance, gyrocommutative, gyroassociative binary operations in gyrogroups and gyrovector spaces. Owing to the gyrocommutative law in (10.13), Thomas gyration is recognized as the familiar Thomas precession. The gyrocommutative law was already known to Silberstein in 1914 [Silberstein (1914)] in the following sense. The Thomas precession generated by $\mathbf{u},\mathbf{v} \in \mathbb{R}^3_c$ is the unique rotation that takes $\mathbf{v}\oplus\mathbf{u}$ into $\mathbf{u}\oplus\mathbf{v}$ about an axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ in $\mathbb{R}^3$ [Mocanu (1992)], thus giving rise to the gyrocommutative law. Obviously, Silberstein did not use the terms "Thomas precession" and "gyrocommutative law". These terms have been coined later, respectively, following Thomas' 1926 paper [Thomas (1926)], and in 1991 [Ungar (1991); Ungar (1997)].

A description of the 3-space rotation, which since 1926 is named after Thomas, is found in Silberstein's 1914 book [Silberstein (1914)]. In 1914 Thomas precession did not have a name, and Silberstein called it in his 1914 book a "certain space-rotation" [Silberstein (1914), p. 169].

An early study of Thomas rotation, made by the famous mathematician Emile Borel in 1913, is described in his 1914 book [Borel (1914)] and, more recently, in [Stachel (1995)]. According to Belloni and Reina [Belloni and Reina (1986)], Sommerfeld's route to Thomas precession dates back to 1909. However, prior to Thomas discovery the relativistic peculiar 3-space rotation had a most uncertain physical status [Walter (1999b), p. 119]. The only knowledge Thomas had in 1925 about the peculiar relativistic gyroscopic precession, however, came from De Sitter's formula describing the relativistic corrections for the motion of the moon, found in Eddington's book [Eddington (1924)], which was just published at that time [Ungar (2001), Sec. 1, Chap. 1].

The physical significance of the peculiar rotation in special relativity emerged in 1925 when Thomas relativistically re-computed the precessional frequency of the doublet separation in the fine structure of the atom, and thus rectified a missing factor of 1/2. This correction has come to be known as the *Thomas half*. Thomas' discovery of the relativistic precession of the electron spin on Christmas 1925 thus led to the understanding of the significance of the relativistic effect which became known as *Thomas precession*. Llewellyn Hilleth Thomas died in Raleigh, NC, on April 20, 1992. A paper [Chen and Ungar (2002a)] dedicated to the centenary of the birth of Llewellyn H. Thomas (1902–1992) describes the Bloch gyrovector of Chap. 9.

Once identified as $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, it is clear from its definition in (10.6) that Thomas precession owes its existence solely to the nonassociativity of Einstein addition of Einsteinian velocities. Accordingly, Thomas precession has no classical counterpart since the addition of classical, Newtonian velocities is associative.

It is widely believed that special relativistic effects are negligible when the velocities involved are much less than the vacuum speed of light c. Yet, Thomas precession effect in the orbital motion of spinning electrons in atoms is clearly observed in resulting spectral lines despite the speed of electrons in atoms being small compared with the speed of light. One may, therefore, ask whether it is possible to furnish a classical background to Thomas precession [MacKeown (1997)]. Hence, it is important to realize that Thomas precession stems from the nonassociativity of Einsteinian velocities, so that it has no echo in Newtonian velocities.

In 1966, Ehlers, Rindler and Robinson [Ehlers, Rindler and Robinson (1966)] proposed a new formalism for dealing with the Lorentz group. Their formalism, however, did not find its way to the mainstream literature.

Therefore, thirty three years later, two of them suggested considering the "notorious Thomas precession formula" (in their words, p. 431 in [Rindler and Robinson (1999)]) as an indicator of the quality of a formalism for dealing with the Lorentz group. The idea of Rindler and Robinson to use the "notorious Thomas precession formula" as an indicator works fine in the analytic hyperbolic geometric viewpoint of special relativity, where the ugly duckling of special relativity, the "notorious Thomas precession formula", becomes the beautiful swan of analytic hyperbolic geometry. The resulting gyro-algebra of the Lorentz group will be presented in Sec. 10.15.

10.6 The Relativistic Gyrovector Space

If integer scalar multiplication $n\otimes\mathbf{v}$ is defined in the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus)$ by the equation

$$n\otimes\mathbf{v} = \mathbf{v}\oplus \ldots \oplus\mathbf{v} \qquad (n \text{ gyroadditions}) \tag{10.15}$$

for any positive integer n and $\mathbf{v}\in\mathbb{R}^3_c$ then it follows from Einstein addition law, (10.3), that

$$n\otimes\mathbf{v} = c\frac{(1+\|\mathbf{v}\|/c)^n - (1-\|\mathbf{v}\|/c)^n}{(1+\|\mathbf{v}\|/c)^n + (1-\|\mathbf{v}\|/c)^n}\frac{\mathbf{v}}{\|\mathbf{v}\|} \tag{10.16}$$

for any positive integer n and $\mathbf{v}\in\mathbb{R}^3_c$. Suggestively, the scalar multiplication that Einstein addition admits in the relativistic velocity gyrogroup $(\mathbb{R}^3_c, \oplus)$ is defined by the equation, (6.234),

$$\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r - (1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r + (1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\tanh\left(r\tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned} \tag{10.17}$$

where r is any real number, $r\in\mathbb{R}$, $\mathbf{v}\in\mathbb{R}^3_c$, $\mathbf{v}\neq\mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

Einstein scalar multiplication turns the Einstein gyrogroup $(\mathbb{R}^3_c, \oplus)$ of relativistically admissible velocities into a gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ that possesses the following properties.

$$
\begin{aligned}
1\otimes\mathbf{v} &= \mathbf{v} && \\
(r_1 + r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\
(r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law}
\end{aligned}
$$

for all real numbers $r, r_1, r_2 \in\mathbb{R}$ and admissible velocities $\mathbf{v}\in\mathbb{R}^3_c$.

Unlike vector spaces, the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ does not possess a distributive law since, in general,

$$r\otimes(\mathbf{u}\oplus\mathbf{v}) \neq r\otimes\mathbf{u}\oplus r\otimes\mathbf{v} \tag{10.18}$$

for $r\in\mathbb{R}$ and $\mathbf{u}, \mathbf{v}\in\mathbb{R}^3_c$.

Remarkably, the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ of Einsteinian velocities with its gyrodistance function given by the equation

$$d_\oplus(\mathbf{u}, \mathbf{v}) = \|\mathbf{u}\ominus\mathbf{v}\| \tag{10.19}$$

$\mathbf{u}, \mathbf{v}\in\mathbb{R}^3_c$, forms the setting for the Beltrami ball model of 3-dimensional hyperbolic geometry just as the vector space $(\mathbb{R}^3, +, \cdot)$ of Newtonian velocities with its Euclidean distance function

$$d_+(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| \tag{10.20}$$

forms the setting for the standard model of 3-dimensional Euclidean geometry.

The connection between Einstein velocity addition and the Beltrami ball model of 3-dimensional hyperbolic geometry, already noted by Fock [Fock (1964), p. 39], follows from the Einstein gyrodistance function (10.19) between two neighboring points in $\mathbb{R}^3_c$, the square of which is given by the equation

$$ds^2 = \|\mathbf{x}\ominus(\mathbf{x} + d\mathbf{x})\|^2 \tag{10.21}$$

The squared Einsteinian distance ds^2 in (10.21) between a point $\mathbf{x}$ and its infinitesimally close neighboring point $\mathbf{x} + d\mathbf{x}$ in $\mathbb{R}^3_c$ turns out to be a Riemannian line element. The resulting Riemannian line element, presented in (7.55), is called the *Beltrami–Riemannian line element* and is recognized as the Riemannian line element of the Beltrami ball model of hyperbolic geometry.

The Euclidean rigid motions of the Euclidean space $\mathbb{R}^3$ are the transformations of $\mathbb{R}^3$ that keep the Euclidean distance function (10.20) invariant.

These transformations, called isometries, are translations $\mathbf{x} \to \mathbf{v} + \mathbf{x}$, and rotations, $\mathbf{x} \to V\mathbf{x}$, where $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3$ and $V \in SO(3)$. Similarly, The hyperbolic rigid motions of the hyperbolic ball space $\mathbb{R}^3_c$ are the transformations of $\mathbb{R}^3_c$ that keep the hyperbolic distance function (10.19) invariant. These isometries are left gyrotranslations $\mathbf{x} \to \mathbf{v} \oplus \mathbf{x}$, and rotations, $\mathbf{x} \to V\mathbf{x}$, where $\mathbf{v}, \mathbf{x} \in \mathbb{R}^3_c$ and $V \in SO(3)$.

10.7 Gyrogeodesics, Gyromidpoints and Gyrocentroids

A point $\mathbf{v}$ in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$, or the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$ shown in Fig. 10.1, represents all the inertial frames $\Sigma_{\mathbf{v}}$ with relativistically admissible velocity $\mathbf{v}$ relative to a rest frame $\Sigma_{\mathbf{0}}$. Accordingly, the relativistic velocity of frame $\Sigma_{\mathbf{v}}$ relative to frame $\Sigma_{\mathbf{u}}$ is $\ominus\mathbf{u} \oplus \mathbf{v}$, and the relativistic velocity of frame $\Sigma_{\mathbf{u}}$ relative to frame $\Sigma_{\mathbf{v}}$ is $\ominus\mathbf{v} \oplus \mathbf{u}$. Remarkably, these two Einsteinian reciprocal velocities are not reciprocal in the classical sense since they are related by the identity

$$\ominus\mathbf{u} \oplus \mathbf{v} = \ominus(\mathbf{u} \ominus \mathbf{v}) = \ominus \mathrm{gyr}[\mathbf{u}, \ominus\mathbf{v}](\ominus\mathbf{v} \oplus \mathbf{u}) \tag{10.22}$$

that involves a Thomas precession. Identity (10.22) is obtained by employing the *gyroautomorphic inverse property*, Def. 3.1 and Theorem 3.2, of Einstein addition, according to which $\ominus(\mathbf{a} \oplus \mathbf{b}) = \ominus\mathbf{a} \ominus \mathbf{b}$, and the gyrocommutative law of Einstein addition. The classical counterpart of (10.22), $-\mathbf{u} + \mathbf{v} = -(-\mathbf{v} + \mathbf{u})$, is known as the principle of reciprocity.

For any points $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, and the real parameter $t \in \mathbb{R}$ that may represent "time", $-\infty < t < \infty$, the gyroline

$$\mathbf{u} \oplus (\ominus\mathbf{u} \oplus \mathbf{v}) \otimes t \tag{10.23}$$

traces a geodesic line in the Beltrami ball model of hyperbolic geometry. The gyrosegment $\mathbf{uv}$, given by (10.23) with $0 \le t \le 1$, is shown in Fig. 10.1.The gyroline (10.23) is the unique gyrogeodesic passing through the points $\mathbf{u}$ and $\mathbf{v}$. It passes through the point $\mathbf{u}$ at "time" $t = 0$, and through the point $\mathbf{v}$ at "time" $t = 1$. Geodesics in the Beltrami ball model of hyperbolic geometry are the analogue of straight lines in Euclidean geometry, representing paths that minimize arc lengths measured by the distance function (10.19).

For $t = 1/2$ in (10.23) we have the hyperbolic midpoint, or the gyro-

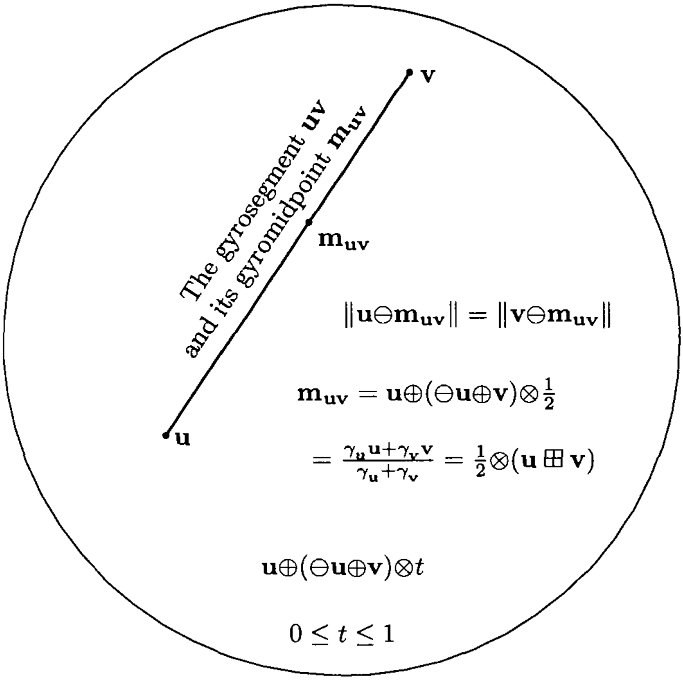


Fig. 10.1 The Einstein Gyrovector Plane. Points of the disc $\mathbb{R}_c^2 = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < c\}$ represent two-dimensional relativistically admissible velocities $\mathbf{v}$ relative to some inertial rest frame with velocity $\mathbf{0}$. The latter, accordingly, is represented by the origin of the disc. While the disc origin is distinguished in the Euclidean geometry of the disc, it is indistinguishable in its hyperbolic geometry. The disc $\mathbb{R}_c^2$ is endowed with Einstein addition $\oplus$ and scalar multiplication $\otimes$, giving rise to the Einstein gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$, that turns out in Section 7.5 to be the Beltrami disc model of two-dimensional hyperbolic geometry. The geodesic segment, or gyrosegment, $\mathbf{uv}$ joining the points $\mathbf{u}$ and $\mathbf{v}$ in the Einstein relativistic gyrovector plane $(\mathbb{R}_c^2, \oplus, \otimes)$ and the gyromidpoint $\mathbf{m_{uv}} = \mathbf{m}_{\mathbf{uv}}^{hyperbolic}$ between $\mathbf{u}$ and $\mathbf{v}$ are shown. The expressions that generate the gyrosegment $\mathbf{uv}$ and its gyromidpoint $\mathbf{m_{uv}}$ exhibit obvious analogies with their Euclidean counterparts.

midpoint, $\mathbf{m_{uv}}$,

$$\mathbf{m_{uv}} = \mathbf{u} \oplus (\ominus \mathbf{u} \oplus \mathbf{v}) \otimes \tfrac{1}{2} \tag{10.24}$$

Figure 10.1, satisfying, (6.86),

$$\mathbf{m_{uv}} = \mathbf{m_{vu}} \tag{10.25}$$

and, (6.87),

$$d_\oplus(\mathbf{u}, \mathbf{m_{uv}}) = d_\oplus(\mathbf{v}, \mathbf{m_{uv}}) \tag{10.26}$$

The gyromidpoint $\mathbf{m_{uv}}$ can also be written as, (6.300),

$$\mathbf{m_{uv}} = \frac{\gamma_\mathbf{u}\mathbf{u} + \gamma_\mathbf{v}\mathbf{v}}{\gamma_\mathbf{u} + \gamma_\mathbf{v}} \tag{10.27}$$

and as, (6.91),

$$\mathbf{m_{uv}} = \tfrac{1}{2}\otimes(\mathbf{u} \boxplus \mathbf{v}) \tag{10.28}$$

so that in the Newtonian limit, $c \to \infty$, the gyromidpoint in (10.24)–(10.28) reduces to its Euclidean counterpart, the midpoint $(\mathbf{u} + \mathbf{v})/2$.

10.8 The Midpoint and the Gyromidpoint – Newtonian and Einsteinian Mechanical Interpretation

The relativistic mass is the key to unlocking the secret of an old problem in hyperbolic geometry, the determination of various hyperbolic centroids, called gyrocentroids, like the gyrocentroid of the hyperbolic triangle [Bottema (1958)].

Let us consider two particles with equal masses m moving with Newtonian velocities $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Their respective momenta are $m\mathbf{u}$ and $m\mathbf{v}$ so that their classical CM velocity is the point, Fig. 10.2,

$$\mathbf{c}_\mathbf{uv}^{newtonian} = \frac{m\mathbf{u} + m\mathbf{v}}{m + m} = \frac{\mathbf{u} + \mathbf{v}}{2} \tag{10.29}$$

in the Newtonian velocity space $\mathbb{R}^3$.

It turns out that the Newtonian CM velocity (10.29) coincides with the Euclidean midpoint, Fig. 10.2,

$$\mathbf{m}_\mathbf{uv}^{euclidean} = \frac{\mathbf{u} + \mathbf{v}}{2} \tag{10.30}$$

of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. The Euclidean midpoint (10.30) thus has the Newtonian classical mechanical interpretation (10.29) as a classical center of mass [Hausner (1998); Berger (1987)] or, equivalently, a classical CM velocity,

$$\mathbf{m}_\mathbf{uv}^{euclidean} = \mathbf{c}_\mathbf{uv}^{newtonian} \tag{10.31}$$

By analogy with (10.29) and (10.30) let us consider two particles with equal rest masses m moving with Einsteinian velocities $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$. Their

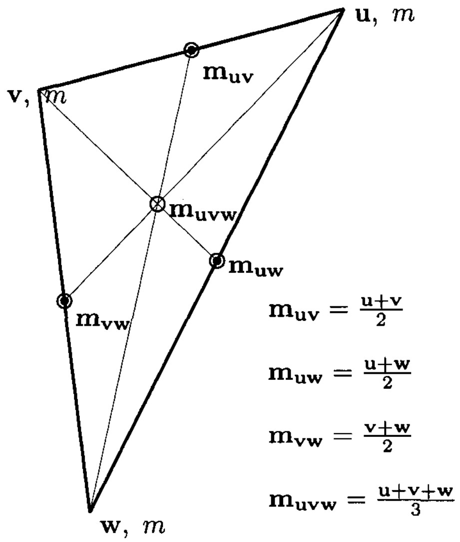


Fig. 10.2 A particle with mass m is located at each of the three vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ of triangle $\mathbf{uvw}$ in the Euclidean 3-space $\mathbb{R}^3$ of Newtonian velocities. The midpoints of the sides of triangle $\mathbf{uvw}$ are $\mathbf{m_{uv}}$, $\mathbf{m_{uw}}$ and $\mathbf{m_{vw}}$. The centroid $\mathbf{m_{uvw}}$ of triangle $\mathbf{uvw}$ is equal to the CM velocity of the three particles that are moving with Newtonian velocities $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ relative to some inertial rest frame. This mechanical interpretation of the Euclidean triangle centroid is well-known [Hausner (1998); Krantz (2003)][Berger (1987), Fig. 3.4.10.1, p. 79]. A straightforward extension of the interpretation to the relativistic regime, based on the concept of the relativistic mass, is shown in Fig. 10.3.

respective momenta must be relativistically corrected so that they are, respectively, $m\gamma_{\mathbf{u}}\mathbf{u}$ and $m\gamma_{\mathbf{v}}\mathbf{v}$ [Tsai (1986)]. Their CM velocity in the Einsteinian velocity space $\mathbb{R}^3_c$ is, accordingly, the point

$$\begin{aligned} \mathbf{c}_{\mathbf{uv}}^{einsteinian} &= \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \end{aligned} \tag{10.32}$$

shown in Fig. 10.3.

It turns out that the Einsteinian CM velocity (10.32) coincides with the gyromidpoint (10.27),

$$\mathbf{m}_{\mathbf{uv}}^{hyperbolic} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \tag{10.33}$$

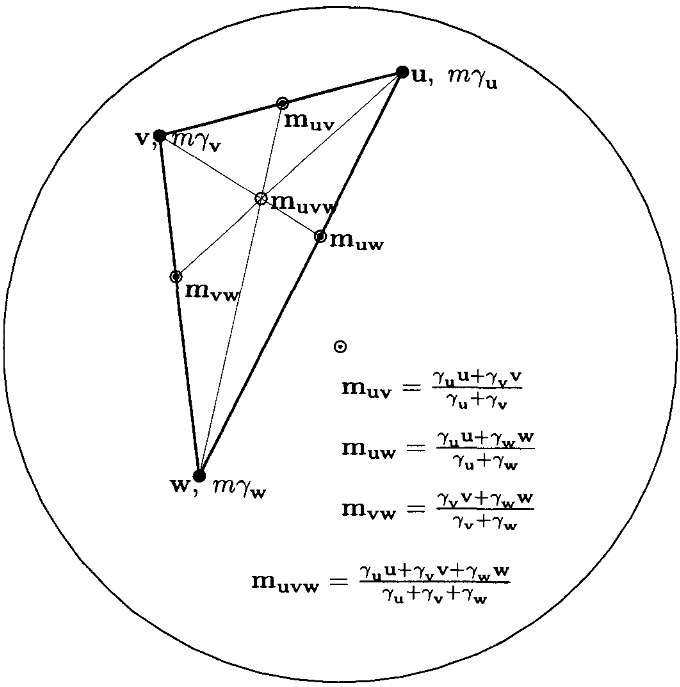


Fig. 10.3 Putting to rest the relativistic mass misconceptions. The hyperbolic centroid, gyrocentroid, $\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \mathbf{m}_{\mathbf{uvw}}$ of a gyrotriangle $\mathbf{uvw}$ in the Einsteinian velocity gyrovector space $(\mathbb{R}_c^2, \oplus, \otimes)$ coincides with the CM velocity of three particles with equal rest masses m situated at the vertices $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$, of the triangle $\mathbf{uvw}$, as shown in (10.37). The analogous Newtonian counterpart is obvious, Fig. 10.2, and is recovered in the Newtonian limit, $c \to \infty$.

of $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$, shown in Fig. 10.1. The gyromidpoint (10.33) thus has the Einsteinian relativistic mechanical interpretation (10.32) as a relativistic CM velocity,

$$\mathbf{m}_{\mathbf{uv}}^{hyperbolic} = \mathbf{c}_{\mathbf{uv}}^{einsteinian} \tag{10.34}$$

in full analogy with the Newtonian classical mechanical interpretation (10.29) of the Euclidean midpoint (10.30), presented in (10.31).

The analogy (10.31) $\leftrightarrow$ (10.34) between the pairs (10.1) is shown in Figs. 10.2 and 10.3. It demonstrates that the relativistic mass possesses hyperbolic geometric, or gyrogeometric, significance along with its well-known physical significance.

Figure 10.3 illustrates the hyperbolic/Einsteinian interpretation of the gyrotriangle gyrocentroid in a way fully analogous to the Euclidean/Newtonian interpretation of the triangle centroid in Fig. 10.2. In Fig. 10.3 we extend the observations made in Fig. 10.1 from the gyromidpoint $\mathbf{m_{uv}}$ of a gyrosegment $\mathbf{uv}$, determined by its two endpoints $\mathbf{u}$ and $\mathbf{v}$, to the gyrocentroid of a gyrotriangle $\mathbf{uvw}$, determined by its three vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

The three vertices of the gyrotriangle $\mathbf{uvw}$ in Fig. 10.3 represent three particles with equal rest masses m moving with relativistic velocities $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^3$ relative to a rest frame Σ_0. Relativistically corrected, the relativistic momenta of the three particles are, respectively, $m\gamma_{\mathbf{u}}\mathbf{u}$, $m\gamma_{\mathbf{v}}\mathbf{v}$, and $m\gamma_{\mathbf{w}}\mathbf{w}$, so that their relativistic CM velocity is

$$\begin{aligned}\mathbf{c}_{\mathbf{uvw}}^{einsteinian} &= \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v} + m\gamma_{\mathbf{w}}\mathbf{w}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}} + m\gamma_{\mathbf{w}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}}\end{aligned} \tag{10.35}$$

This turns out, by Theorem 6.88, to be the gyrocentroid $\mathbf{m_{uvw}} = \mathbf{m}_{\mathbf{uvw}}^{hyperbolic}$,

$$\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}}} \tag{10.36}$$

of the gyrotriangle $\mathbf{uvw}$, Fig. 10.3.

The gyrosegment in gyrotriangle $\mathbf{uvw}$, Fig. 10.3, joining a gyromidpoint of a side with its opposite vertex is called a *gyromedian.* As in Euclidean geometry, the three gyrotriangle gyromedians are concurrent [Greenberg (1993)], the point of concurrency being the gyrotriangle gyrocentroid. Figure 10.3 shows that, as expected, the three gyromedians of the gyrotriangle $\mathbf{uvw}$ are concurrent, giving rise to the gyrocentroid of the gyrotriangle $\mathbf{uvw}$.

The gyrotriangle gyrocentroid $\mathbf{m}_{\mathbf{uvw}}^{hyperbolic}$, (10.36), of gyrotriangle $\mathbf{uvw}$ in the Einstein gyrovector space $(\mathbb{R}_c^3, \oplus, \otimes)$ is calculated by standard methods of elementary linear algebra for determining line intersections, owing to the result that gyrolines in the Beltrami model are Euclidean straight lines.

Following (10.35) and (10.36) we have

$$\mathbf{m}_{\mathbf{uvw}}^{hyperbolic} = \mathbf{c}_{\mathbf{uvw}}^{einsteinian} \tag{10.37}$$

Equation (10.37) extends the identity in (10.34) from a system of two particles with equal rest masses and velocities $\mathbf{u}$ and $\mathbf{v}$ to a system of three particles with equal rest masses and velocities $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$. It demonstrates that the gyrotriangle gyrocentroid can be interpreted as a relativistic CM velocity of three moving particles with equal rest masses.

The Euclidean analogue of (10.36) is clearly the Euclidean centroid [Hausner (1998); Krantz (2003)],

$$\mathbf{m}_{\mathbf{uvw}}^{euclidean} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3} \tag{10.38}$$

of the Euclidean triangle $\mathbf{uvw}$, also known as the triangle barycenter [Berger (1987), p. 79].

Moreover, the Newtonian-classical analogue of the Einsteinian-relativistic CM velocity (10.35) is the Newtonian-classical CM velocity of three particles with equal masses, moving with Newtonian velocities $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ relative to a rest frame Σ_0,

$$\mathbf{c}_{\mathbf{uvw}}^{newtonian} = \frac{\mathbf{u} + \mathbf{v} + \mathbf{w}}{3} \tag{10.39}$$

Hence, by (10.38) and (10.39), we have

$$\mathbf{m}_{\mathbf{uvw}}^{euclidean} = \mathbf{c}_{\mathbf{uvw}}^{newtonian} \tag{10.40}$$

thus obtaining the classical counterpart of (10.37). Identity (10.40) demonstrates that the Euclidean triangle centroid can be interpreted as a classical CM velocity. The analogy (10.40) $\leftrightarrow$ (10.37) extends the analogy (10.31) $\leftrightarrow$ (10.34) between the pairs in (10.1) from a system of two particles to a system of three particles.

Further extension to four particles is shown in Figs. 10.4 and 10.5. In these figures we see four particles with equal rest masses m, represented by the points $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ of the Einstein gyrovector space $(\mathbb{R}_c^3, \oplus, \otimes)$ of Einsteinian velocities. The four particles, accordingly, have the respective relativistic velocities $\mathbf{u}, \mathbf{v}, \mathbf{w}$, and $\mathbf{x}$. The relativistic CM velocity is given by the equation

$$\begin{aligned} \mathbf{c}_{\mathbf{uvwx}}^{einsteinian} &= \frac{m\gamma_{\mathbf{u}}\mathbf{u} + m\gamma_{\mathbf{v}}\mathbf{v} + m\gamma_{\mathbf{w}}\mathbf{w} + m\gamma_{\mathbf{x}}\mathbf{x}}{m\gamma_{\mathbf{u}} + m\gamma_{\mathbf{v}} + m\gamma_{\mathbf{w}} + m\gamma_{\mathbf{x}}} \\ &= \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{x}}\mathbf{x}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{x}}} \end{aligned} \tag{10.41}$$

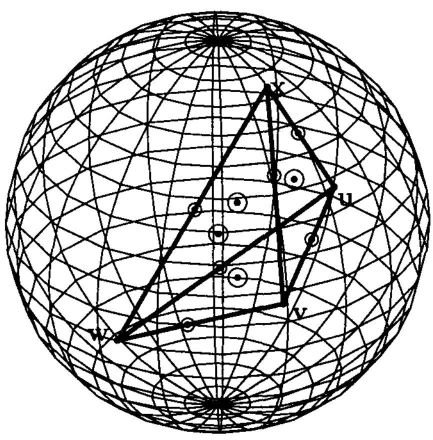


Fig. 10.4 The hyperbolic tetrahedron **uvwx**, called a gyrotetrahedron, is shown in the Einstein gyrovector space $\mathbb{R}^3_c = (\mathbb{R}^3_c, \oplus, \otimes)$, which underlies the Beltrami ball model of hyperbolic geometry, as explained in Sec. 7.5. The gyrotetrahedron **uvwx** is shown inside the c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$ where it lives. Its vertices are $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x} \in \mathbb{R}^3_c$; and its faces are gyrotriangles. Gyromidpoints, gyromedians, and gyrocentroids in the gyrotetrahedron are shown in Fig. 10.5.

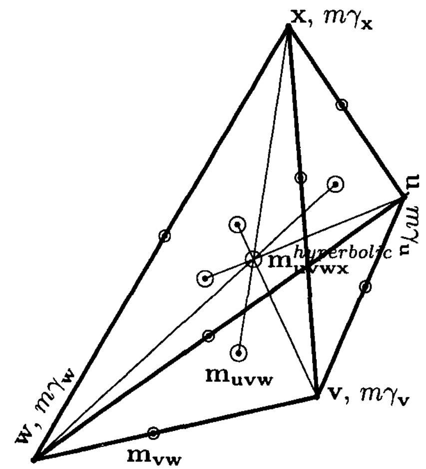


Fig. 10.5 The gyromidpoints of the 6 sides and the gyrocentroids of the four faces of the gyrotetrahedron in Fig. 10.4 are shown. The gyroline joining a vertex of a gyrotetrahedron and the gyrocentroid of the opposite face is called a gyrotetrahedron gyromedian. The four gyromedians of the gyrotetrahedron **uvwx**, determined by the four indicated gyrocentroids, are concurrent. The point of concurrency is the gyrotetrahedron gyrocentroid $\mathbf{m}_{\mathbf{uvwx}}$ given by (10.41).

and the hyperbolic centroid of the hyperbolic tetrahedron **uvwx** is

$$\mathbf{m}^{hyperbolic}_{\mathbf{uvwx}} = \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w} + \gamma_{\mathbf{x}}\mathbf{x}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}} + \gamma_{\mathbf{w}} + \gamma_{\mathbf{x}}} \tag{10.42}$$

so that, Fig. 10.5,

$$\mathbf{m}^{hyperbolic}_{\mathbf{uvwx}} = \mathbf{c}^{einsteinian}_{\mathbf{uvwx}} \tag{10.43}$$

Understanding the hyperbolic geometry that underlies (10.42) enables us to improve our understanding of the physics of the relativistic CM velocity (10.41).

As an example illustrating the physical understanding that we gain from geometry, we note that it follows from the geometric significance of the hyperbolic triangle centroid that it remains covariant under the hyperbolic rigid motions. Let us therefore left gyrotranslate the points in (10.42) by $\ominus\mathbf{y}\in\mathbb{R}^3_c$, obtaining

$$\begin{aligned}\ominus\mathbf{y}\oplus\mathbf{m}_{\mathbf{uvwx}}^{hyperbolic}&\\ =\frac{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}(\ominus\mathbf{y}\oplus\mathbf{u})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}(\ominus\mathbf{y}\oplus\mathbf{v})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}(\ominus\mathbf{y}\oplus\mathbf{w})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}(\ominus\mathbf{y}\oplus\mathbf{x})}{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}}&\end{aligned} \tag{10.44}$$

as explained in Figs. 6.16–6.17, pp. 204–206.

Hence, by means of the equality (10.43) between hyperbolic triangle centroids, gyrocentroids, and the relativistic CM velocity we have from (10.44),

$$\begin{aligned}\ominus\mathbf{y}\oplus\mathbf{c}_{\mathbf{uvwx}}^{einsteinian}&\\ =\frac{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}(\ominus\mathbf{y}\oplus\mathbf{u})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}(\ominus\mathbf{y}\oplus\mathbf{v})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}(\ominus\mathbf{y}\oplus\mathbf{w})+\gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}(\ominus\mathbf{y}\oplus\mathbf{x})}{\gamma_{\ominus\mathbf{y}\oplus\mathbf{u}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{v}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{w}}+\gamma_{\ominus\mathbf{y}\oplus\mathbf{x}}}&\end{aligned} \tag{10.45}$$

so that (hyperbolic) geometric significance implies (relativistic) physical significance.

Clearly, (10.41) gives the relativistic CM velocity as measured by an observer who is at rest relative to the rest frame Σ_0 and, similarly, (10.45) gives the relativistic CM velocity as seen by an observer who is at rest relative to the inertial frame $\Sigma_{\mathbf{y}}$. The relativistic CM velocity, as a result, is observer covariant in the same way that the hyperbolic centroid, or gyrocentroid, of the hyperbolic tetrahedron, the gyrotetrahedron in Figs. 10.4 and 10.5, is covariant under the hyperbolic rigid motions of the gyrotetrahedron. Contrasting the opinion in [Adler (1987), pp. 742–743], the hyperbolic geometric interpretation of the relativistic CM velocity demonstrates that Einstein's relativistic mass is an asset rather than a liability. Hence, it will be no longer easy to dismiss Einstein's relativistic mass as an artifact. Hyperbolic geometric interpretation of diverse physical phenomena are found in [Dubrovskii, Smorodinskii and Surkov (1984)].

10.9 The Einstein Gyroparallelogram

An Einstein gyroparallelogram is a gyroparallelogram, Def. 6.40, in an Einstein gyrovector space.

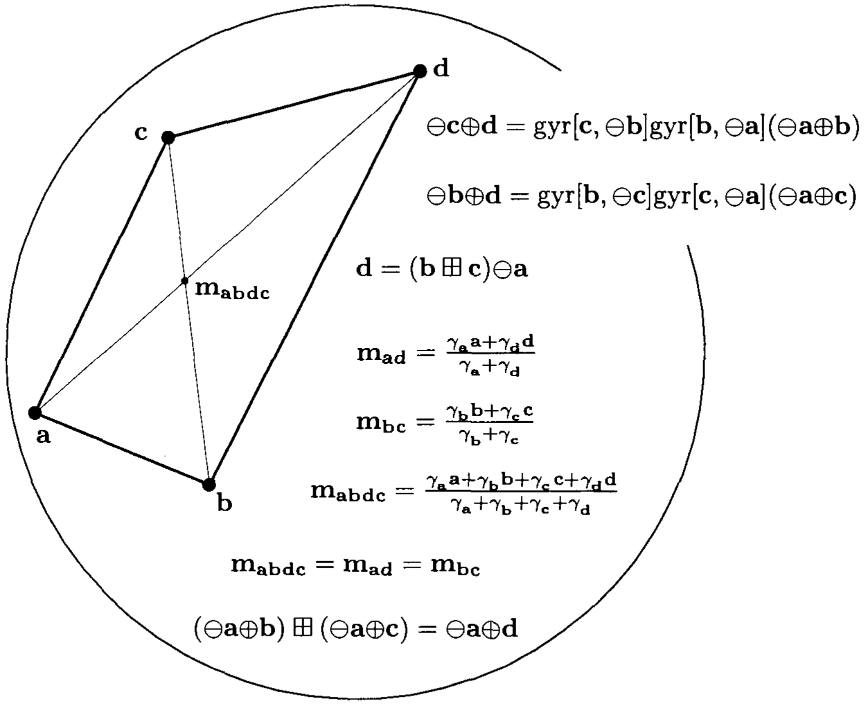


Fig. 10.6 The Einstein gyroparallelogram. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three nongyrocollinear points in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$, $\mathbb{V}_s$ being the s-ball of the real inner product space $(\mathbb{V}, +, \cdot)$, and let $\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a}$. Then the four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of the Einstein gyroparallelogram $\mathbf{abdc}$, Def. 6.40, and, by Theorem 6.45, *opposite sides are equal modulo gyrations. Shown are three expressions for the gyrocenter* $\mathbf{m_{abdc}} = \mathbf{m_{ad}} = \mathbf{m_{bc}}$ of the Einstein gyroparallelogram $\mathbf{abdc}$, which can be obtained by CM velocity considerations.

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three nongyrocollinear points in an Einstein gyrovector space $(G, \oplus, \otimes)$, and let $\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a}$. Then the four points $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are the vertices of the Einstein gyroparallelogram $\mathbf{abdc}$, Def. 6.40, with gyrocenter $\mathbf{m_{abdc}}$, Fig. 10.6. The two diagonals, $\mathbf{ad}$ and $\mathbf{bc}$, of the gyroparallelogram intersect at their gyromidpoints $\mathbf{m_{ad}}$ and $\mathbf{m_{bc}}$, Fig. 10.6,

$$\begin{aligned}\mathbf{m_{ad}} &= \tfrac{1}{2}\otimes(\mathbf{a} \boxplus \mathbf{d})\\ \mathbf{m_{bc}} &= \tfrac{1}{2}\otimes(\mathbf{b} \boxplus \mathbf{c})\\ \mathbf{m_{ad}} &= \mathbf{m_{bc}} = \mathbf{m_{abdc}}\end{aligned} \tag{10.46}$$

By CM velocity considerations similar to those shown in Fig. 10.3, the

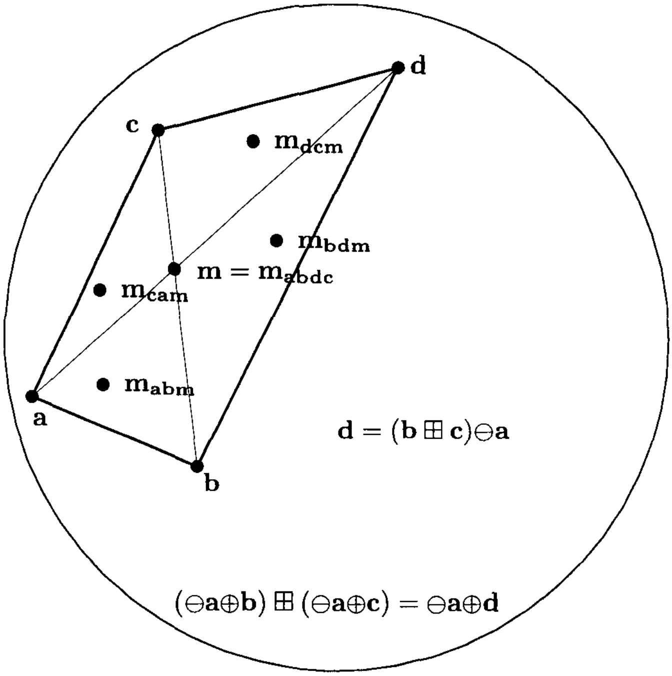


Fig. 10.7 The two diagonals **ad** and **bc** of a gyroparallelogram **abdc** intersect at the gyroparallelogram gyrocenter $\mathbf{m} = \mathbf{m_{abdc}}$ and divide the gyroparallelogram into the four gyrotriangles **bcd**, **acd**, **abd** and **abc**. The gyrocentroids of these gyrotriangles, which are, respectively, $\mathbf{m_{abm}}$, $\mathbf{m_{bdm}}$, $\mathbf{m_{dcm}}$ and $\mathbf{m_{cam}}$, form a gyroparallelogram. Interestingly, the gyroparallelogram gyrocenter $\mathbf{m}$ is the midpoint of the two opposite gyrotriangle gyrocentroids $\mathbf{m_{abm}}$ and $\mathbf{m_{dcm}}$ ($\mathbf{m_{bdm}}$ and $\mathbf{m_{cam}}$).

gyrocenter $\mathbf{m_{abcd}}$ of the Einstein gyroparallelogram **abdc** is given by each of the following three expressions, Fig. 10.6.

$$
\begin{aligned}
\mathbf{m_{ad}} &= \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{d}}} \\
\mathbf{m_{bc}} &= \frac{\gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c}}{\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}} \\
\mathbf{m_{abdc}} &= \frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c} + \gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{b}} + \gamma_{\mathbf{c}} + \gamma_{\mathbf{d}}}
\end{aligned}
\tag{10.47}
$$

The gyroparallelogram gyrocenter $\mathbf{m} = \mathbf{m_{abdc}}$ in Fig. 10.7 is (i) the gyromidpoint of the two opposite gyrotriangle gyrocentroids $\mathbf{m_{abm}}$ and

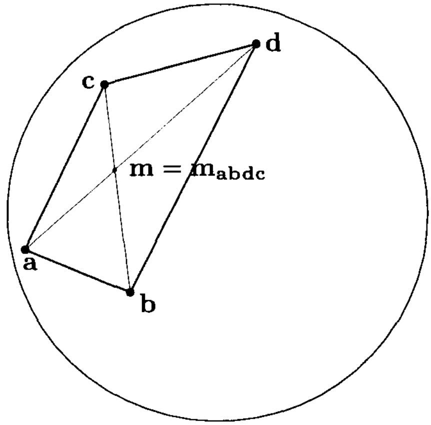


Fig. 10.8 The gyroparallelogram of the previous figure, Fig. 10.7, before left gyrotranslating it by $\ominus\mathbf{m}$ to the origin of its Einstein gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ in Fig. 10.9. To the Euclidean eye the gyroparallelogram does not look like a Euclidean parallelogram.

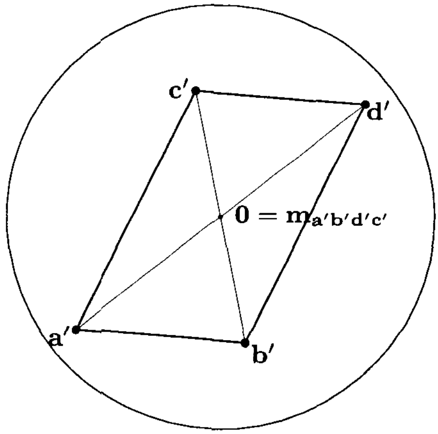


Fig. 10.9 The gyroparallelogram of the previous figure, Fig. 10.8, has been left gyrotranslated in this figure so that its gyrocenter coincides with the origin of its Einstein gyrovector plane. As such, the gyroparallelogram now looks as a Euclidean parallelogram.

$\mathbf{m_{dcm}}$, and (ii) the gyromidpoint of the two opposite gyrotriangle gyrocentroids $\mathbf{m_{bdm}}$ and $\mathbf{m_{cam}}$ as shown in Fig. 10.7. Hence, it follows from the Einstein gyromidpoint identity (10.27) that the gyroparallelogram gyrocenter and the four gyrotriangle gyrocentroids in Fig. 10.7 are related by the two equations

$$\mathbf{m_{abdc}} = \frac{\gamma_{\mathbf{m_{abm}}}\mathbf{m_{abm}} + \gamma_{\mathbf{m_{dcm}}}\mathbf{m_{dcm}}}{\gamma_{\mathbf{m_{abm}}} + \gamma_{\mathbf{m_{dcm}}}} \tag{10.48}$$

and

$$\mathbf{m_{abdc}} = \frac{\gamma_{\mathbf{m_{bdm}}}\mathbf{m_{bdm}} + \gamma_{\mathbf{m_{cam}}}\mathbf{m_{cam}}}{\gamma_{\mathbf{m_{bdm}}} + \gamma_{\mathbf{m_{cam}}}} \tag{10.49}$$

As a result, the four gyrotriangle gyrocentroids, $\mathbf{m_{abm}}$, $\mathbf{m_{bdm}}$, $\mathbf{m_{dcm}}$ and $\mathbf{m_{cam}}$, in Fig. 10.7 form a gyroparallelogram that shares its gyrocenter with the original gyroparallelogram $\mathbf{abdc}$.

To recognize graphically an Einstein gyroparallelogram with gyrocenter $\mathbf{m}$ in an Einstein gyrovector plane one may left gyrotranslate it by $\ominus\mathbf{m}$. The resulting gyroparallelogram has gyrocenter $\mathbf{m} = \mathbf{0}$. The left gyrotranslated gyroparallelogram then looks like a Euclidean parallelogram, as shown in

Figs. 10.8 and 10.9, where $\mathbf{m} = \mathbf{m_{abdc}}$ is given by (10.47), and

$$\begin{aligned} \mathbf{a}' &= \ominus\mathbf{m}\oplus\mathbf{a} \\ \mathbf{b}' &= \ominus\mathbf{m}\oplus\mathbf{b} \\ \mathbf{c}' &= \ominus\mathbf{m}\oplus\mathbf{c} \\ \mathbf{d}' &= \ominus\mathbf{m}\oplus\mathbf{d} \end{aligned} \tag{10.50}$$

In the following theorem we will find that any vertex of an Einstein gyroparallelogram is a linear combination of the other three vertices, with coefficients that are expressed in terms of the gamma factors of the three vertices and their gyrodifferences.

Theorem 10.1 *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_s$ *be any three points of an Einstein gyrovector space* $(\mathbb{V}_s, \oplus, \otimes)$ *of the s-ball of an inner product space* $(\mathbb{V}, +, \cdot)$, *and let* $\mathbf{abdc}$ *be their gyroparallelogram, Fig. 10.6. Then*

$$\frac{\gamma_{\mathbf{a}}\mathbf{a} + \gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{a}} + \gamma_{\mathbf{d}}} = \frac{\gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c}}{\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}} \tag{10.51}$$

$$\mathbf{d} = \frac{\gamma_{\mathbf{a}}(1 + \gamma_{\mathbf{b}\ominus\mathbf{c}})\mathbf{a} - (\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{a}\ominus\mathbf{c}})(\gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c})}{\gamma_{\mathbf{a}}(1 + \gamma_{\mathbf{b}\ominus\mathbf{c}}) - (\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{a}\ominus\mathbf{c}})(\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}})} \tag{10.52}$$

and

$$\gamma_{\mathbf{d}} = -\gamma_{\mathbf{a}} + \frac{(\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{a}\ominus\mathbf{c}})(\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}})}{1 + \gamma_{\mathbf{b}\ominus\mathbf{c}}} \tag{10.53}$$

Proof. By the gyroparallelogram condition, Def. 6.40, we have

$$\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c})\ominus\mathbf{a} \tag{10.54}$$

implying, by a right cancellation,

$$\mathbf{a} \boxplus \mathbf{d} = \mathbf{b} \boxplus \mathbf{c} \tag{10.55}$$

Identity (10.51) follows from (10.55) by (3.156).

Solving (10.51) for $\mathbf{d}$ as a function of $\mathbf{a}, \mathbf{b}, \mathbf{c}$, $\gamma_{\mathbf{a}}, \gamma_{\mathbf{b}}, \gamma_{\mathbf{c}}$ and $\gamma_{\mathbf{d}}$, and employing the identity $\|\mathbf{d}\|^2 = (\gamma_{\mathbf{d}}^2 - 1)/\gamma_{\mathbf{d}}^2$ and similar ones for $\mathbf{a}, \mathbf{b}, \mathbf{c}$, gives an equation for $\gamma_{\mathbf{d}}$ in terms of $\gamma_{\mathbf{a}}, \gamma_{\mathbf{b}}, \gamma_{\mathbf{c}}$, $\mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{b}\cdot\mathbf{c}$. The inner product $\mathbf{a}\cdot\mathbf{b}$ can be expressed by (3.145) in terms of $\gamma_{\mathbf{a}}, \gamma_{\mathbf{b}}$ and $\gamma_{\mathbf{a}\ominus\mathbf{b}}$ and similarly for $\mathbf{a}\cdot\mathbf{c}$ and $\mathbf{b}\cdot\mathbf{c}$. These allow one to express $\gamma_{\mathbf{d}}$ in terms of $\gamma_{\mathbf{a}}, \gamma_{\mathbf{b}}, \gamma_{\mathbf{c}}$, $\gamma_{\mathbf{a}\ominus\mathbf{b}}$, $\gamma_{\mathbf{a}\ominus\mathbf{c}}$ and $\gamma_{\mathbf{b}\ominus\mathbf{c}}$, obtaining (10.53). Finally, (10.52) follows from (10.51) and (10.53). $\square$

One should note that Identities (10.52) and (10.53) of Theorem 10.1 can be viewed as the unique solution of (10.51) for the two dependent unknowns $\mathbf{d}$ and its gamma factor $\gamma_{\mathbf{d}}$.

Identity (10.53) can be written as

$$\gamma_{\mathbf{a}} + \gamma_{\mathbf{d}} = \frac{\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{a}\ominus\mathbf{c}}}{1 + \gamma_{\mathbf{b}\ominus\mathbf{c}}}(\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}}) \tag{10.56}$$

Owing to the gyroparallelogram symmetry between the two pairs of opposite vertices, (10.56) implies its symmetric copy

$$\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}} = \frac{\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{b}\ominus\mathbf{d}}}{1 + \gamma_{\mathbf{a}\ominus\mathbf{d}}}(\gamma_{\mathbf{a}} + \gamma_{\mathbf{d}}) \tag{10.57}$$

so that

$$\frac{\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{a}\ominus\mathbf{c}}}{1 + \gamma_{\mathbf{b}\ominus\mathbf{c}}} \frac{\gamma_{\mathbf{a}\ominus\mathbf{b}} + \gamma_{\mathbf{b}\ominus\mathbf{d}}}{1 + \gamma_{\mathbf{a}\ominus\mathbf{d}}} = 1 \tag{10.58}$$

Theorem 10.2 *Let $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{V}_s$ be the four vertices of a gyroparallelogram* $\mathbf{abdc}$ *in an Einstein gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$ of the s-ball of an inner product space $(\mathbb{V}, +, \cdot)$, Fig. 10.6. Then the two diagonal gyrolengths, $\|\ominus\mathbf{a}\oplus\mathbf{d}\|$ and $\|\ominus\mathbf{b}\oplus\mathbf{c}\|$ of the gyroparallelogram are related to the gyroparallelogram side gyrolengths by the identity*

$$\sqrt{1 + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}}\sqrt{1 + \gamma_{\ominus\mathbf{b}\oplus\mathbf{c}}} = \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}} \tag{10.59}$$

Proof. The gyroparallelogram is a gyrogeometric object and its vertices are, accordingly, gyrocovariant. Hence, identities between the vertices of a gyroparallelogram, like (10.51) – (10.58), are gyrocovariant as well. Thus, in particular, the left gyrotranslation of Identity (10.53) by $\ominus\mathbf{a}$ gives the following new identity.

$$\begin{aligned} \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}} &= -\gamma_{\ominus\mathbf{a}\oplus\mathbf{a}} + \frac{(\gamma_{(\ominus\mathbf{a}\oplus\mathbf{a})\ominus(\ominus\mathbf{a}\oplus\mathbf{b})} + \gamma_{(\ominus\mathbf{a}\oplus\mathbf{a})\ominus(\ominus\mathbf{a}\oplus\mathbf{c})})(\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}})}{1 + \gamma_{(\ominus\mathbf{a}\oplus\mathbf{b})\ominus(\ominus\mathbf{a}\oplus\mathbf{c})}} \\ &= -1 + \frac{(\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}})(\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}})}{1 + \gamma_{\ominus\mathbf{b}\oplus\mathbf{c}}} \end{aligned} \tag{10.60}$$

implying

$$(1 + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}})(1 + \gamma_{\ominus\mathbf{b}\oplus\mathbf{c}}) = (\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}})^2 \tag{10.61}$$

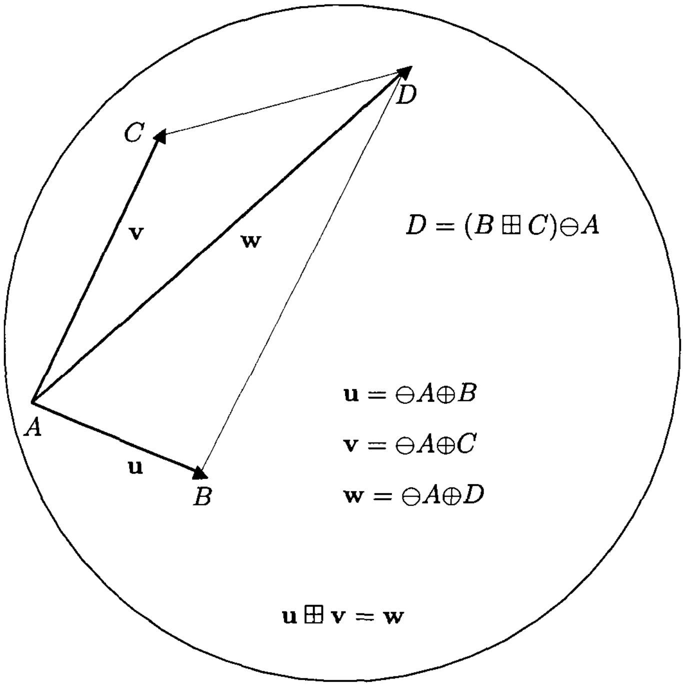


Fig. 10.10 The Einstein gyroparallelogram addition law of relativistically admissible velocities. Let $A, B, C \in \mathbb{R}_s^n$ be any three nongyrocollinear points of an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, giving rise to the two gyrovectors $\mathbf{u} = \ominus A \oplus B$ and $\mathbf{v} = \ominus A \oplus C$. Furthermore, let D be a point of the gyrovector space such that $ABDC$ is a gyroparallelogram, that is, $D = (B \boxplus C) \ominus A$. Then, Einstein coaddition of $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \boxplus \mathbf{v} = \mathbf{w}$, obeys the gyroparallelogram law, $\mathbf{w} = \ominus A \oplus D$. Einstein coaddition, $\boxplus$, thus gives rise to the gyroparallelogram addition law of Einsteinian velocities, which is commutative and fully analogous to the parallelogram addition law of Newtonian velocities.

as desired.

The first identity in (10.60) is obtained by a left gyrotranslation of Identity (10.53) by $\ominus\mathbf{a}$. The second identity in (10.60) is obtained by noting that $\gamma_{\ominus\mathbf{a}\oplus\mathbf{a}} = \gamma_0 = 1$, and by employing the Gyrotranslation Theorem 3.13, noting that a gamma factor is invariant under gyrations. $\square$

We may note that Identity (10.59) of Theorem 10.2 can be obtained from Identity (8.217) of Theorem 8.60 by translation from Möbius gyrovector spaces to Einstein gyrovector spaces; see an exercise in Sec. 8.24, p. 327.

10.10 The Relativistic Gyroparallelogram Law

By the gyroparallelogram law, Theorem 6.42, in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ we have, Fig. 10.10,

$$\mathbf{u} \boxplus \mathbf{v} = \mathbf{w} \tag{10.62}$$

where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_s$ are relativistically admissible velocities determined by the vertices of the parallelogram $ABDC$, Fig. 10.10, according to the equations

$$\begin{aligned} \mathbf{u} &= \ominus A \oplus B \\ \mathbf{v} &= \ominus A \oplus C \\ \mathbf{w} &= \ominus A \oplus D \end{aligned} \tag{10.63}$$

Here $\mathbf{u}$ and $\mathbf{v}$ are two gyrovectors with a common tail A and respective heads B and C, forming the gyroparallelogram $ABDC$, Fig. 10.10. The gyrovector $\mathbf{w}$ with tail A and head D forms the diagonal $\ominus A \oplus D$ of the gyroparallelogram.

Similarly, the diagonal gyrovector $\ominus B \oplus C$ of gyroparallelogram $ABDC$ satisfies the identity

$$(\ominus B \oplus A) \boxplus (\ominus B \oplus D) = \ominus B \oplus C \tag{10.64}$$

Since Einstein coaddition in (10.62) turns out to be the gyroparallelogram law of addition of *two* relativistically admissible velocities, it would be useful to rewrite it in a form that admits extension to k summands, $k > 2$. The extension, accordingly, should uncover the k-dimensional gyroparallelepiped law of addition of k relativistically admissible velocities. Following (3.156) we thus rewrite the gyroparallelogram addition law (10.62) as

$$\mathbf{v}_1 \boxplus_2 \mathbf{v}_2 = \frac{\gamma_{\mathbf{v}_1}\mathbf{v}_1 + \gamma_{\mathbf{v}_2}\mathbf{v}_2}{\gamma_{\mathbf{v}_1} + \gamma_{\mathbf{v}_2} - \frac{\gamma_{\ominus \mathbf{v}_1 \oplus \mathbf{v}_2} + 1}{\gamma_{\mathbf{v}_1} + \gamma_{\mathbf{v}_2} + 0}} \tag{10.65}$$

where we use the notation $\boxplus_{\text{E}} = \boxplus = \boxplus_2$ to emphasize that Einstein coaddition is a *binary* operation, that is, it is an operation between *two* gyrovectors.

The extension of the gyroparallelogram addition law (10.65) from Einstein coaddition $\boxplus_2$ between two summands to higher ordered Einstein coadditions $\boxplus_k$, $k > 2$, between k summands in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is given by the *gyroparallelepiped addition law*

$$\mathbf{v}_1 \boxplus_k \mathbf{v}_2 \boxplus_k \ldots \boxplus_k \mathbf{v}_k = \frac{\sum_{i=1}^{k} \gamma_{\mathbf{v}_i} \mathbf{v}_i}{\sum_{i=1}^{k} \gamma_{\mathbf{v}_i} - \frac{\sum_{ij} \gamma_{\ominus \mathbf{v}_i \oplus \mathbf{v}_j} + N_k}{\sum_{i=1}^{k} \gamma_{\mathbf{v}_i} + M_k}} \tag{10.66}$$

$\mathbf{v}_k \in \mathbb{R}^n_s$, $k = 2, 3, 4, \ldots$, where $\sum_{ij}$ is the sum over all pairs (i, j) such that $1 \leq i < j \leq k$.

The integers M_k and N_k in (10.66) are uniquely determined by the *compatibility condition* that $\boxplus_k$ reduces to $\boxplus_{k-1}$, for all $k > 3$, when one of its k summands vanishes. Thus, for instance, the compatibility condition for $k = 3$ is

$$\mathbf{v}_1 \boxplus_3 \mathbf{v}_2 \boxplus_3 \mathbf{0} = \mathbf{v}_1 \boxplus_2 \mathbf{v}_2 \tag{10.67}$$

It follows from the compatibility condition that (i) the integer M_k is given by the equation

$$M_k = 2 - k \tag{10.68}$$

and that (ii) the integer N_k is given by the recursive equation

$$\begin{aligned} N_1 &= 0 \\ N_k &= N_{k-1} - k + 3 \end{aligned} \tag{10.69}$$

$k = 2, 3, \ldots$

As an example, it follows from (10.66) and (10.68) – (10.69) that Einstein coaddition of order three, $\boxplus_3$, is given by the equation

$$\mathbf{v}_1 \boxplus_3 \mathbf{v}_2 \boxplus_3 \mathbf{v}_3 = \frac{\gamma_{\mathbf{v}_1} \mathbf{v}_1 + \gamma_{\mathbf{v}_2} \mathbf{v}_2 + \gamma_{\mathbf{v}_3} \mathbf{v}_3}{\gamma_{\mathbf{v}_1} + \gamma_{\mathbf{v}_2} + \gamma_{\mathbf{v}_3} - \frac{\gamma_{\ominus \mathbf{v}_1 \oplus \mathbf{v}_2} + \gamma_{\ominus \mathbf{v}_1 \oplus \mathbf{v}_3} + \gamma_{\ominus \mathbf{v}_2 \oplus \mathbf{v}_3} + 1}{\gamma_{\mathbf{v}_1} + \gamma_{\mathbf{v}_2} + \gamma_{\mathbf{v}_3} - 1}} \tag{10.70}$$

10.11 The Parallelepiped

We present the parallelepiped definition as a guide for the definition of the gyroparallelepiped that we will uncover in Sec. 10.13.

Definition 10.3 (The Parallelepiped). *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ *be any four points of the Euclidean n-space* $\mathbb{R}^n$, $n \geq 3$, *such that the three vectors* $-\mathbf{a} + \mathbf{b}$, $-\mathbf{a} + \mathbf{c}$ *and* $-\mathbf{a} + \mathbf{d}$ *are linearly independent. The points* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ *in* $\mathbb{R}^n$ *are the vertices of the parallelepiped* $\mathbf{abcda'b'c'd'}$, *Fig. 10.11, if* (**a**$'$) *The point* $\mathbf{a}'$ *is given by the equation*

$$-\mathbf{a} \oplus \mathbf{a}' = (-\mathbf{a} \oplus \mathbf{b}) \oplus (-\mathbf{a} \oplus \mathbf{c}) \oplus (-\mathbf{a} \oplus \mathbf{d}) \tag{10.71}$$

(known as the parallelepiped addition law);

(**b**$'$) *The point* $\mathbf{b}'$ *is given by the equation*

$$-\mathbf{b} \oplus \mathbf{b}' = (-\mathbf{b} \oplus \mathbf{a}) \oplus (-\mathbf{b} \oplus \mathbf{a}') \tag{10.72}$$

(that is, equivalently, $\mathbf{aba'b'}$ *is a parallelogram in Fig. 10.11, and* (10.72) *is a parallelogram addition in that parallelogram);*

(**c**$'$) *The point* $\mathbf{c}'$ *is given by the equation*

$$-\mathbf{c} \oplus \mathbf{c}' = (-\mathbf{c} \oplus \mathbf{b}) \oplus (-\mathbf{c} \oplus \mathbf{b}') \tag{10.73}$$

(that is, equivalently, $\mathbf{bc'b'c}$ *is a parallelogram in Fig. 10.11, and* (10.73) *is a parallelogram addition in that parallelogram);*

(**d**$'$) *The point* $\mathbf{d}'$ *is given by the equation*

$$-\mathbf{d} \oplus \mathbf{d}' = (-\mathbf{d} \oplus \mathbf{a}) \oplus (-\mathbf{d} \oplus \mathbf{a}') \tag{10.74}$$

(that is, equivalently, $\mathbf{ada'd'}$ *is a parallelogram in Fig. 10.11, and* (10.74) *is a parallelogram addition in that parallelogram).*

We call the parallelograms $\mathbf{aba'b'}$, $\mathbf{bc'b'c}$, $\mathbf{ada'd'}$, *etc., diagonal (as opposed to face) parallelograms of the parallelepiped* $\mathbf{abcda'b'c'd'}$ *in Fig. 10.11.*

It is well-known that the parallelepiped possesses the following properties $(PD_1)-(PD_6)$. The eight vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ of a parallelepiped $\mathbf{abcda'b'c'd'}$ in a Euclidean 3-space, Fig. 10.11, form 12 parallelograms. These are

(PD_1) the 6 diagonal-parallelograms of the parallelepiped: (i) $\mathbf{aba'b'}$, (ii) $\mathbf{dc'd'c}$, (iii) $\mathbf{aca'c'}$, (iv) $\mathbf{db'd'b}$, (v) $\mathbf{ada'd'}$, (vi) $\mathbf{bcb'c'}$; and

(PD_2) the 6 face-parallelograms of the parallelepiped: (i) $\mathbf{abc'd}$, (ii) $\mathbf{a'b'cd'}$, (iii) $\mathbf{bd'a'c'}$, (iv) $\mathbf{b'dac}$, (v) $\mathbf{a'b'dc'}$, (vi) $\mathbf{abd'c}$.

(PD_3) Each vertex of the parallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.11, admits

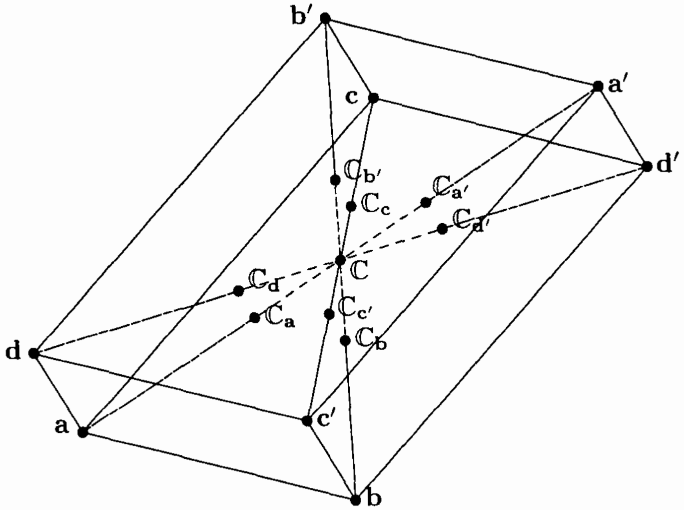


Fig. 10.11 The Euclidean parallelepiped $\mathbf{abcda'b'c'd'}$. Any three linearly independent vectors $-\mathbf{a}+\mathbf{b}$, $-\mathbf{a}+\mathbf{b}$, $-\mathbf{a}+\mathbf{d}$, that emanate from a common point $\mathbf{a}$ in the Euclidean 3-space $\mathbb{R}^3$ form a parallelepiped by Definition 10.3. The parallelepiped gives rise to the parallelepiped (addition) law, (10.71). Faces of the parallelepiped are parallelograms. Hence, for instance, $-\mathbf{a}+\mathbf{b}' = (-\mathbf{a}+\mathbf{c})+(-\mathbf{a}+\mathbf{d})$ by the parallelogram (addition) law. The parallelepiped also contains 6 diagonal parallelograms as, for instance, $\mathbf{ada'd'}$.
Each vertex of the parallelepiped has an opposite triangle in the parallelepiped the centroid of which lies on the segment that joins the vertex and its opposite one. Thus, for instance, the centroid $\mathbb{C}_{\mathbf{a}}$ of the triangle $\mathbf{bcd}$ opposite to the vertex $\mathbf{a}$ lies on the diagonal segment $\mathbf{aa}'$. The four diagonal segments are concurrent, the concurrency point, $\mathbb{C}$, being the midpoint of each diagonal segment. The point $\mathbb{C}$ is called the center of the parallelepiped.

a parallelepiped vector addition. These are, Fig. 10.11,

$$
\begin{aligned}
-\mathbf{a}+\mathbf{a}' &= (-\mathbf{a}+\mathbf{b})+(-\mathbf{a}+\mathbf{c})+(-\mathbf{a}+\mathbf{d})\\
-\mathbf{b}+\mathbf{b}' &= (-\mathbf{b}+\mathbf{a})+(-\mathbf{b}+\mathbf{c}')+(-\mathbf{b}+\mathbf{d}')\\
-\mathbf{c}+\mathbf{c}' &= (-\mathbf{c}+\mathbf{a})+(-\mathbf{c}+\mathbf{b}')+(-\mathbf{c}+\mathbf{d}')\\
-\mathbf{d}+\mathbf{d}' &= (-\mathbf{d}+\mathbf{a})+(-\mathbf{d}+\mathbf{b}')+(-\mathbf{d}+\mathbf{c}')\\
-\mathbf{a}'+\mathbf{a} &= (-\mathbf{a}'+\mathbf{b}')+(-\mathbf{a}'+\mathbf{c}')+(-\mathbf{a}'+\mathbf{d}')\\
-\mathbf{b}'+\mathbf{b} &= (-\mathbf{b}'+\mathbf{a}')+(-\mathbf{b}'+\mathbf{c})+(-\mathbf{b}'+\mathbf{d})\\
-\mathbf{c}'+\mathbf{c} &= (-\mathbf{c}'+\mathbf{a}')+(-\mathbf{c}'+\mathbf{b})+(-\mathbf{c}'+\mathbf{d})\\
-\mathbf{d}'+\mathbf{d} &= (-\mathbf{d}'+\mathbf{a}')+(-\mathbf{d}'+\mathbf{b})+(-\mathbf{d}'+\mathbf{c})
\end{aligned}
\tag{10.75}
$$

The first equation in (10.75) is valid by Identity (10.71) of Def. 10.3 of the parallelepiped. The validity of all the equations in (10.75) is a significant result that expresses the symmetries of the parallelepiped addition law.

(PD_4) The 4 diagonals $\mathbf{aa'}$, $\mathbf{bb'}$, $\mathbf{cc'}$ and $\mathbf{dd'}$, Fig. 10.11, of the parallelepiped $\mathbf{abcda'b'c'd'}$ are concurrent, the concurrency point being the midpoint of each of the diagonals.

(PD_5) Let $\mathbf{a}$ be one of the parallelepiped vertices, and let $\mathbb{C}_\mathbf{a}$ be the centroid of its opposite triangle $\mathbf{bcd}$ in the parallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.11. Then $\mathbb{C}_\mathbf{a}$ is given by the equation

$$\mathbb{C}_\mathbf{a} = \frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3} \tag{10.76}$$

which, equivalently, can be written as

$$-\mathbf{a}+\mathbb{C}_\mathbf{a} = \frac{(-\mathbf{a}+\mathbf{b})+(-\mathbf{a}+\mathbf{c})+(-\mathbf{a}+\mathbf{d})}{3} \tag{10.77}$$

(PD_6) Furthermore, $\mathbb{C}_\mathbf{a}$ lies on the segment $\mathbf{aa'}$ that connects the vertex $\mathbf{a}$ to its opposite vertex $\mathbf{a'}$. In fact, it follows from (10.77) and the first equation in (10.75) that

$$-\mathbf{a}+\mathbb{C}_\mathbf{a} = \frac{1}{3}(-\mathbf{a}+\mathbf{a'}) \tag{10.78}$$

The other vertices of the parallelepiped possess centroids similar to that of vertex $\mathbf{a}$ in (10.76)–(10.78), as shown graphically in Fig. 10.11.

The centroids associated with the parallelepiped vertices, mentioned in item (PD_5) and shown in Fig. 10.11, are

$$\begin{aligned} &\mathbb{C}_\mathbf{a} = \tfrac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}, \quad \mathbb{C}_\mathbf{b} = \tfrac{\mathbf{a}+\mathbf{c'}+\mathbf{d'}}{3}, \quad \mathbb{C}_\mathbf{c} = \tfrac{\mathbf{a}+\mathbf{b'}+\mathbf{d'}}{3}, \quad \mathbb{C}_\mathbf{d} = \tfrac{\mathbf{a}+\mathbf{b'}+\mathbf{c'}}{3} \\ &\mathbb{C}_\mathbf{a'} = \tfrac{\mathbf{b'}+\mathbf{c'}+\mathbf{d'}}{3}, \quad \mathbb{C}_\mathbf{b'} = \tfrac{\mathbf{a'}+\mathbf{c}+\mathbf{d}}{3}, \quad \mathbb{C}_\mathbf{c'} = \tfrac{\mathbf{a'}+\mathbf{b}+\mathbf{d}}{3}, \quad \mathbb{C}_\mathbf{d'} = \tfrac{\mathbf{a'}+\mathbf{b}+\mathbf{c}}{3} \end{aligned} \tag{10.79}$$

Pairs of opposite centroids lie on diagonals of their parallelepiped: (1) $\mathbb{C}_\mathbf{a}$ and $\mathbb{C}_\mathbf{a'}$ lie on the segment $\mathbf{aa'}$; (2) $\mathbb{C}_\mathbf{b}$ and $\mathbb{C}_\mathbf{b'}$ lie on the segment $\mathbf{bb'}$; (3) $\mathbb{C}_\mathbf{c}$ and $\mathbb{C}_\mathbf{c'}$ lie on the segment $\mathbf{cc'}$; and (4) $\mathbb{C}_\mathbf{d}$ and $\mathbb{C}_\mathbf{d'}$ lie on the segment $\mathbf{dd'}$.

10.12 The Pre-Gyroparallelepiped

Guided by Def. 10.3 of the parallelepiped in terms of parallelograms, and having the gyroparallelogram definition and properties in hand, Secs. 10.9 – 10.10, we are now in a position to define the pre-gyroparallelepiped. The latter will, in turn, lead us to the definition of the gyroparallelepiped.

Items ($\mathbf{b}'$) – ($\mathbf{d}'$) of Def. 10.3 of the parallelepiped involve parallelograms. Hence, they can readily be gyro-translated into gyrolanguage, as we did in Secs. 10.9 – 10.10, and as we do in Def. 10.4 below.

In contrast, the gyro-translation into gyrolanguage of item ($\mathbf{a}'$) of Def. 10.3 of the parallelepiped, which involves the parallelepiped addition law, is yet unknown. In order to uncover the gyro-translation of the parallelepiped addition law (10.71) into gyrolanguage we initially select the vertex $\mathbf{a}'$ in item ($\mathbf{a}'$) of Def. 10.3 arbitrarily, obtaining the definition of the pre-gyroparallelepiped. The following Def. 10.4 of the pre-gyroparallelepiped is, accordingly, analogous to Def. 10.3 of the parallelepiped with one exception. While items ($\mathbf{b}'$), ($\mathbf{c}'$), and ($\mathbf{d}'$) of Def. 10.4 share obvious gyro-analogies with their respective counterparts in Def. 10.3, item ($\mathbf{a}'$) is based on arbitrariness.

Definition 10.4 (The Pre-Gyroparallelepiped). *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^n_s$ *be any four points of the Einstein gyrovector space* $(\mathbb{R}^n_s, \oplus, \otimes)$ *of the ball* $\mathbb{R}^n_s$ *of the Euclidean n-space* $\mathbb{R}^n$, $n \geq 3$, *such that the three gyrovectors* $\ominus\mathbf{a}\oplus\mathbf{b}$, $\ominus\mathbf{a}\oplus\mathbf{c}$ *and* $\ominus\mathbf{a}\oplus\mathbf{d}$ *are linearly independent in* $\mathbb{R}^n$. *The points* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ *in* $\mathbb{R}^n_s$ *are the vertices of the pre-gyroparallelepiped* $\mathbf{abcda'b'c'd'}$, *Fig. 10.12, if*

($\mathbf{a}'$) *The point* $\mathbf{a}'$ *is selected arbitrarily, so that*

$$\ominus\mathbf{a}\oplus\mathbf{a}' = \textit{An Arbitrary Point in } \mathbb{R}^n_s \tag{10.80}$$

($\mathbf{b}'$) *The point* $\mathbf{b}'$ *is given by the equation*

$$\ominus\mathbf{b}\oplus\mathbf{b}' = (\ominus\mathbf{b}\oplus\mathbf{a}) \boxplus_2 (\ominus\mathbf{b}\oplus\mathbf{a}') \tag{10.81}$$

(that is, equivalently, $\mathbf{aba'b'}$ *is a gyroparallelogram in Fig. 10.12, and* (10.81) *is a gyroparallelogram addition in that gyroparallelogram);*

($\mathbf{c}'$) *The point* $\mathbf{c}'$ *is given by the equation*

$$\ominus\mathbf{c}\oplus\mathbf{c}' = (\ominus\mathbf{c}\oplus\mathbf{b}) \boxplus_2 (\ominus\mathbf{c}\oplus\mathbf{b}') \tag{10.82}$$

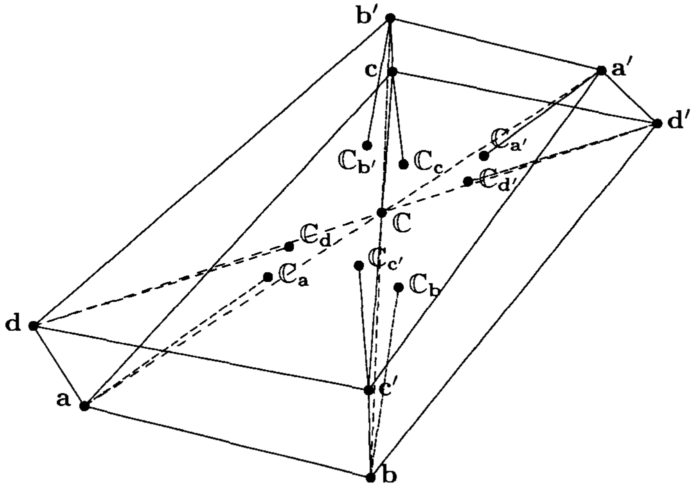


Fig. 10.12 The Einstein pre-gyroparallelepiped $\mathbf{abcda'b'c'd'}$ is a first attempt to extend the gyroparallelogram to a gyroparallelepiped. For any given four points, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, in the Einstein 3-gyrovector space $(\mathbb{R}^3_s, \oplus, \otimes)$, an additional vertex, $\mathbf{a'}$, is selected arbitrarily since the appropriate way to determine it is yet unknown.
Three additional vertices of the pre-gyroparallelepiped $\mathbf{abcda'b'c'd'}$ are constructed such that $\mathbf{aba'b'}$, $\mathbf{bcb'c'}$, and $\mathbf{ada'd'}$ are diagonal gyroparallelograms. Hence, the pre-gyroparallelepiped has 6 diagonal gyroparallelograms that share some diagonals, so that the 4 diagonal gyrosegments, $\mathbf{aa'}$, $\mathbf{bb'}$, $\mathbf{cc'}$, and $\mathbf{dd'}$ are concurrent. The point of concurrency, $\mathbb{C}$, coincides with the midpoint of each of the 4 diagonal gyrosegments. It is therefore called the gyrocenter of the pre-gyroparallelepiped.
Each vertex of the pre-gyroparallelepiped has an opposite gyrotriangle in the pre-gyroparallelepiped the gyrocentroid of which is shown. Thus, for instance, $\mathbb{C}_\mathbf{a}$ is the gyrocentroid of the gyrotriangle $\mathbf{bcd}$ that lies opposite to the vertex $\mathbf{a}$. Note that, in general, the two opposite gyrocentroids $\mathbb{C}_\mathbf{a}$ and $\mathbb{C}_\mathbf{a'}$ ($\mathbb{C}_\mathbf{b}$ and $\mathbb{C}_\mathbf{b'}$, $\mathbb{C}_\mathbf{c}$ and $\mathbb{C}_\mathbf{c'}$, $\mathbb{C}_\mathbf{d}$ and $\mathbb{C}_\mathbf{d'}$) do not lie on their associated diagonal $\mathbf{aa'}$ ($\mathbf{bb'}$, $\mathbf{cc'}$, $\mathbf{dd'}$, respectively). Could an appropriate determination of the arbitrary vertex $\mathbf{a'}$ simultaneously force each of the 8 gyrocentroids to lie on its associated diagonal?

(that is, equivalently, $\mathbf{bcb'c'}$ *is a gyroparallelogram in Fig. 10.12, and* (10.82) *is a gyroparallelogram addition in that gyroparallelogram);*

($\mathbf{d'}$) *The point* $\mathbf{d'}$ *is given by the equation*

$$\ominus\mathbf{d}\oplus\mathbf{d'} = (\ominus\mathbf{d}\oplus\mathbf{a}) \boxplus_2 (\ominus\mathbf{d}\oplus\mathbf{a'}) \tag{10.83}$$

(that is, equivalently, $\mathbf{ada'd'}$ *is a gyroparallelogram in Fig. 10.12, and* (10.83) *is a gyroparallelogram addition in that gyroparallelo-*

gram).

Let $\mathbf{a}$ be one of the gyroparallelepiped vertices, and let $\mathbb{C}_{\mathbf{a}}$ be the gyrocentroid of its opposite gyrotriangle $\mathbf{bcd}$ in the pre-gyroparallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.12, in full analogy with (10.76). Then $\mathbb{C}_{\mathbf{a}}$ is given by the equation

$$\mathbb{C}_{\mathbf{a}} = \frac{\gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c} + \gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}} + \gamma_{\mathbf{d}}} \tag{10.84}$$

which, equivalently, can be written as

$$\ominus\mathbf{a}\oplus\mathbb{C}_{\mathbf{a}} = \frac{\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}(\ominus\mathbf{a}\oplus\mathbf{b}) + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}}(\ominus\mathbf{a}\oplus\mathbf{c}) + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}(\ominus\mathbf{a}\oplus\mathbf{d})}{\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}} \tag{10.85}$$

Other vertices of the pre-gyroparallelogram possess centroids similar to that of vertex $\mathbf{a}$ in (10.84) – (10.85), as shown in Fig. 10.12. These eight gyrocentroids, analogous to (10.79), are listed in (10.93).

Comparing Figs. 10.11 and 10.12 we see that a remarkable disanalogy emerges. In general, the opposite gyrocentroids $\mathbb{C}_{\mathbf{a}}$ and $\mathbb{C}_{\mathbf{a'}}$ ($\mathbb{C}_{\mathbf{b}}$ and $\mathbb{C}_{\mathbf{b'}}$, $\mathbb{C}_{\mathbf{c}}$ and $\mathbb{C}_{\mathbf{c'}}$, $\mathbb{C}_{\mathbf{d}}$ and $\mathbb{C}_{\mathbf{d'}}$) do not lie on the gyrosegment $\mathbf{aa'}$ ($\mathbf{bb'}$, $\mathbf{cc'}$, $\mathbf{dd'}$, respectively) that joins opposite vertices of the pre-gyroparallelepiped. One may hope that the disanalogy stems solely from the arbitrariness in the selection of the vertex $\mathbf{a}$, so that there exists a unique vertex $\mathbf{a}$ of the pre-gyroparallelepiped in Fig. 10.12 that *simultaneously* repairs the breakdown of analogy in the position of each of the eight gyrocentroids. This is indeed the case, as we will see in Sec. 10.13.

10.13 The Gyroparallelepiped

A unique candidate for the arbitrarily selected vertex $\mathbf{a'}$ in item ($\mathbf{a'}$) of Def. 10.4 of the pre-gyroparallelepiped in Fig. 10.12, which repairs the breakdown of analogy between Figs. 10.11 and 10.12, is already in hand.

(1) In the same way that Einstein coaddition of order two, $\boxplus_2$ in (10.65), gives rise in Sec. 10.9 to a gyroparallelogram addition law, Fig. 10.6, analogous to the parallelogram addition law,
(2) we may expect that Einstein coaddition of order three, $\boxplus_3$ in (10.70), gives rise to a gyroparallelepiped addition law analogous to the parallelepiped addition law, (10.71), in Fig. 10.11.
(3) Figure 10.13 shows that this is indeed the case.

Figure 10.13 is generated by the same way that Fig. 10.12 is generated with one exception. In Fig. 10.12 the vertex $\mathbf{a}'$ is selected arbitrarily, while in Fig. 10.13 the vertex $\mathbf{a}'$ is determined by employing Einstein coaddition of order three along with other gyroanalogies; see (10.86) below.

As a result, in Fig. 10.12 none of the gyrocentroids lies on its diagonal gyrosegment (for instance, the gyrocentroids $\mathbb{C}_{\mathbf{a}}$ and $\mathbb{C}_{\mathbf{a}'}$ do not lie on their diagonal gyrosegment $\mathbf{aa}'$), in disanalogy with their Euclidean counterpart in Fig. 10.11. However, an appropriate determination of the vertex $\mathbf{a}'$ in Fig. 10.13 simultaneously forces all the eight gyrocentroids in the figure to lie on their respective diagonal gyrosegments, as their Euclidean counterparts do in Fig. 10.11. Such a remarkable simultaneous fit between all the eight gyrocentroids and their respective diagonal gyrosegments in Fig. 10.13 cannot be fortuitous. Hence, we reach the conclusion that

(1) in the same way that Einstein coaddition of order two gives rise to the gyro-analogue of the 2-dimensional Euclidean parallelogram addition law in Fig. 10.7 and in (10.81) – (10.83),
(2) Einstein coaddition of order three gives rise to the gyro-analogue of the 3-dimensional Euclidean parallelepiped addition law in (10.86) below.

Accordingly, we obtain the following Def. 10.5 of the gyroparallelepiped, shown in Fig. 10.13. Definition 10.5 is a copy of Def. 10.4 of the pre-gyroparallelepiped with one exception. The vertex $\mathbf{a}'$ in the gyroparallelepiped definition 10.5 is not selected arbitrarily but, rather, it is determined by Einstein coaddition of order three, $\boxplus_3$.

Definition 10.5 (The Gyroparallelepiped). *Let* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}_s^n$ *be any four points of the Einstein gyrovector space* $(\mathbb{R}_s^n, \oplus, \otimes)$ *of the ball* $\mathbb{R}_s^n$ *of the Euclidean n-space* $\mathbb{R}^n$, $n \geq 3$, *such that the three gyrovectors* $\ominus\mathbf{a}\oplus\mathbf{b}$, $\ominus\mathbf{a}\oplus\mathbf{c}$ *and* $\ominus\mathbf{a}\oplus\mathbf{d}$ *are linearly independent in* $\mathbb{R}^n$. *The points* $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ *and* $\mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ *in* $\mathbb{R}_s^n$ *are the vertices of the gyroparallelepiped* $\mathbf{abcda'b'c'd'}$, *Fig. 10.13, if*

($\mathbf{a}'$) *The point* $\mathbf{a}'$ *is given by the equation*

$$\ominus\mathbf{a}\oplus\mathbf{a}' = (\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus_3 (\ominus\mathbf{a}\oplus\mathbf{c}) \boxplus_3 (\ominus\mathbf{a}\oplus\mathbf{d}) \qquad (10.86)$$

called the 3-dimensional gyroparallelepiped addition law;

($\mathbf{b}'$) *The point* $\mathbf{b}'$ *is given by the equation*

$$\ominus\mathbf{b}\oplus\mathbf{b}' = (\ominus\mathbf{b}\oplus\mathbf{a}) \boxplus_2 (\ominus\mathbf{b}\oplus\mathbf{a}') \qquad (10.87)$$

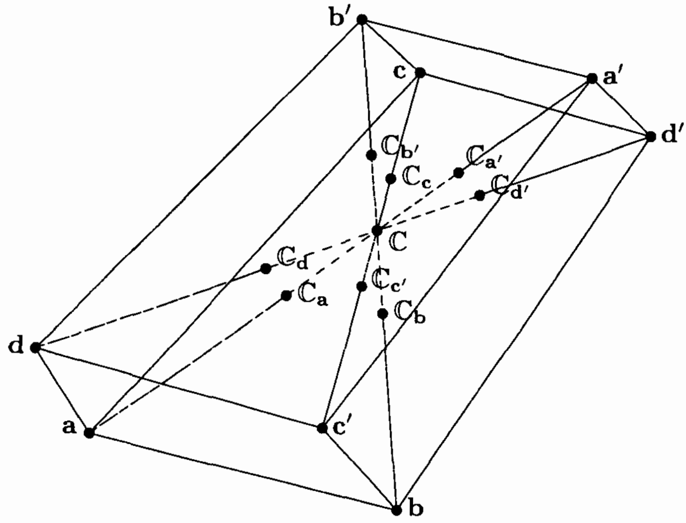


Fig. 10.13 The Einstein Gyroparallelepiped $\mathbf{abcda'b'c'd'}$. Any three gyrovectors $\ominus\mathbf{a}\oplus\mathbf{b}$, $\ominus\mathbf{a}\oplus\mathbf{c}$, $\ominus\mathbf{a}\oplus\mathbf{d}$, that emanate from a common point $\mathbf{a}$ in the Einstein 3-gyrovector space $(\mathbb{R}^3, \oplus, \otimes)$ form a gyroparallelepiped by Def. 10.5. The gyroparallelepiped gives rise to the gyroparallelepiped (addition) law, (10.86). In general, faces of the gyroparallelepiped are not gyroparallelograms, thus forming the only disanalogy with the (Euclidean) parallelepiped. Hence, for instance, the gyrovector $\ominus\mathbf{a}\oplus\mathbf{b}'$ of a face of the gyroparallelepiped is given by (10.94) rather than by a gyroparallelogram addition. But, in full analogy with the parallelepiped, the gyroparallelepiped contains 6 diagonal gyroparallelograms as, for instance, $\mathbf{ada'd'}$.
Each vertex of the gyroparallelepiped has an opposite gyrotriangle in the gyroparallelepiped the gyrocentroid of which lies on the gyrosegment that joins the vertex and its opposite one. Thus, for instance, the gyrocentroid $\mathbb{C}_{\mathbf{a}}$ of the gyrotriangle $\mathbf{bcd}$ opposite to the vertex $\mathbf{a}$ lies on the diagonal gyrosegment $\mathbf{aa'}$. Similarly, the two opposite gyrocentroids $\mathbb{C}_{\mathbf{a}}$ and $\mathbb{C}_{\mathbf{a}'}$ ($\mathbb{C}_{\mathbf{b}}$ and $\mathbb{C}_{\mathbf{b}'}$, $\mathbb{C}_{\mathbf{c}}$ and $\mathbb{C}_{\mathbf{c}'}$, $\mathbb{C}_{\mathbf{d}}$ and $\mathbb{C}_{\mathbf{d}'}$) lie on their associated diagonal $\mathbf{aa'}$ ($\mathbf{bb'}$, $\mathbf{cc'}$, $\mathbf{dd'}$, respectively).
The four diagonal gyrosegments are concurrent, the concurrency point, $\mathbb{C}$, being the gyromidpoint of each diagonal gyrosegment. The point $\mathbb{C}$ is called the gyrocenter of the gyroparallelepiped.

(that is, equivalently, $\mathbf{aba'b'}$ *is a gyroparallelogram in Fig. 10.12, and* (10.87) *is a gyroparallelogram addition in that gyroparallelogram);*

($\mathbf{c}'$) *The point* $\mathbf{c}'$ *is given by the equation*

$$\ominus\mathbf{c}\oplus\mathbf{c}' = (\ominus\mathbf{c}\oplus\mathbf{b}) \boxplus_2 (\ominus\mathbf{c}\oplus\mathbf{b}') \tag{10.88}$$

(that is, equivalently, $\mathbf{bc'b'c}$ *is a gyroparallelogram in Fig. 10.12,*

and (10.88) *is a gyroparallelogram addition in that gyroparallelogram);*

($\mathbf{d}'$) *The point* $\mathbf{d}'$ *is given by the equation*

$$\ominus\mathbf{d}\oplus\mathbf{d}' = (\ominus\mathbf{d}\oplus\mathbf{a}) \boxplus_2 (\ominus\mathbf{d}\oplus\mathbf{a}') \tag{10.89}$$

(that is, equivalently, $\mathbf{ada'd'}$ *is a gyroparallelogram in Fig. 10.12, and* (10.89) *is a gyroparallelogram addition in that gyroparallelogram).*

The gyroparallelepiped possesses properties, $(GD_1) - (GD_6)$, which are analogous to the parallelepiped properties, $(PD_1) - (PD_6)$, with one exception, (GP_2). The eight vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{a}', \mathbf{b}', \mathbf{c}', \mathbf{d}'$ of a gyroparallelepiped $\mathbf{abcda'b'c'd'}$ in a Euclidean 3-space, Fig. 10.11, form 6 gyroparallelograms. These are

(GD_1) the 6 diagonal-gyroparallelograms of the gyroparallelepiped: (i) $\mathbf{aba'b'}$, (ii) $\mathbf{dc'd'c}$, (iii) $\mathbf{aca'c'}$, (iv) $\mathbf{db'd'b}$, (v) $\mathbf{ada'd'}$, (vi) $\mathbf{bcb'c'}$.

(GD_2) (A disanalogy) Contrasting the parallelepiped faces, in general, each of the gyroparallelepiped faces: (i) $\mathbf{abc'd}$, (ii) $\mathbf{a'b'cd'}$, (iii) $\mathbf{bd'a'c'}$, (iv) $\mathbf{b'dac}$, (v) $\mathbf{a'b'dc'}$, (vi) $\mathbf{abd'c}$, does not form a gyroparallelogram.

(GD_3) Each vertex of the gyroparallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.13, admits a gyroparallelepiped vector addition. These are

$$\begin{aligned}
\ominus\mathbf{a}\oplus\mathbf{a}' &= (\ominus\mathbf{a}\oplus\mathbf{b}) \boxplus_3 (\ominus\mathbf{a}\oplus\mathbf{c}) \boxplus_3 (\ominus\mathbf{a}\oplus\mathbf{d})\\
\ominus\mathbf{b}\oplus\mathbf{b}' &= (\ominus\mathbf{b}\oplus\mathbf{a}) \boxplus_3 (\ominus\mathbf{b}\oplus\mathbf{c}') \boxplus_3 (\ominus\mathbf{b}\oplus\mathbf{d}')\\
\ominus\mathbf{c}\oplus\mathbf{c}' &= (\ominus\mathbf{c}\oplus\mathbf{a}) \boxplus_3 (\ominus\mathbf{c}\oplus\mathbf{b}') \boxplus_3 (\ominus\mathbf{c}\oplus\mathbf{d}')\\
\ominus\mathbf{d}\oplus\mathbf{d}' &= (\ominus\mathbf{d}\oplus\mathbf{a}) \boxplus_3 (\ominus\mathbf{d}\oplus\mathbf{b}') \boxplus_3 (\ominus\mathbf{d}\oplus\mathbf{c}')\\
\ominus\mathbf{a}'\oplus\mathbf{a} &= (\ominus\mathbf{a}'\oplus\mathbf{b}') \boxplus_3 (\ominus\mathbf{a}'\oplus\mathbf{c}') \boxplus_3 (\ominus\mathbf{a}'\oplus\mathbf{d}')\\
\ominus\mathbf{b}'\oplus\mathbf{b} &= (\ominus\mathbf{b}'\oplus\mathbf{a}') \boxplus_3 (\ominus\mathbf{b}'\oplus\mathbf{c}) \boxplus_3 (\ominus\mathbf{b}'\oplus\mathbf{d})\\
\ominus\mathbf{c}'\oplus\mathbf{c} &= (\ominus\mathbf{c}'\oplus\mathbf{a}') \boxplus_3 (\ominus\mathbf{c}'\oplus\mathbf{b}) \boxplus_3 (\ominus\mathbf{c}'\oplus\mathbf{d})\\
\ominus\mathbf{d}'\oplus\mathbf{d} &= (\ominus\mathbf{d}'\oplus\mathbf{a}') \boxplus_3 (\ominus\mathbf{d}'\oplus\mathbf{b}) \boxplus_3 (\ominus\mathbf{d}'\oplus\mathbf{c})
\end{aligned} \tag{10.90}$$

The gyroparallelepiped identities in (10.90) share obvious analogies with the parallelepiped identities in (10.75).

The first equation in (10.90) is valid by Identity (10.86) of Def. 10.5 of the gyroparallelepiped. The validity of all the equations in (10.90) is a significant result that expresses the symmetries of the gyroparallelepiped addition law.

(GD_4) The 4 diagonals $\mathbf{aa}'$, $\mathbf{bb}'$, $\mathbf{cc}'$, and $\mathbf{dd}'$ of the gyroparallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.13, are concurrent, the concurrency point being the gyromidpoint of each of the diagonals.

(GD_5) Let $\mathbf{a}$ be one of the gyroparallelepiped vertices, and let $\mathbb{C}_{\mathbf{a}}$ be the centroid of its opposite gyrotriangle $\mathbf{bcd}$ in the gyroparallelepiped $\mathbf{abcda'b'c'd'}$, Fig. 10.13. Then $\mathbb{C}_{\mathbf{a}}$ is given by the equation

$$\mathbb{C}_{\mathbf{a}} = \frac{\gamma_{\mathbf{b}}\mathbf{b} + \gamma_{\mathbf{c}}\mathbf{c} + \gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{b}} + \gamma_{\mathbf{c}} + \gamma_{\mathbf{d}}} \tag{10.91}$$

which, equivalently, can be written as

$$\ominus\mathbf{a}\oplus\mathbb{C}_{\mathbf{a}} = \frac{\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}(\ominus\mathbf{a}\oplus\mathbf{b}) + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}}(\ominus\mathbf{a}\oplus\mathbf{c}) + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}(\ominus\mathbf{a}\oplus\mathbf{d})}{\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}} \tag{10.92}$$

as shown in Figs. 6.16–6.17.

(GD_6) Furthermore, $\mathbb{C}_{\mathbf{a}}$ lies on the segment $\mathbf{aa}'$ that connects the vertex $\mathbf{a}$ to its opposite vertex $\mathbf{a}'$. The other vertices of the gyroparallelepiped possess properties similar to that of vertex $\mathbf{a}$ in (10.91)–(10.92), as shown graphically in Fig. 10.13.

The gyrotriangle gyrocentroids associated with the gyroparallelepiped vertices, mentioned in item (GD_5) and shown in Fig. 10.13, are

$$\begin{aligned}
\mathbb{C}_{\mathbf{a}} &= \frac{\gamma_{\mathbf{b}}\mathbf{b}+\gamma_{\mathbf{c}}\mathbf{c}+\gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{b}}+\gamma_{\mathbf{c}}+\gamma_{\mathbf{d}}}, & \mathbb{C}_{\mathbf{b}} &= \frac{\gamma_{\mathbf{a}}\mathbf{a}+\gamma_{\mathbf{c}'}\mathbf{c}'+\gamma_{\mathbf{d}'}\mathbf{d}'}{\gamma_{\mathbf{a}}+\gamma_{\mathbf{c}'}+\gamma_{\mathbf{d}'}} \\
\mathbb{C}_{\mathbf{c}} &= \frac{\gamma_{\mathbf{a}}\mathbf{a}+\gamma_{\mathbf{b}'}\mathbf{b}'+\gamma_{\mathbf{d}'}\mathbf{d}'}{\gamma_{\mathbf{a}}+\gamma_{\mathbf{b}'}+\gamma_{\mathbf{d}'}}, & \mathbb{C}_{\mathbf{d}} &= \frac{\gamma_{\mathbf{a}}\mathbf{a}+\gamma_{\mathbf{a}'}\mathbf{b}'+\gamma_{\mathbf{c}'}\mathbf{c}'}{\gamma_{\mathbf{a}}+\gamma_{\mathbf{a}'}+\gamma_{\mathbf{c}'}} \\
\mathbb{C}_{\mathbf{a}'} &= \frac{\gamma_{\mathbf{b}'}\mathbf{b}'+\gamma_{\mathbf{c}'}\mathbf{c}'+\gamma_{\mathbf{d}'}\mathbf{d}'}{\gamma_{\mathbf{b}'}+\gamma_{\mathbf{c}'}+\gamma_{\mathbf{d}'}}, & \mathbb{C}_{\mathbf{b}'} &= \frac{\gamma_{\mathbf{a}'}\mathbf{a}'+\gamma_{\mathbf{c}}\mathbf{c}+\gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{a}'}+\gamma_{\mathbf{c}}+\gamma_{\mathbf{d}}} \\
\mathbb{C}_{\mathbf{c}'} &= \frac{\gamma_{\mathbf{a}'}\mathbf{a}'+\gamma_{\mathbf{b}}\mathbf{b}+\gamma_{\mathbf{d}}\mathbf{d}}{\gamma_{\mathbf{a}'}+\gamma_{\mathbf{b}}+\gamma_{\mathbf{d}}}, & \mathbb{C}_{\mathbf{d}'} &= \frac{\gamma_{\mathbf{a}'}\mathbf{a}'+\gamma_{\mathbf{b}}\mathbf{b}+\gamma_{\mathbf{c}}\mathbf{c}}{\gamma_{\mathbf{a}'}+\gamma_{\mathbf{b}}+\gamma_{\mathbf{c}}}
\end{aligned} \tag{10.93}$$

in full analogy with (10.79).

In the extension from the parallelepiped to the gyroparallelepiped only one property is lost. The 6 faces of a parallelepiped are parallelograms while, in general, the 6 faces of a gyroparallelepiped are not gyroparallelograms. This indicates that the importance of the gyroparallelepiped rests on the fact that (i) it contains 6 diagonal gyroparallelograms (but no face gyroparallelograms), and that (ii) it gives rise to a gyroparallelepiped addition law which is fully analogous to the common parallelepiped addition law.

Since, in general, a face of a gyroparallelepiped is not a gyroparallelogram, face diagonals cannot be obtained from the gyroparallelogram law. Thus, for instance, the gyrovector $\ominus\mathbf{a}\oplus\mathbf{b}'$ is a diagonal of the face $\mathbf{acb'd}$ of the gyroparallelepiped $\mathbf{abcda'b'c'd'}$ in Fig. 10.13. It is not given by a gyroparallelogram law but, rather, by the equation

$$\ominus\mathbf{a}\oplus\mathbf{b}' = \frac{A_b\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}(\ominus\mathbf{a}\oplus\mathbf{b}) + A_c\gamma_{\ominus\mathbf{a}\oplus\mathbf{c}}(\ominus\mathbf{a}\oplus\mathbf{c}) + A_d\gamma_{\ominus\mathbf{a}\oplus\mathbf{d}}(\ominus\mathbf{a}\oplus\mathbf{d})}{\gamma_{\ominus\mathbf{a}\oplus\mathbf{b}}(B_3+1) + \gamma_{\ominus\mathbf{c}\oplus\mathbf{d}}(A_3-1) - (A_3-1)(B_3+1)} \tag{10.94a}$$

where

$$\begin{aligned} A_3 &= \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{c}} + \gamma_{\ominus\mathbf{a}\oplus\mathbf{d}} \\ B_3 &= \gamma_{\ominus\mathbf{b}\oplus\mathbf{c}} + \gamma_{\ominus\mathbf{b}\oplus\mathbf{d}} + \gamma_{\ominus\mathbf{c}\oplus\mathbf{d}} \end{aligned} \tag{10.94b}$$

and

$$\begin{aligned} A_b &= \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{c}\oplus\mathbf{d}} - A_3 + 1 \\ A_c &= \gamma_{\ominus\mathbf{a}\oplus\mathbf{b}} + \gamma_{\ominus\mathbf{c}\oplus\mathbf{d}} - B_3 - 1 \\ A_d &= A_c \end{aligned} \tag{10.94c}$$

In (10.94), the point $\mathbf{b}'$ of the gyroparallelepiped in Fig. 10.13 is represented relative to the point $\mathbf{a}$, that is, the gyrovector $\ominus\mathbf{a}\oplus\mathbf{b}'$, as a linear combination of the three gyrovectors, $\ominus\mathbf{a}\oplus\mathbf{b}$, $\ominus\mathbf{a}\oplus\mathbf{c}$ and $\ominus\mathbf{a}\oplus\mathbf{d}$, that generate the gyroparallelepiped. Similar to the representation of the point $\mathbf{b}'$ in (10.94), all other points of the gyroparallelepiped in Fig. 10.13 can be represented relative to the point $\mathbf{a}$ as a linear combination of the three gyrovectors that generate the gyroparallelepiped.

10.14 The Relativistic Gyroparallelepiped Law

Einstein velocity addition is neither commutative nor associative. There are attempts in the literature to repair the breakdown of both commutativity and associativity.

Owing to its noncommutativity, Einstein velocity addition is such that "velocity parallelograms do not close" (in Sommerfeld's words, quoted in Sec. 10.3). According to Scott Walter, in 1913 Émile Borel adopted a symmetric form for the addition of relativistic velocities:

> [Émile] Borel fixed the 'defective' assertion that the orientation of the relative velocity of a point with respect to two inertial systems is noncommutative. ...Noting with pleasure the Japanese mathematician Kimonsuke Ogura's adoption of the term 'kinematic space' [coined by Borel] ([Ogura (1913)]), Borel deplored the latter's presentation of the law of velocity addition in its original, noncommutative form. Apparently, Ogura had 'not seen all the advantages' of the symmetric form of the law adopted by Borel [Borel (1913), note 4].
>
> Scott Walter [Walter (1999b), pp. 117–118]

Realizing that Einstein's special theory of relativity is regulated by gyrogeometry, the noncommutativity of Einstein velocity addition for which "velocity parallelograms do not close" poses no problem. In gyrogeometry Einstein addition, $\oplus$, naturally comes with Einstein coaddition, $\boxplus$, which is commutative and for which "velocity gyroparallelograms do close", as shown in Fig. 10.10.

It should be emphasized here that coaddition was introduced into the theory of gyrogroups in Def. 2.7 in order to capture analogies with group theory. Yet, further useful properties of the coaddition get unexpectedly discovered in diverse situations, one of which reveals the relativistic gyroparallelogram velocity addition law in hyperbolic geometry, Fig. 10.10, which is fully analogous to the classical parallelogram velocity addition law in Euclidean geometry. Thus, the breakdown of commutativity in Einstein addition, which remained stubbornly intractable for a century, has been put to rest by the natural emergence of Einstein gyroparallelogram addition law of gyrovectors.

Some effects in quantum mechanics are regulated by hyperbolic geometry as shown, for instance, in Chap. 9 and in [Lévay (2004a); Lévay (2004b)]. The resulting breakdown of associativity in some translation operators of quantum mechanics is discussed by several authors; see, for instance, [Nesterov (2004)] and [Jackiw (1985)]. It is therefore interesting to realize that hyperbolic geometry offers a gyroparallelepiped addition law of hyperbolic translations, which is both associative and commutative in a sense that is fully analogous to the parallelepiped addition law of vectors.

In three dimensions the relativistic gyroparallelepiped law of addition of three relativistically admissible velocities is demonstrated in (10.86), and illustrated graphically in Fig. 10.13. It involves the Einstein coaddition of order three, $\boxplus_3$, given by (10.70). Similarly, a commutative and associative addition of k relativistically admissible velocities or gyrovectors, $k \geq 2$, is accomplished by employing the Einstein gyroparallelepiped law, which involves the Einstein coaddition of order k, $\boxplus_k$, given by (10.66).

10.15 The Lorentz Transformation and its Gyro-Algebra

In 1904 Lorentz reduced the electromagnetic equations for a moving system to the form of those that hold for a system at rest, thus discovering the transformation group that was later named after him by Einstein. Replacing Einstein's three-vector formalism and its seemingly structureless Einstein velocity addition by a four-vector formalism and the group structure of its Lorentz transformation, Minkowski has reformulated Einstein's special theory of relativity.

Starting with Einstein velocity addition as a fundamental, we will derive the Lorentz transformation group as a consequence. We will thus find that Einstein's three-vector formalism, with Einstein velocity addition and its gyrogroup structure, provides insight to the Minkowskian four-vector formalism, with the Lorentz transformation and its group structure. We will see that the resulting gyro-algebra of the Lorentz group captures analogies that the Lorentz transformation shares with its Galilean counterpart. Furthermore, we will see that the resulting analogies unlock the mystery of Einstein's relativistic mass, giving rise to the gyrobarycentric coordinates that are fully analogous to the barycentric coordinates that were first conceived by Möbius in 1827 [Mumford, Series and Wright (2002)].

Let $(t, \mathbf{x})^t \in \mathbb{R}^{\geq 0} \times \mathbb{R}^3$, where exponent t denotes transposition, be a spacetime event. A Lorentz transformation of spacetime coordinates is a coordinate transformation that leaves the norm, (4.55),

$$\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \sqrt{t^2 - \frac{\|\mathbf{x}\|^2}{c^2}} \tag{10.95}$$

of a spacetime event $(t, \mathbf{x})^t$ invariant.

A Lorentz transformation without rotation is known in the jargon as a *boost*. Let $L(\mathbf{u})$ be a Lorentz boost parameterized by a relativistically admissible velocity $\mathbf{u} \in \mathbb{R}^3_c$. Its application to spacetime coordinates $(t, \mathbf{x})^t =$

$(t, \mathbf{v}t)^t$ is given by the equation, (4.60),

$$
\begin{aligned}
L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} &= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ (\mathbf{u}\oplus\mathbf{v})\gamma_{\mathbf{u}}(1 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v})t \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{\mathbf{u}}(t + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{x}) \\ \gamma_{\mathbf{u}}\mathbf{u}t + \mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}^2}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{x})\mathbf{u} \end{pmatrix} \\
&= \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix}
\end{aligned}
\tag{10.96}
$$

leaving the spacetime norm (10.95) invariant,

$$
\left\| \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \right\| = \left\| \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} \right\| \tag{10.97}
$$

The boost $L(\mathbf{u})$ in (10.96) is a linear transformation of spacetime coordinates. Hence, it has a matrix representation $L_m(\mathbf{v})$, which is given by the equation [Møller (1952)],

$$
L_m(\mathbf{v}) =
$$

$$
\begin{pmatrix}
\gamma_{\mathbf{v}} & c^{-2}\gamma_{\mathbf{v}}v_1 & c^{-2}\gamma_{\mathbf{v}}v_2 & c^{-2}\gamma_{\mathbf{v}}v_3 \\
\gamma_{\mathbf{v}}v_1 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 \\
\gamma_{\mathbf{v}}v_2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_2 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2^2 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 \\
\gamma_{\mathbf{v}}v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_1v_3 & c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_2v_3 & 1 + c^{-2}\frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1}v_3^2
\end{pmatrix}
\tag{10.98}
$$

Employing the matrix representation (10.98) of the Lorentz transformation, the application of the Lorentz transformation in (10.96) can be written as

$$
L(\mathbf{v})\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L_m(\mathbf{v})\begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{x}' \end{pmatrix} \tag{10.99}
$$

where $\mathbf{v} = (v_1, v_2, v_3)^t \in \mathbb{R}_c^3$, $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$, $\mathbf{x}' = (x_1', x_2', x_3')^t \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}^{\geq 0}$.

In the Newtonian limit of large vacuum speed of light, $c \to \infty$, the Lorentz boost $L(\mathbf{v})$, (10.98) – (10.99), reduces to the Galilei boost $G(\mathbf{v})$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$,

$$G(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \lim_{c\to\infty} L(\mathbf{v}) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ v_1 & 1 & 0 & 0 \\ v_2 & 0 & 1 & 0 \\ v_3 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{10.100}$$

$$= \begin{pmatrix} t \\ x_1 + v_1 t \\ x_2 + v_2 t \\ x_3 + v_3 t \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{x} + \mathbf{v}t \end{pmatrix}$$

where $\mathbf{x} = (x_1, x_2, x_3)^t \in \mathbb{R}^3$ and $t \in \mathbb{R}^{\geq 0}$.

The composition of two Galilei boosts is equivalent to a single Galilei boost, and is given by parameter composition according to the equation

$$G(\mathbf{u})G(\mathbf{v}) = G(\mathbf{u} + \mathbf{v}) \tag{10.101}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, as it is clear from (10.100).

The composition of Lorentz boosts is more complicated than that of Galilei boosts since, in general, the composition of two Lorentz boosts is not a boost but, rather, a boost preceded (or followed) by the space rotation called Thomas precession.

Let $V \in SO(3)$ be a space rotation, and let $E(V)$ be the space rotation of a spacetime event, extended from V by the equation, (4.25),

$$E(V) \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ V\mathbf{x} \end{pmatrix} \tag{10.102}$$

$t \in \mathbb{R}^{>0}$, $\mathbf{x} \in \mathbb{R}^3$. Similarly, let $\mathrm{Gyr}[\mathbf{u}, \mathbf{v}]$ be the space gyration of a spacetime event, extended from $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ by the equation, (4.27),

$$\mathrm{Gyr}[\mathbf{u}, \mathbf{v}] = E(\mathrm{gyr}[\mathbf{u}, \mathbf{v}]) \tag{10.103}$$

Then, the relativistic analogue of (10.101) is, (4.32),

$$\begin{aligned} L(\mathbf{u})L(\mathbf{v}) &= L(\mathbf{u}\oplus\mathbf{v})\mathrm{Gyr}[\mathbf{u},\mathbf{v}] \\ &= \mathrm{Gyr}[\mathbf{u},\mathbf{v}]L(\mathbf{v}\oplus\mathbf{u}) \end{aligned} \tag{10.104}$$

Hence, the composition of two successive Lorentz boosts, $L(\mathbf{v})$ followed by $L(\mathbf{u})$,

(1) is equivalent to a single boost, $L(\mathbf{u}\oplus\mathbf{v})$, preceded by the Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$; and equally well, it
(2) is equivalent to a single boost, $L(\mathbf{v}\oplus\mathbf{u})$, followed by the same Thomas gyration generated by $\mathbf{u}$ and $\mathbf{v}$.

The Lorentz transformation $L(\mathbf{v},V)$ of spacetime events $(t,\mathbf{x})^t\in\mathbb{R}^{>0}\times\mathbb{R}^3$ is a boost preceded by a space rotation,

$$L(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = L(\mathbf{v})E(V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} \tag{10.105}$$

Hence, explicitly, the Lorentz transformation $L(\mathbf{v},V)$ takes the form

$$L(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{v}}(t+\frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x}) \\ \gamma_{\mathbf{v}}\mathbf{v}t + V\mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v} \end{pmatrix} \tag{10.106}$$

$\mathbf{v}\in\mathbb{R}_c^3$, $V\in SO(3)$, $\mathbf{x}\in\mathbb{R}^3$, $t\in\mathbb{R}^{\geq 0}$, as we see from (10.96).

In the Newtonian limit of large vacuum speed of light, $c\to\infty$, the Lorentz transformation $L(\mathbf{v},V)$ in (10.106) reduces to its Galilean counterpart $G(\mathbf{v},V)$,

$$G(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{v}t + V\mathbf{x} \end{pmatrix} \tag{10.107}$$

$\mathbf{v}\in\mathbb{R}^3$, $V\in SO(3)$, $\mathbf{x}\in\mathbb{R}^3$, $t\in\mathbb{R}$.

Clearly, the Lorentz transformation $L(\mathbf{v},I)$ with no rotation, I being the identity element of $SO(3)$, is a Lorentz boost,

$$L(\mathbf{v},I) = L(\mathbf{v}) \tag{10.108}$$

for all $\mathbf{v}\in\mathbb{R}_c^3$, shown in (10.99). Similarly, for all $\mathbf{v}\in\mathbb{R}^3$

$$G(\mathbf{v},I) = G(\mathbf{v}) \tag{10.109}$$

is a Galilei boost, shown in (10.100).

The composition of two Galilei transformations is given by parameters composition according to the equation

$$G(\mathbf{u},U)G(\mathbf{v},V) = G(\mathbf{u}+U\mathbf{v},UV) \tag{10.110}$$

where $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$, and $U,V\in SO(n)$.

In full analogy, the composition of two Lorentz transformations is given by parameters composition according to the equation, (4.41),

$$L(\mathbf{u},U)L(\mathbf{v},V) = L(\mathbf{u}\oplus U\mathbf{v}, \mathrm{gyr}[\mathbf{u},U\mathbf{v}]UV) \tag{10.111}$$

where $\mathbf{u},\mathbf{v}\in\mathbb{R}^3_c$, and $U,V\in SO(3)$.

The Galilei transformation composition law (10.110) is obvious, known in group theory as a product called a *semidirect product.* In contrast, the Lorentz transformation composition law (10.111) is a more general product, called a *gyrosemidirect product*, (4.36). Its usefulness stems from the analogies that it shares with its Galilean counterpart (10.110). The standard semidirect product of group theory is a special case of the gyrosemidirect product of gyrogroup theory corresponding to the special case when the associated gyrogroup is a group. Interestingly, Isaacs asserts in his book [Isaacs (1994), p. 83] that the semidirect product is "relevant only to group theory". Here we see, however, that it is relevant to gyrogroup theory as well and, in particular, it is important in understanding the Lorentz group of special relativity and the analogies it shares with the Galilean group.

Owing to the analogies that the Galilei and the Lorentz transformation composition law share in (10.110) – (10.111), one can extend intuitive understanding of the Galilei transformation to intuitive understanding of the Lorentz transformation. The discovery of the Lorentz transformation composition law (10.111) fills a gap at a fundamental level in the development of the basics of special relativity. There are in the literature various attempts to uncover the Lorentz transformation composition law, obtaining results that share no obvious analogies with the Galilei transformation composition law; see, for instance, [Halpern (1968), Chap. 1. Appendix 3] and [Fedorov (1962); Santander (1982); van Wyk (1984); Ungar (1988a); Mocanu (1993); Ferraro and Thibeault (1999); Gough (2000); Lucinda (2001); Vigoureux (2001); Coll and Martínez (2002); Kennedy (2002)].

The Lorentz transformations (10.106) form a subgroup of the so called full Lorentz group. This subgroup is known as the *homogeneous, proper, orthochronous* Lorentz group $SO(3,1)$. It is:

(1) *homogeneous*, since each of its elements is a Lorentz transformation that takes the origin of spacetime coordinates into an origin of spacetime coordinates; it is
(2) *proper*, since each of its elements is a Lorentz transformation that is continuously connected to the identity transformation of spacetime; and it is
(3) *orthochronous*, since each of its elements is a Lorentz transformation that preserves the sign of time, that is, it takes positive (negative) time into positive (negative) time.

Identity (10.111) is thus the group operation of the homogeneous, proper, orthochronous Lorentz group $SO(3,1)$.

The velocity-orientation representation $L(\mathbf{v}, V)$ of the homogeneous, proper, orthochronous Lorentz transformation group in terms of its two parameters, the relative velocity parameter $\mathbf{v}\in\mathbb{R}^3_c$ and the relative orientation parameter $V\in SO(3)$ is not new. It was already used by Silberstein in his 1914 book [Silberstein (1914), p. 168] along with the Lorentz transformation composition law

$$L(\mathbf{u}, U)L(\mathbf{v}, V) = L(\mathbf{w}, W) \tag{10.112}$$

However, unlike (10.111), Silberstein did not express explicitly the composite pair $(\mathbf{w}, W)$ in terms of its generating pairs $(\mathbf{u}, U)$ and $(\mathbf{v}, V)$. Hence, the Lorentz transformation composition law (10.112) that Silberstein used in the early days of special relativity is almost invariably absent in modern texts. Most modern explorers of relativity physics abandon the 1914 Lorentz transformation composition law (10.112), realizing that its study is severely restricted by its complexity, as one can see from the attempt in [Rivas, Valle and Aguirregabiria (1986)]. The complexity, however, effortlessly fades away in (10.111) by the use of gyrogroup theoretic techniques.

The classical way for dealing with the problem of the composition of two successive Lorentz transformations, which is quiet different from that of Silberstein, is well described by Sard [Sard (1970), Chap. 5] and by Halpern [Halpern (1968), Chap. 1]. It involves Lorentz matrices (or tensors) that tell very little about the two underlying physically significant observables involved, that is, the relative velocities and the relative orientations between inertial frames, and their composition laws.

Recently, the need to improve our understanding of the Lorentz transformation composition led Coll and Fernando to present the composition in terms of Lorentz transformation generators, employing the so called Baker–Campbell–Hausdorff formula of the theory of Lie algebras [Coll and Martínez (2002)].

Available methods for the study the Lorentz transformation composition law analytically are thus complicated and far beyond the reach of those who need it for practical use. In contrast, the gyro-formalism enables the Lorentz transformation composition law, (10.111), to be presented in a simple way that provides vivid visual analogies with its well familiar Galilean counterpart, (10.110).

10.16 Galilei and Lorentz Transformation Links

To see another remarkable analogy that the velocity-orientation representation of the Lorentz transformation shares with its Galilean counterpart, let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad and \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.113}$$

be two Galilean spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, with equal time components,

$$t = t' \tag{10.114}$$

Then, the unique Galilei transformation without rotation (boost) that links the two spacetime events in (10.113) – (10.114) has the velocity parameter $\mathbf{v} - \mathbf{u}$, satisfying

$$G(\mathbf{v} - \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.115}$$

as one can readily see from (10.100). We call (10.115) a *Galilei link.*

In the analogous relativistic link, the Galilean negative velocity addition, $-$, in (10.115) becomes the Einstein negative coaddition, $\boxminus$, in (3.156), as we will see in (10.118) below. In full analogy with (10.113)–(10.115), let

$$\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} \qquad and \qquad \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.116}$$

be two relativistic spacetime events, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, and $t, t' \in \mathbb{R}^{>0}$, with equal spacetime norms,

$$t'\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}} = t\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}} \tag{10.117}$$

Then, the unique Lorentz transformation without rotation (boost) that links the two spacetime events in (10.116) – (10.117) has the velocity parameter $\mathbf{v} \boxminus \mathbf{u}$, satisfying, [Ungar (2001), pp. 343 – 348],

$$L(\mathbf{v} \boxminus \mathbf{u}) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.118}$$

We call (10.118) a *Lorentz link.* The analogies that the Lorentz link (10.118) shares with the Galilean link (10.115) are obvious. Remarkably, the analogies that the Lorentz and the Galilei links share are uncovered in terms of Einstein coaddition rather than Einstein addition. This observation demonstrates that in order to capture analogies that Einsteinian velocities share with Newtonian velocities the two mutually dual Einstein additions, $\oplus$ and $\boxplus$, must be invoked.

Furthermore, the most general Galilei transformation that links the simultaneous spacetime events in (10.113) – (10.114) is $G(\mathbf{v} - R\mathbf{u}, R)$, satisfying

$$G(\mathbf{v} - R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.119}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, and $t, t' \in \mathbb{R}$, where $R \in SO(3)$ is a free rotation parameter. Identity (10.119) reduces to (10.115) when the rotation R is trivial, that is, when R reduces to the identity map I.

In full analogy, the most general homogeneous, proper, orthochronous Lorentz transformation that links the equinorm spacetime events in (10.116) – (10.117) is $L(\mathbf{v} \boxminus R\mathbf{u}, R)$ satisfying [Ungar (2001)],

$$L(\mathbf{v} \boxminus R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix} \tag{10.120}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$, and $t, t' \in \mathbb{R}^{>0}$, where $R \in SO(3)$ is a free rotation parameter.

Historically, the problem of determining Lorentz links between given spacetime events was only partially solved by several explorers [van Wyk (1986)]. Recently the problem was also solved by Urbantke [Urbantke (2003)] by studying Lorentz boosts from a geometrical viewpoint, expressing boosts in terms of line reflections.

The use of the gyro-algebra of Lorentz transformation as a tool in the study of the shape of moving objects is presented in [Ungar (2001), Ch. 11] where, in particular, the well-known Doppler effect is recovered from the shape of moving sinusoidal waves.

10.17 $(t_1{:}t_2)$-Gyromidpoints as CM Velocities

Our study of the relativistic CM velocity began in Sec. 10.7 with the study of the Einstein gyromidpoint, Fig. 10.1, and gyrocentroid, Figs. 10.3 and 10.5. The Einstein gyromidpoint and gyrocentroid enable us to uncover the gyrogeometric interpretation of relativistic CM velocities of systems of moving particles with equal relativistic rest masses. In order to extend our study to the relativistic CM velocities of systems of moving particles with relativistic rest masses that need not be equal we explore the generalized gyromidpoint, already studied in Secs. 4.10 and 4.11 of Chap. 4, in the context of the relativistic CM velocity.

Let $\mathbb{R}_c^3 = (\mathbb{R}_c^3, \oplus, \otimes)$ be an Einstein gyrovector space, and let $\mathbf{v}{\in}\mathbb{R}_c^3$ and $t{\in}\mathbb{R}^{>0}$. The pair $(t, \mathbf{v}t)^t$ represents a spacetime event with time t and space $\mathbf{x} = \mathbf{v}t$. The Lorentz boost $L(\mathbf{u})$, $\mathbf{u}{\in}\mathbb{R}_c^3$, of this event is given by (10.96) which, by means of Einstein addition (10.3) and the gamma identity (10.4), can be written in the form

$$L(\mathbf{u})\begin{pmatrix} t \\ \mathbf{v}t \end{pmatrix} = \begin{pmatrix} \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}}t \\ \frac{\gamma_{\mathbf{u}\oplus\mathbf{v}}}{\gamma_{\mathbf{v}}}(\mathbf{u}\oplus\mathbf{v})t \end{pmatrix} \tag{10.121}$$

Hence, in particular, for $t = \gamma_{\mathbf{v}}$ we have the elegant identity

$$L(\mathbf{u})\begin{pmatrix} \gamma_{\mathbf{v}} \\ \gamma_{\mathbf{v}}\mathbf{v} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{u}\oplus\mathbf{v}} \\ \gamma_{\mathbf{u}\oplus\mathbf{v}}(\mathbf{u}\oplus\mathbf{v}) \end{pmatrix} \tag{10.122}$$

that expresses the application of the Lorentz boost parameterized by $\mathbf{u}$ to a *unimodular* spacetime event, (4.54), (4.64), parameterized by $\mathbf{v}$ as a left gyrotranslation $\mathbf{u}\oplus\mathbf{v}$ of $\mathbf{v}$ by $\mathbf{u}$, for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}_c^3$. The unimodular spacetime events are the four-velocities of Minkowskian relativity, Lorentz transformed by an Einstein velocity addition. Einstein velocity addition is, thus, a century-old idea whose time has come back. Following the discovery of the gyrovector space structure to which Einstein velocity addition gives rise, and unleashing the power of its hyperbolic geometry [Ungar (2005)], new analogies with classical results emerge.

The Lorentz boost $L(\mathbf{u})$ is linear, (4.63), as we see from (10.96). To exploit the linearity of the Lorentz boost let us consider the linear combination of two unimodular spacetime events

$$
\begin{aligned}
t_1 \begin{pmatrix} \gamma_{\mathbf{a}_1} \\ \gamma_{\mathbf{a}_1}\mathbf{a}_1 \end{pmatrix} + t_2 \begin{pmatrix} \gamma_{\mathbf{a}_2} \\ \gamma_{\mathbf{a}_2}\mathbf{a}_2 \end{pmatrix} &= \begin{pmatrix} t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2} \\ t_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + t_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 \end{pmatrix} \\
&= t_{12} \begin{pmatrix} \gamma_{\mathbf{a}_{12}} \\ \gamma_{\mathbf{a}_{12}}\mathbf{a}_{12} \end{pmatrix}
\end{aligned}
\tag{10.123}
$$

$t_1, t_2 \ge 0$, $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3_c$, where $t_{12} \ge 0$ and $\mathbf{a}_{12} \in \mathbb{R}^3_c$ are to be determined in (10.124) and (10.125) below.

Comparing ratios between lower and upper entries in (10.123) we have

$$
\mathbf{a}_{12} = \mathbf{a}_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2) = \frac{t_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + t_2\gamma_{\mathbf{a}_2}\mathbf{a}_2}{t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2}} \tag{10.124}
$$

so that, by convexity, $\mathbf{a}_{12} \in \mathbb{R}^3_c$ as desired.

The point $\mathbf{a}_{12} = \mathbf{a}_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2)$ is called the $(t_1{:}t_2)$-gyromidpoint of $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}^3_c$. This term will be justified by Identity (10.136) below.

Comparing upper entries in (10.123) we have

$$
t_{12} = \frac{t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2}}{\gamma_{\mathbf{a}_{12}}} \tag{10.125}
$$

so that $t_{12} \ge 0$, as desired.

The elegant correspondence (10.122) between the Lorentz boost $L(\mathbf{u})$ and Einstein addition $\oplus$ is significant. Unlike Einstein addition, the Lorentz boost is linear. The linearity of the Lorentz boost and its correspondence with Einstein addition enables us to uncover important results about gyrocovariance of gyrogeometric objects with respect left gyrotranslations, culminated in the creation of gyrobarycentric coordinates.

Applying the Lorentz boost $L(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^3_c$, to (10.123) in two different ways, important results follow from (10.122) and the linearity of the Lorentz boost.

Thus, on the one hand

$$
\begin{aligned}
L(\mathbf{x})\left\{t_{12}\begin{pmatrix}\gamma_{\mathbf{a}_{12}}\\ \gamma_{\mathbf{a}_{12}}\mathbf{a}_{12}\end{pmatrix}\right\} &= L(\mathbf{x})\left\{t_1\begin{pmatrix}\gamma_{\mathbf{a}_1}\\ \gamma_{\mathbf{a}_1}\mathbf{a}_1\end{pmatrix}+t_2\begin{pmatrix}\gamma_{\mathbf{a}_2}\\ \gamma_{\mathbf{a}_2}\mathbf{a}_2\end{pmatrix}\right\}\\
&= t_1L(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{a}_1}\\ \gamma_{\mathbf{a}_1}\mathbf{a}_1\end{pmatrix}+t_2L(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{a}_2}\\ \gamma_{\mathbf{a}_2}\mathbf{a}_2\end{pmatrix}\\
&= t_1\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{a}_1}\\ \gamma_{\mathbf{x}\oplus\mathbf{a}_1}(\mathbf{x}\oplus\mathbf{a}_1)\end{pmatrix}+t_2\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{a}_2}\\ \gamma_{\mathbf{x}\oplus\mathbf{a}_2}(\mathbf{x}\oplus\mathbf{a}_2)\end{pmatrix}\\
&= \begin{pmatrix}t_1\gamma_{\mathbf{x}\oplus\mathbf{a}_1}+t_2\gamma_{\mathbf{x}\oplus\mathbf{a}_2}\\ t_1\gamma_{\mathbf{x}\oplus\mathbf{a}_1}(\mathbf{x}\oplus\mathbf{a}_1)+t_2\gamma_{\mathbf{x}\oplus\mathbf{a}_2}(\mathbf{x}\oplus\mathbf{a}_2)\end{pmatrix}
\end{aligned}
\tag{10.126}
$$

and on the other hand

$$
\begin{aligned}
L(\mathbf{x})\left\{t_{12}\begin{pmatrix}\gamma_{\mathbf{a}_{12}}\\ \gamma_{\mathbf{a}_{12}}\mathbf{a}_{12}\end{pmatrix}\right\} &= t_{12}L(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{a}_{12}}\\ \gamma_{\mathbf{a}_{12}}\mathbf{a}_{12}\end{pmatrix}\\
&= t_{12}\begin{pmatrix}\gamma_{\mathbf{x}\oplus\mathbf{a}_{12}}\\ \gamma_{\mathbf{x}\oplus\mathbf{a}_{12}}\mathbf{x}\oplus\mathbf{a}_{12}\end{pmatrix}\\
&= \begin{pmatrix}t_{12}\gamma_{\mathbf{x}\oplus\mathbf{a}_{12}}\\ t_{12}\gamma_{\mathbf{x}\oplus\mathbf{a}_{12}}\mathbf{x}\oplus\mathbf{a}_{12}\end{pmatrix}
\end{aligned}
\tag{10.127}
$$

Comparing ratios between lower and upper entries of (10.126) and (10.127) we have

$$
\mathbf{x}\oplus\mathbf{a}_{12}=\frac{t_1\gamma_{\mathbf{x}\oplus\mathbf{a}_1}(\mathbf{x}\oplus\mathbf{a}_1)+t_2\gamma_{\mathbf{x}\oplus\mathbf{a}_2}(\mathbf{x}\oplus\mathbf{a}_2)}{t_1\gamma_{\mathbf{x}\oplus\mathbf{a}_1}+t_2\gamma_{\mathbf{x}\oplus\mathbf{a}_2}}
\tag{10.128}
$$

so that by (10.124) and (10.128),

$$
\mathbf{x}\oplus\mathbf{a}_{12}(\mathbf{a}_1,\mathbf{a}_2;t_1,t_2)=\mathbf{a}_{12}(\mathbf{x}\oplus\mathbf{a}_1,\mathbf{x}\oplus\mathbf{a}_2;t_1,t_2)
\tag{10.129}
$$

Identity (10.129) demonstrates that the structure of the $(t_1{:}t_2)$-gyromidpoint $\mathbf{a}_{12}$ of $\mathbf{a}_1$ and $\mathbf{a}_2$ as a function of points $\mathbf{a}_1$ and $\mathbf{a}_2$ is not distorted by left gyrotranslations. Similarly, it is not distorted by rotations in the sense that if $R \in SO(3)$ represents a rotation of $\mathbb{R}^3_c$ then

$$R\mathbf{a}_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2) = \mathbf{a}_{12}(R\mathbf{a}_1, R\mathbf{a}_1; t_1, t_2) \tag{10.130}$$

It follows from Identities (10.129) and (10.130) that the $(t_1{:}t_2)$-gyromidpoint $\mathbf{a}_{12} \in \mathbb{R}^3_c$ possesses, as a function of the points $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3_c$, gyrogeometric significance. It is gyrocovariant, being covariant with respect to the gyrovector space motions of the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$. The associated relativistic mechanics significance of the $(t_1{:}t_2)$-gyromidpoint $\mathbf{a}_{12}$ as the relativistic CM velocity will be uncovered in (10.134) below.

Comparing the top entries of (10.126) and (10.127) we have

$$t_{12} = \frac{t_1\gamma_{\mathbf{x}\oplus\mathbf{a}_1} + t_2\gamma_{\mathbf{x}\oplus\mathbf{a}_2}}{\gamma_{\mathbf{x}\oplus\mathbf{a}_{12}}} \tag{10.131}$$

But, we also have from (10.125)

$$t_{12} = \frac{t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2}}{\gamma_{\mathbf{a}_{12}}} \tag{10.132}$$

implying that the nonnegative scalar field $t_{12} = t_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2)$ in (10.131) – (10.132) is invariant under left gyrotranslations of $\mathbf{a}_1$ and $\mathbf{a}_2$. Clearly, it is also invariant under rotations R of $\mathbf{a}_1$ and $\mathbf{a}_2$ so that, being invariant under the group of motions of the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$, it possesses gyrogeometric significance. As such, we call $t_{12} = t_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2)$ a gyrogeometric scalar field.

Substituting $\mathbf{x} = \ominus\mathbf{a}_{12}$ in (10.131) we have

$$t_{12} = t_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1} + t_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2} \tag{10.133}$$

revealing the physical interpretation of the gyrogeometric scalar field t_{12}. It represents the joint relativistic mass of two particles with rest masses t_1 and t_2, relativistically corrected in the $(t_1{:}t_2)$-gyromidpoint frame $\Sigma_{\mathbf{a}_{12}}$. The CM inertial frame $\Sigma_{\mathbf{a}_{12}}$ is represented in Fig. 10.14 by the CM velocity point $\mathbf{a}_{12}$.

Substituting $\mathbf{x} = \ominus\mathbf{a}_{12}$ in (10.128) we obtain the identity

$$t_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1) + t_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2) = 0 \tag{10.134}$$

revealing that $\Sigma_{\mathbf{a}_{12}}$ is the vanishing momentum inertial frame. As in classical mechanics, the frame $\Sigma_{\mathbf{a}_{12}}$ is called the relativistic CM frame since

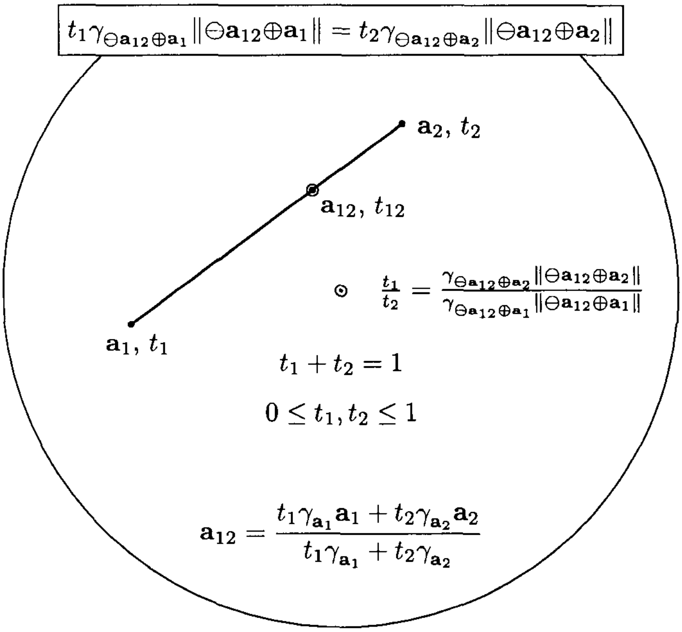


Fig. 10.14 The Relativistic Law of the Lever. The Einstein $(t_1{:}t_2)$-gyromidpoint $\mathbf{a}_{12} = \mathbf{a}_{12}(\mathbf{a}_1, \mathbf{a}_2; t_1, t_2)$ in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ is the point of the gyrosegment $\mathbf{a}_1\mathbf{a}_2$ with gyrobarycentric coordinates $(t_1{:}t_2)$ relative to the set $\{\mathbf{a}_1, \mathbf{a}_2\}$ in $\mathbb{R}^3_c$. When t_1 (t_2) varies from 0 to 1, the $(t_1{:}t_2)$-gyromidpoint $\mathbf{a}_{12}$ slides along the gyrosegment $\mathbf{a}_1\mathbf{a}_2$ from $\mathbf{a}_2$ to $\mathbf{a}_1$ (from $\mathbf{a}_1$ to $\mathbf{a}_2$). The gyromidpoint $\mathbf{a}_{12}$ is the barycenter in velocity space of the masses t_1 and t_2 with respective velocities $\mathbf{a}_1$ and $\mathbf{a}_2$.

the total momentum in that frame, (10.134), vanishes. Accordingly, the point $\mathbf{a}_{12}$ in Fig. 10.14 represents the CM velocity of two particles with rest masses t_1 and t_2, and velocities $\mathbf{a}_1$ and $\mathbf{a}_2$ relative to a rest frame Σ_0.

We may note that (10.134) can be written, equivalently, as

$$t_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1) = \ominus t_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2) \tag{10.135}$$

Taking norms of both sides of (10.135) we have

$$t_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1\| = t_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2\| \tag{10.136}$$

so that

$$\frac{t_1}{t_2} = \frac{\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2\|}{\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1\|} \tag{10.137}$$

The classical analogue of (10.137) is well-known [Coxeter (1961),

Fig. 13.7a, p. 217]. Owing to its property (10.137), the relativistic CM velocity $\mathbf{a}_{12}$, given by (10.124), is called the $(t_1{:}t_2)$-gyromidpoint of $\mathbf{a}_1$ and $\mathbf{a}_2$ in $\mathbb{R}_c^3$. Property (10.136) of the $(t_1{:}t_2)$-gyromidpoint means that the ratio between the "proper speed" $\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2\|$ of frame $\Sigma_{\mathbf{a}_2}$ relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$ and the "proper speed" $\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1\|$ of frame $\Sigma_{\mathbf{a}_1}$ relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$ is $t_1{:}t_2$, as shown in Fig. 10.14. Proper speed is the magnitude, or norm, of a proper velocity. Proper velocities in Einsteinian special relativity and their proper Lorentz transformation law will be studied in Sec. 10.24.

Rewriting (10.132) as

$$t_{12}\gamma_{\mathbf{a}_{12}} = t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2} \tag{10.138}$$

we obtain the two identities (10.137) – (10.138) that form the *relativistic law of the lever*, illustrated in Fig. 10.14. It is fully analogous to the classical law of the lever, to which it reduces in the Newtonian limit $c \to \infty$.

The origin $\mathbf{0}$ of an Einstein gyrovector space $\mathbb{R}_c^3 = (\mathbb{R}_c^3, \oplus, \otimes)$ represents the vanishing velocity of a rest frame Σ_0. Two moving objects with rest masses $t_1, t_2 > 0$ and respective velocities $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}_c^3$ relative to Σ_0, as well as their CM velocity $\mathbf{a}_{12}$, are shown in Fig. 10.14 for $\mathbb{R}_c^2$. The relativistically corrected masses in Σ_0 are $t_1\gamma_{\mathbf{a}_1}$ and $t_2\gamma_{\mathbf{a}_2}$ so that the total relativistic mass relative to Σ_0 of the two objects in Fig. 10.14 is $t_1\gamma_{\mathbf{a}_1} + t_2\gamma_{\mathbf{a}_2}$. This, by (10.138), is equal to $t_{12}\gamma_{\mathbf{a}_{12}}$, that is, the relativistically corrected mass of an object with rest mass t_{12} moving with the CM velocity $\mathbf{a}_{12}$ relative to Σ_0.

The $(t_1{:}t_2)$-gyromidpoint is *homogeneous* in the sense that it depends on the ratio $t_1{:}t_2$ of the masses t_1 and t_2, as we see from (10.124). Since it is the ratio $t_1{:}t_2$ that is of interest, we call $(t_1{:}t_2)$ the *homogeneous gyrobarycentric coordinates* of $\mathbf{a}_{12}$ relative to the set $A = \{\mathbf{a}_1, \mathbf{a}_2\}$, Fig. 10.14. Under the normalization condition $t_1 + t_2 = 1$, the homogeneous gyrobarycentric coordinates $(t_1{:}t_2)$ of $\mathbf{a}_{12}$ relative to the set A are called *gyrobarycentric coordinates.* Their classical counterpart, known as barycentric coordinates [Yiu (2000)] (also known as trilinear coordinates, [Weisstein (2003)]), were first conceived by Möbius in 1827 [Mumford, Series and Wright (2002)], where various aspects of hyperbolic geometry are attractively presented.

When $t = t_1 = t_2$ the $(t_1{:}t_2)$-gyromidpoint of $\mathbf{a}_1$ and $\mathbf{a}_2$ reduces to the gyromidpoint $\mathbf{m}_{\mathbf{a}_1\mathbf{a}_2}$, (10.33), as we see from (10.124) and (10.27),

$$\mathbf{m}_{\mathbf{a}_1\mathbf{a}_2} = \mathbf{m}(\mathbf{a}_1, \mathbf{a}_2; t{:}t) = \frac{\gamma_{\mathbf{a}_1}\mathbf{a}_1 + \gamma_{\mathbf{a}_2}\mathbf{a}_2}{\gamma_{\mathbf{a}_1} + \gamma_{\mathbf{a}_2}} \tag{10.139}$$

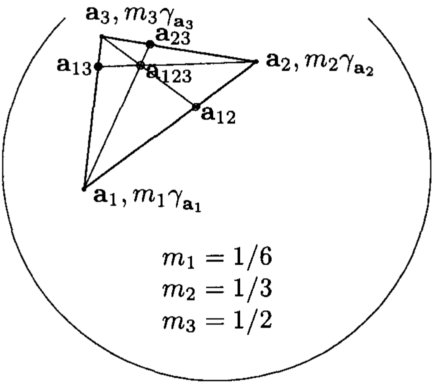


Fig. 10.15 The Hyperbolic Theorem of Ceva. When the normalized relativistic masses m_1, m_2, m_3 at relativistic velocity points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in an Einstein gyrovector plane are positive, their barycenter, which in velocity space is their CM velocity, is represented by the point $\mathbf{a}_{123}$ that lies inside gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$.

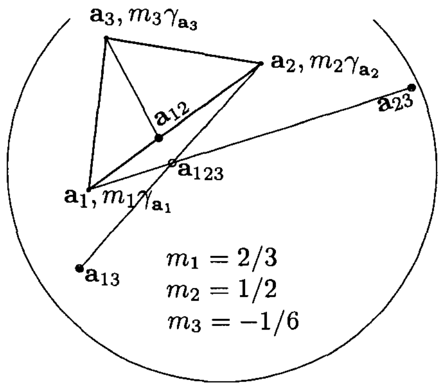


Fig. 10.16 The Hyperbolic Theorem of Ceva. When the normalized "relativistic masses" m_1, m_2, m_3 at relativistic velocity points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in an Einstein gyrovector plane are not all positive, their barycenter, which in velocity space is their CM velocity, is represented by the point $\mathbf{a}_{123}$ that lies outside of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$.

10.18 The Hyperbolic Theorems of Ceva and Menelaus

As we see from Fig. 10.14, if $\mathbf{a}_{12}$ is a point on gyrosegment $\mathbf{a}_1\mathbf{a}_2$, different from $\mathbf{a}_1$ and $\mathbf{a}_2$ in an Einstein gyrovector space $(\mathbb{V}, \oplus, \otimes)$, there are positive "relativistic masses" m_1 and m_2 (the t_1 and t_2 in Fig. 10.14) such that, (10.124),

$$(m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2})\mathbf{a}_{12} = m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 \tag{10.140}$$

and, (10.137),

$$\frac{m_1}{m_2} = \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|} \tag{10.141}$$

The masses can be normalized by the condition $m_1+m_2 = 1$. If normalized, the masses m_1 and m_2 are determined uniquely by the points $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_{12}$.

Interestingly,

$$\gamma_{\mathbf{a}_{12}} = \frac{m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2}}{m_1\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}} + m_2\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}} \tag{10.142}$$

as we see from (10.138) and (10.133) with t replaced by m. Hence, (10.142) can be written as

$$\gamma_{\mathbf{a}_{12}} = \frac{m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2}}{m_{12}} \tag{10.143}$$

where the mass m_{12} in (10.143) is defined by the equation

$$m_{12} = m_1\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}} + m_2\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}} \tag{10.144}$$

Clearly, the mass m_{12} is the sum of the two relativistic masses m_1 and m_2 relativistically corrected relative to their CM velocity frame $\Sigma_{\mathbf{a}_{12}}$.

Following (10.143) – (10.144), Identities (10.140) and (10.142) can be written, respectively, in the symmetric form

$$m_{12}\gamma_{\mathbf{a}_{12}}\mathbf{a}_{12} = m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 \tag{10.145}$$

and

$$m_{12}\gamma_{\mathbf{a}_{12}} = m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2} \tag{10.146}$$

thus revealing their relativistic significance as relativistically corrected masses and their relativistic momenta.

The Relativistic Mass Identities (10.142) – (10.146) possess hyperbolic geometric significance as well. Identities (10.140) – (10.141) will prove useful in the proof of the Hyperbolic Theorem of Ceva 10.6, and the Relativistic Mass Identities (10.142) – (10.145) will prove useful in the proof of the Hyperbolic Theorem of Menelaus 10.7.

If "masses" are allowed to be negative then Identity (10.141) must be modified (i) either by imposing an absolute value sign on m_1/m_2,

$$\left|\frac{m_1}{m_2}\right| = \frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|} \tag{10.147}$$

(ii) or by incorporating the ambiguous sign $\pm$,

$$\frac{m_1}{m_2} = \pm\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|} \tag{10.148}$$

The point $\mathbf{a}_{12}$ in (10.147) – (10.148) is different from $\mathbf{a}_1$ and $\mathbf{a}_2$. The ambiguous sign in (10.148) is "+" if $\mathbf{a}_{12}$ lies between $\mathbf{a}_1$ and $\mathbf{a}_2$, and is "-" otherwise.

Theorem 10.6 (The Hyperbolic Theorem of Ceva, in Einstein Gyrovector Spaces). *Let* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *be three non-gyrocollinear points* *in an Einstein gyrovector space* $(\mathbb{V}, \oplus, \otimes)$. *Furthermore, let* $\mathbf{a}_{123}$ *be a point in their gyroplane (that is, the intersection of the plane of* $\mathbf{a}_1$, $\mathbf{a}_2$ *and* $\mathbf{a}_3$ *in* $\mathbb{V}$ *and the ball* $\mathbb{V}_s$*), which is off the gyrolines* $\mathbf{a}_1\mathbf{a}_2$, $\mathbf{a}_2\mathbf{a}_3$ *and* $\mathbf{a}_3\mathbf{a}_1$. *If* $\mathbf{a}_1\mathbf{a}_{123}$ *meets* $\mathbf{a}_2\mathbf{a}_3$ *at* $\mathbf{a}_{23}$, *etc., as in Figs. 10.15 – 10.16, then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\,\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\,\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = 1 \tag{10.149}$$

Proof. Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{V}_s$ be non-gyrocollinear, and let m_1, m_2 and m_3 be normalized masses, $m_1 + m_2 + m_3 = 1$, such that

$$\mathbf{a}_{123} = \frac{m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 + m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3}{m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2} + m_3\gamma_{\mathbf{a}_3}} \tag{10.150}$$

for a given point $\mathbf{a}_{123}$ in the gyroplane of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, which lies inside gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ (the reader may prove that m_1, m_2 and m_3 exist and are positive and unique).

Let us define $\mathbf{a}_{23}$ by the equation

$$\mathbf{a}_{23} = \frac{m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 + m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3}{m_2\gamma_{\mathbf{a}_2} + m_3\gamma_{\mathbf{a}_3}} \tag{10.151}$$

Then, (i) $\mathbf{a}_{23}$ lies on $\mathbf{a}_2\mathbf{a}_3$ and (ii) $\mathbf{a}_{123}$ lies on $\mathbf{a}_1\mathbf{a}_{23}$ by (10.150) – (10.151). Hence, (10.151) defines $\mathbf{a}_{23}$ correctly, Figs. 10.15 – 10.16, and by (10.151) we have for $\mathbf{a}_{23}$

$$\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|} = \frac{m_3}{m_2} \tag{10.152}$$

as we see from (10.140) – (10.141).

Similarly, for $\mathbf{a}_{12}$ we have

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|} = \frac{m_2}{m_1} \tag{10.153}$$

and for $\mathbf{a}_{13}$ we have

$$\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = \frac{m_1}{m_3} \tag{10.154}$$

Finally, by multiplying (10.152), (10.153), (10.154) we get the result (10.149) (in full analogy with the classical result; see [Hausner (1998), pp. 91 – 92]). □

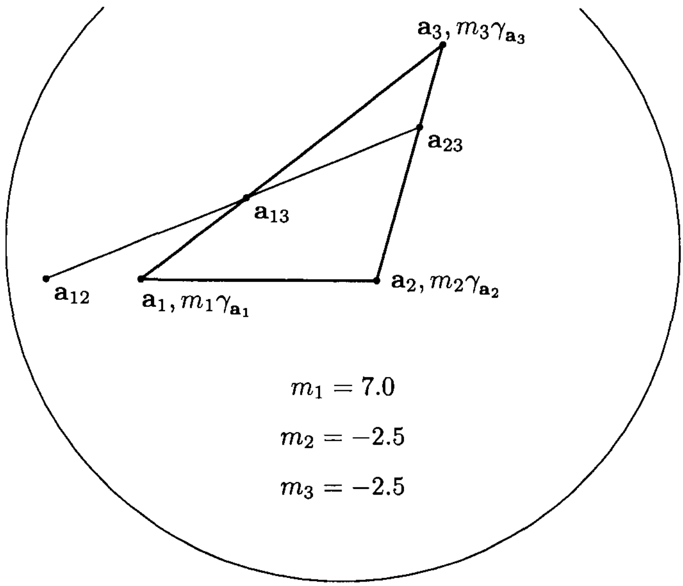


Fig. 10.17 The Hyperbolic Theorem of Menelaus by the Einstein relativistic mass.

With appropriate care one may remove the condition that point $\mathbf{a}_{123}$ must lie inside gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$, allowing "negative masses", as shown in Fig. 10.16, where point $\mathbf{a}_{13}$ lies on the gyroplane of points $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, outside of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$.

Theorem 10.7 (The Hyperbolic Theorem of Menelaus, in Einstein Gyrovector Spaces). *Let $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ be three non-gyrocollinear points in an Einstein gyrovector space $(\mathbb{V}, \oplus, \otimes)$. If a gyroline meets the sides of gyrotriangle $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_3$ at points $\mathbf{a}_{12}$, $\mathbf{a}_{13}$, $\mathbf{a}_{23}$, as in Fig. 10.17, then*

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|}\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|}\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = 1 \tag{10.155}$$

Proof. Following (10.145) and Fig. 10.17 we write

$$m_{12}\gamma_{\mathbf{a}_{12}}\mathbf{a}_{12} = m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 \tag{10.156}$$

$$m_{23}\gamma_{\mathbf{a}_{23}}\mathbf{a}_{23} = m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 + m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3 \tag{10.157}$$

for some real numbers m_1, m_2, m_3, normalized by the conditions $m_1+m_2 =$

$m_2 + m_3 = 1$, where

$$\begin{aligned} m_{12}\gamma_{\mathbf{a}_{12}} &= m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2} \\ m_{23}\gamma_{\mathbf{a}_{23}} &= m_2\gamma_{\mathbf{a}_2} + m_3\gamma_{\mathbf{a}_3} \end{aligned} \tag{10.158}$$

Eliminating $\mathbf{a}_2$ between (10.156) and (10.157) we obtain

$$m_1m_{12}\gamma_{\mathbf{a}_{12}}\mathbf{a}_{12} - m_2m_{23}\gamma_{\mathbf{a}_{23}}\mathbf{a}_{23} = m_1m_2\gamma_{\mathbf{a}_1}\mathbf{a}_1 - m_2m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3 \tag{10.159}$$

In accordance with (10.140) and (10.159) we define $\mathbf{a}_{13}$, Figs. 10.15–10.16, by the equation

$$(m_1m_2\gamma_{\mathbf{a}_1} - m_2m_3\gamma_{\mathbf{a}_3})\mathbf{a}_{13} = m_1m_2\gamma_{\mathbf{a}_1}\mathbf{a}_1 - m_2m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3 \tag{10.160}$$

so that by (10.160) and (10.159),

$$\begin{aligned} (m_1m_2\gamma_{\mathbf{a}_1} - m_2m_3\gamma_{\mathbf{a}_3})\mathbf{a}_{13} &= m_1m_2\gamma_{\mathbf{a}_1}\mathbf{a}_1 - m_2m_3\gamma_{\mathbf{a}_3}\mathbf{a}_3 \\ &= m_1m_{12}\gamma_{\mathbf{a}_{12}}\mathbf{a}_{12} - m_2m_{23}\gamma_{\mathbf{a}_{23}}\mathbf{a}_{23} \end{aligned} \tag{10.161}$$

The coefficient of $\mathbf{a}_{13}$ in (10.160) and (10.161) is not zero, otherwise (10.161) would show that $\mathbf{a}_1\mathbf{a}_3$ and $\mathbf{a}_{12}\mathbf{a}_{23}$ are parallel. Hence, $\mathbf{a}_{13}$ is the unique point lying on both $\mathbf{a}_1\mathbf{a}_3$ and $\mathbf{a}_{12}\mathbf{a}_{23}$, as shown in Figs. 10.15–10.16.

Following (10.140) and (10.147), (10.156) implies

$$\frac{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{12}\|}{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{12}\|} = \left|\frac{m_2}{m_1}\right| \tag{10.162}$$

Similarly, (10.157) implies

$$\frac{\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_2\oplus\mathbf{a}_{23}\|}{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{23}\|} = \left|\frac{m_3}{m_2}\right| \tag{10.163}$$

and (10.160) implies

$$\frac{\gamma_{\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_3\oplus\mathbf{a}_{13}\|}{\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}}\|\ominus\mathbf{a}_1\oplus\mathbf{a}_{13}\|} = \left|\frac{m_1m_2}{-m_2m_3}\right| = \left|-\frac{m_1}{m_3}\right| \tag{10.164}$$

Finally, by multiplying (10.162), (10.163), (10.164) we get the result (10.155). □

With appropriate care one can remove the absolute value signs in (10.162)–(10.164) by employing (10.148) instead of (10.147). The right-hand side of (10.155) will then become -1 instead of 1, thus restoring the classical case.

Einstein's interest in the (Euclidean) Theorem of Menelaus is described in [Luchins and Luchins (1990)]. Unfortunately, he did not know that his relativistic mass can capture remarkable analogies between Euclidean and hyperbolic geometry that include the Hyperbolic Theorem of Menelaus as a special case. A most important goal of analytic hyperbolic geometry is, indeed, to extend Einstein's unfinished symphony.

10.19 Relativistic Two-Particle Systems

Let S be a system of two particles, p_1 and p_2, with respective rest masses m_1 and m_2. The particles move relativistically with velocities $\mathbf{a}_1$ and $\mathbf{a}_2$, respectively, relative to a rest frame Σ_0, $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}_c^3$, Fig. 10.18. The relativistic momenta of the particles relative to Σ_0 are $m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1$ and $m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2$. The relativistic momentum of the system S relative to Σ_0 is the sum of the momenta of its particles, that is,

$$m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2 \tag{10.165}$$

The CM velocity $\mathbf{a}_{12}$ of S is, by (10.124) and similar to (10.32),

$$\mathbf{a}_{12} = \frac{m_1\gamma_{\mathbf{a}_1}\mathbf{a}_1 + m_2\gamma_{\mathbf{a}_2}\mathbf{a}_2}{m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2}} \tag{10.166}$$

$\mathbf{a}_{12} \in \mathbb{R}_c^3$. The CM frame relativistic mass relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$ is, (10.138),

$$m_{12}\gamma_{\mathbf{a}_{12}} = m_1\gamma_{\mathbf{a}_1} + m_2\gamma_{\mathbf{a}_2} \tag{10.167}$$

and, (10.137),

$$\frac{m_1}{m_2} = \frac{\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2\|}{\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}\|\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1\|} \tag{10.168}$$

Both (10.166) and (10.167) are covariant. Hence, they satisfy the identities

$$\ominus\mathbf{x}\oplus\mathbf{a}_{12} = \frac{m_1\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_1}(\ominus\mathbf{x}\oplus\mathbf{a}_1) + m_2\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_2}(\ominus\mathbf{x}\oplus\mathbf{a}_2)}{m_1\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_1} + m_2\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_2}} \tag{10.169}$$

and

$$m_{12}\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_{12}} = m_1\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_1} + m_2\gamma_{\ominus\mathbf{x}\oplus\mathbf{a}_2} \tag{10.170}$$

for all $\mathbf{x} \in \mathbb{R}_c^3$, as we see from (10.128) and (10.131). Identities (10.169) and (10.170) demonstrate that both $\mathbf{a}_{12}$ and m_{12} are independent of the choice of origin for their Einstein gyrovector space $(\mathbb{R}_c^3, \oplus, \otimes)$.

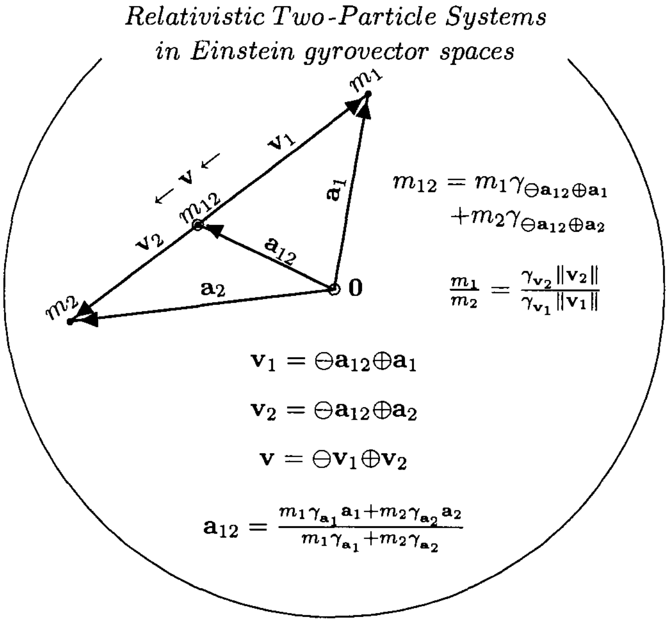


Fig. 10.18 A relativistic two-particle system consisting of two particles with rest masses m_1 and m_2 moving with respective velocities $\mathbf{a}_1$ and $\mathbf{a}_2$ relative to an inertial rest frame Σ_0 is shown in an Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. The velocity relative to Σ_0 of the relativistic center of momentum (CM) frame of the two-particle system is $\mathbf{a}_{12}$, and the velocities of the particles with masses m_1 and m_2 relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$ are, respectively, $\mathbf{v}_1$ and $\mathbf{v}_2$. The mass m_{12} of a fictitious particle at rest relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$ is equal to the sum of the masses m_1 and m_2, each of which is relativistically corrected in the CM frame $\Sigma_{\mathbf{a}_{12}}$.

The three masses of the two particles and their system S, relativistically corrected relative to the rest frame Σ_0 are, respectively, $m_1\gamma_{\mathbf{a}_1}$, $m_2\gamma_{\mathbf{a}_2}$ and $m_{12}\gamma_{\mathbf{a}_{12}}$ as we see from (10.167). The resulting three rest masses, m_1, m_2 and m_{12}, are shown in Fig. 10.18 along with their respective velocities, $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_{12}$, relative to Σ_0.

Particle p_1, with rest mass m_1 in the frame $\Sigma_{\mathbf{a}_1}$, moves with velocity $\mathbf{a}_1$ relative to Σ_0, and velocity $\mathbf{v}_1 = \ominus\mathbf{a}_{12}\oplus\mathbf{a}_1$ relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$, as shown in Fig. 10.18. Hence, its mass in the CM frame is

$$m_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1} \tag{10.171}$$

Similarly, particle p_2, with rest mass m_2 in the frame $\Sigma_{\mathbf{a}_2}$, moves with velocity $\mathbf{a}_2$ relative to Σ_0, and velocity $\mathbf{v}_2 = \ominus\mathbf{a}_{12}\oplus\mathbf{a}_2$ relative to the CM frame $\Sigma_{\mathbf{a}_{12}}$, as shown in Fig. 10.18. Hence, its mass in the CM frame is

$$m_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2} \tag{10.172}$$

To understand the meaning of the rest mass m_{12} in (10.167) we substitute $\mathbf{x} = \mathbf{a}_{12}$ in (10.170), obtaining

$$\begin{aligned} m_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1} + m_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2} &= m_{12}\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_{12}} \\ &= m_{12}\gamma_{\mathbf{0}} \\ &= m_{12} \end{aligned} \tag{10.173}$$

It follows from (10.171)–(10.173) that m_{12} is the sum of the masses m_1 and m_2, each of which is relativistically corrected in the CM frame $\Sigma_{\mathbf{a}_{12}}$. As such m_{12} is the rest mass of the system S in the CM frame.

The three substitutions, $\mathbf{x} = \mathbf{a}_1$, $\mathbf{x} = \mathbf{a}_2$, and $\mathbf{x} = \mathbf{a}_{12}$, in (10.169) give, respectively, the following three identities

$$\ominus\mathbf{a}_1\oplus\mathbf{a}_{12} = \frac{m_2\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_2}(\ominus\mathbf{a}_1\oplus\mathbf{a}_2)}{m_1 + m_2\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_2}} \tag{10.174}$$

$$\begin{aligned} \ominus\mathbf{a}_2\oplus\mathbf{a}_{12} &= \frac{m_1\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_1}(\ominus\mathbf{a}_2\oplus\mathbf{a}_1)}{m_1\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_1} + m_2} \\ &= \ominus\frac{m_1\gamma_{\ominus\mathbf{a}_1\oplus\mathbf{a}_2}\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{a}_2)}{m_1\gamma_{\ominus\mathbf{a}_2\oplus\mathbf{a}_1} + m_2} \end{aligned} \tag{10.175}$$

and

$$\mathbf{0} = m_1\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1) + m_2\gamma_{\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2}(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2) \tag{10.176}$$

Identity (10.176) demonstrates that the CM frame $\Sigma_{\mathbf{a}_{12}}$ is the vanishing momentum frame.

Let us introduce the notation suggested by Fig. 10.18,

$$\begin{aligned} \mathbf{v}_1 &= \ominus\mathbf{a}_{12}\oplus\mathbf{a}_1 \\ \mathbf{v}_2 &= \ominus\mathbf{a}_{12}\oplus\mathbf{a}_2 \\ \mathbf{v} &= \ominus\mathbf{v}_1\oplus\mathbf{v}_2 \end{aligned} \tag{10.177}$$

It follows from (10.177), the gyroautomorphic inverse property, and the gyrotranslation theorem 3.13 that

$$\begin{aligned}\mathbf{v} &= \ominus\mathbf{v}_1\oplus\mathbf{v}_2 \\ &= \ominus(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1)\oplus(\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2) \\ &= (\mathbf{a}_{12}\ominus\mathbf{a}_1)\ominus(\mathbf{a}_{12}\ominus\mathbf{a}_2) \\ &= \mathrm{gyr}[\mathbf{a}_{12},\ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{a}_2)\end{aligned} \tag{10.178}$$

Hence, by gyroautomorphism inversion, we have

$$\ominus\mathbf{a}_1\oplus\mathbf{a}_2 = \mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}]\mathbf{v} \tag{10.179}$$

The velocity points $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_{12}$ are gyrocollinear in their gyrovector space, Fig. 10.18. Hence, by the gyration transitive law, Theorem 6.29, we have

$$\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}] = \mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_{12}] \tag{10.180}$$

It follows from (10.179), by gyrocommutativity, the gyroautomorphic inverse property, and (10.180), that

$$\begin{aligned}\ominus\mathbf{a}_2\oplus\mathbf{a}_1 &= \mathrm{gyr}[\ominus\mathbf{a}_2,\mathbf{a}_1](\mathbf{a}_1\ominus\mathbf{a}_2) \\ &= \ominus\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{a}_2) \\ &= \ominus\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_1]\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}]\mathbf{v} \\ &= \ominus\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_{12}]\mathbf{v}\end{aligned} \tag{10.181}$$

We thus uncover from (10.179) and (10.181) the two related identities

$$\begin{aligned}\ominus\mathbf{a}_1\oplus\mathbf{a}_2 &= \mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}]\mathbf{v} \\ \ominus\mathbf{a}_2\oplus\mathbf{a}_1 &= \ominus\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_{12}]\mathbf{v}\end{aligned} \tag{10.182}$$

By the gyrocommutative law and the gyroautomorphic inverse property we have

$$\begin{aligned}\ominus\mathbf{a}_1\oplus\mathbf{a}_{12} &= \mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}](\mathbf{a}_{12}\ominus\mathbf{a}_1) \\ &= \ominus\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_{12}](\ominus\mathbf{a}_{12}\oplus\mathbf{a}_1)\end{aligned} \tag{10.183}$$

and similarly,

$$\ominus\mathbf{a}_2\oplus\mathbf{a}_{12} = \ominus\mathrm{gyr}[\mathbf{a}_2,\ominus\mathbf{a}_{12}](\ominus\mathbf{a}_{12}\oplus\mathbf{a}_2) \tag{10.184}$$

Substituting the notation (10.177) in (10.168) we have

$$\frac{m_1}{m_2} = \frac{\gamma_{\mathbf{v}_2}\|\mathbf{v}_2\|}{\gamma_{\mathbf{v}_1}\|\mathbf{v}_1\|} \tag{10.185}$$

Substituting the notation (10.177) in (10.174)–(10.176), noting (10.177)–(10.184), we obtain respectively the following three equations

$$\ominus\text{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_{12}]\mathbf{v}_1 = \frac{m_2\gamma_{\mathbf{v}}\text{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_{12}]\mathbf{v}}{m_1 + m_2\gamma_{\mathbf{v}}} \tag{10.186}$$

$$\begin{aligned} \ominus\text{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_{12}]\mathbf{v}_2 &= \frac{m_1\gamma_{\mathbf{v}}\text{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1]\text{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_{12}]\mathbf{v}}{m_1\gamma_{\mathbf{v}} + m_2} \\ &= \ominus\frac{m_1\gamma_{\mathbf{v}}\text{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_{12}]\mathbf{v}}{m_1\gamma_{\mathbf{v}} + m_2} \end{aligned} \tag{10.187}$$

and

$$\mathbf{0} = m_1\gamma_{\mathbf{v}_1}\mathbf{v}_1 + m_2\gamma_{\mathbf{v}_2}\mathbf{v}_2 \tag{10.188}$$

The equal gyroautomorphisms on both extreme sides of each of (10.186) and (10.187) can be dropped, obtaining from (10.186)–(10.188) the following system of three equations

$$\begin{aligned} \mathbf{v}_1 &= \ominus\frac{m_2\gamma_{\mathbf{v}}}{m_1 + m_2\gamma_{\mathbf{v}}}\mathbf{v} \\ \mathbf{v}_2 &= \frac{m_1\gamma_{\mathbf{v}}}{m_1\gamma_{\mathbf{v}} + m_2}\mathbf{v} \\ m_1\gamma_{\mathbf{v}_1}\mathbf{v}_1 &= \ominus m_2\gamma_{\mathbf{v}_2}\mathbf{v}_2 \end{aligned} \tag{10.189}$$

The third equation in (10.177) and the first two equations in (10.189) imply

$$\frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}} + t}\oplus\frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}} + t^{-1}} = \mathbf{v} \tag{10.190}$$

for all $t \in \mathbb{R}^{>0}$. Interestingly, Identity (10.190) generalizes the last identity in (6.269), which corresponds to $t = 1$.

The system of three equations (10.189) describes the internal structure of a relativistic two-particle system in terms of the relative velocity $\mathbf{v}$ between the two particles and their velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to their CM frame $\Sigma_{\mathbf{v}_{12}}$, Fig. 10.18. The system (10.189) is gyrocovariant, that is, it is covariant under rotations $\tau \in SO(3)$ and left gyrotranslations of $\mathbb{R}^3_c$. Furthermore, it is fully analogous to its Newtonian counterpart [Hestenes (1999), pp. 230–231], to which it reduces in the Newtonian limit $c \to \infty$.

In the Newtonian limit, $c \to \infty$, the system (10.189) reduces to the system of three equations

$$\begin{aligned} \mathbf{v}_1 &= -\frac{m_2}{m_1+m_2}\mathbf{v} \\ \mathbf{v}_2 &= \frac{m_1}{m_1+m_2}\mathbf{v} \\ m_1\mathbf{v}_1 &= -m_2\mathbf{v}_2 \end{aligned} \tag{10.191}$$

where the third equation follows immediately from the first two equations. The first two equations in (10.191) can be found, for instance, in [Hestenes (1999), Eqs. 611a–611b, p. 231].

Substituting the first two equations of (10.191) in the third we obtain the trivial identity

$$\frac{m_1m_2}{m_1+m_2} = \frac{m_1m_2}{m_1+m_2} \tag{10.192}$$

The relativistic counterpart (10.189) of (10.191), however, gives rise to an unexpected, nontrivial identity. Substituting the first two equations of (10.189) in the third we obtain the relativistic counterpart of (10.192),

$$\frac{m_1m_2\gamma_{\mathbf{v}}\gamma_{\mathbf{v}_1}}{m_1+m_2\gamma_{\mathbf{v}}} = \frac{m_1m_2\gamma_{\mathbf{v}}\gamma_{\mathbf{v}_2}}{m_1\gamma_{\mathbf{v}}+m_2} \tag{10.193}$$

which reduces to (10.192) in the Newtonian limit $c \to \infty$, when the various gamma factors reduce to 1. Identity (10.193) uncovers the elegant identity

$$\frac{\gamma_{\mathbf{v}_1}}{m_1+m_2\gamma_{\mathbf{v}}} = \frac{\gamma_{\mathbf{v}_2}}{m_1\gamma_{\mathbf{v}}+m_2} \tag{10.194}$$

which, unlike (10.192), is not trivial. Identity (10.194) is valid for any $m_1, m_2 \in \mathbb{R}^{>0}$, with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}$ linked to each other by the third equation in (10.177) and by (10.188).

Identity (10.194) implies, for instance, that if $m_1 = m_2$ then $\gamma_{\mathbf{v}_1} = \gamma_{\mathbf{v}_2}$. Indeed, it is clear from Fig. 10.18 that if $m_1 = m_2$ then $\mathbf{v}_1 = -\mathbf{v}_2$ so that $\gamma_{\mathbf{v}_1} = \gamma_{\mathbf{v}_2}$ as expected. Contrasting the interesting, unexpected identity in (10.194) and (10.193) with its trivial classical mechanics counterpart (10.192), the rich structure of gyrogeometry and its application in relativistic mechanics is demonstrated once again.

10.20 The Covariant Relativistic Center of Momentum (CM) Velocity

Definition 10.8 (CM Velocities And Gyrobarycentric Coordinates). *Let S be an isolated system of n noninteracting material particles the k-th particle of which has rest mass $m_k > 0$ and velocity $\mathbf{v}_k \in \mathbb{R}^3_c$ relative to a rest frame Σ_0, $k = 1, \ldots, n$.*

(1) *The relativistic CM velocity of the system S is,* (4.85),

$$\mathbf{v}_0 = \frac{\sum_{k=1}^{n} m_k \gamma_k \mathbf{v}_k}{\sum_{k=1}^{n} m_k \gamma_k} \tag{10.195}$$

relative to the rest frame Σ_0, and

(2) *the CM frame rest mass is,* (4.94),

$$m = \frac{\sum_{k=1}^{n} m_k \gamma_{\mathbf{v}_k}}{\gamma_{\mathbf{v}_0}} \tag{10.196}$$

Furthermore,

(3) *the homogeneous gyrobarycentric coordinates of the relativistic CM velocity $\mathbf{v}_0$ with respect to the set*

$$A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$$

are

$$(m_1 : m_2 : \ \ldots \ : m_n)$$

These homogeneous gyrobarycentric coordinates become gyrobarycentric coordinates when they are normalized by the condition

$$\sum_{k=1}^{n} m_k = 1 \tag{10.197}$$

(4) *The relativistic CM velocity $\mathbf{v}_0$ in an Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ is called the*

$$(m_1 : m_2 : \ \ldots \ : m_n)\text{-}gyromidpoint$$

of the set $A = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$.

The relativistic CM velocity $\mathbf{v}_0$, (10.195), is well-known. It can be found, for instance, in [Rindler (1982), Eq. 30.2, p. 88], where Rindler remarks that the relativistic CM velocity definition can be justified in terms of a space diagram such as [Rindler (1982), Fig. 13, p. 61]. Paradoxically, "In relativity, in contrast to Newtonian mechanics, the centre of mass of a system is not uniquely determined" [Rindler (1982), p. 89], as Rindler explains by example. We will now justify Def. 10.8 by gyrocovariance considerations.

Let $(\gamma_{\mathbf{v}_k}, \gamma_{\mathbf{v}_k}\mathbf{v}_k)^t$, $\mathbf{v}_k \in \mathbb{R}_c^3$, $k = 1, \ldots, n$, be n unimodular spacetime events, that is n four-velocities, and let

$$\sum_{k=1}^{n} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \tag{10.198}$$

$m_k \geq 0$, be a generic linear combination of these spacetime events, where $m \geq 0$ and $\mathbf{v}_0 \in \mathbb{R}_c^3$ are to be determined in (10.199) and (10.206) below.

Comparing ratios between lower and upper entries in (10.198) we have

$$\mathbf{v}_0 = \mathbf{v}_0(\mathbf{v}_1, \ldots, \mathbf{v}_n; m_1, \ldots, m_n) = \frac{\sum_{k=1}^{n} m_k \gamma_{\mathbf{v}_k} \mathbf{v}_k}{\sum_{k=1}^{n} m_k \gamma_{\mathbf{v}_k}} \tag{10.199}$$

so that $\mathbf{v}_0$ lies in the convex hull of the set of the points $\mathbf{v}_k$ of $\mathbb{R}_c^3$, $k = 1, \ldots, n$, in $\mathbb{R}^3$. The convex hull of a set of points in an Einstein gyrovector space (where gyrosegments are Euclidean segments) is the smallest convex set in $\mathbb{R}^3$ that includes the points. In a two-dimensional Einstein gyrovector space $(\mathbb{R}_c^2, \oplus, \otimes)$ it is a convex gyropolygon. Hence, $\mathbf{v}_0 \in \mathbb{R}_c^3$ as desired.

Applying the Lorentz boost $L(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}_c^3$, to (10.198) in two different ways, it follows from (10.122) and from the linearity of the Lorentz boost that, on the one hand

$$\begin{aligned} L(\mathbf{x}) \left\{ m \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \right\} &= \sum_{k=1}^{n} m_k L(\mathbf{x}) \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} \\ &= \sum_{k=1}^{n} m_k \begin{pmatrix} \gamma_{\mathbf{x}\oplus\mathbf{v}_k} \\ \gamma_{\mathbf{x}\oplus\mathbf{v}_k}(\mathbf{x}\oplus\mathbf{v}_k) \end{pmatrix} \\ &= \begin{pmatrix} \sum_{k=1}^{n} m_k \gamma_{\mathbf{x}\oplus\mathbf{v}_k} \\ \sum_{k=1}^{n} m_k \gamma_{\mathbf{x}\oplus\mathbf{v}_k}(\mathbf{x}\oplus\mathbf{v}_k) \end{pmatrix} \end{aligned} \tag{10.200}$$

and on the other hand,

$$\begin{aligned} L(\mathbf{x})\left\{m\begin{pmatrix}\gamma_{\mathbf{v}_0}\\ \gamma_{\mathbf{v}_0}\mathbf{v}_0\end{pmatrix}\right\} &= mL(\mathbf{x})\begin{pmatrix}\gamma_{\mathbf{v}_0}\\ \gamma_{\mathbf{v}_0}\mathbf{v}_0\end{pmatrix} \\ &= \begin{pmatrix} m\gamma_{\mathbf{x}\oplus\mathbf{v}_0}\\ m\gamma_{\mathbf{x}\oplus\mathbf{v}_0}(\mathbf{x}\oplus\mathbf{v}_0)\end{pmatrix} \end{aligned} \tag{10.201}$$

Comparing ratios between lower and upper entries of (10.200) and (10.201) we have

$$\mathbf{x}\oplus\mathbf{v}_0 = \frac{\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{v}_k}(\mathbf{x}\oplus\mathbf{v}_k)}{\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{v}_k}} \tag{10.202}$$

so that, by (10.199) and (10.202),

$$\begin{aligned} &\mathbf{x}\oplus\mathbf{v}_0(\mathbf{v}_1,\ldots,\mathbf{v}_n;m_1,\ldots,m_n) \\ &\quad= \mathbf{v}_0(\mathbf{x}\oplus\mathbf{v}_1,\ldots,\mathbf{x}\oplus\mathbf{v}_n;m_1,\ldots,m_n) \end{aligned} \tag{10.203}$$

Identity (10.203) demonstrates that the structure of $\mathbf{v}_0$ as a function of points $\mathbf{v}_k\in\mathbb{R}_c^3$, $k=1,\ldots,n$, is preserved by a left gyrotranslation of the points by any $\mathbf{x}\in\mathbb{R}_c^3$.

Similarly, the structure is preserved by rotations in the sense that if $\tau\in SO(3)$ represents a rotation of $\mathbb{R}_c^3$ then

$$\begin{aligned} &\tau\mathbf{v}_0(\mathbf{v}_1,\ldots,\mathbf{v}_n;m_1,\ldots,m_n) \\ &\quad= \mathbf{v}_0(\tau\mathbf{v}_1,\ldots,\tau\mathbf{v}_n;m_1,\ldots,m_n) \end{aligned} \tag{10.204}$$

Hence the point $\mathbf{v}_0\in\mathbb{R}_c^3$ is gyrocovariant, being covariant under the hyperbolic rigid motions of its generating points in the Einstein gyrovector space $(\mathbb{R}_c^3,\oplus,\otimes)$. It possesses, as a function of its generating points $\mathbf{v}_1,\ldots,\mathbf{v}_n\in\mathbb{R}_c^3$, gyrogeometric significance. Following Def. 3.23, and in accordance with the vision of Felix Klein in his *Erlangen Program* [Mumford, Series and Wright (2002)], $\mathbf{v}_0$ is a gyrogeometric object.

Comparing the top entries of (10.200) and (10.201) we have

$$m = \frac{\sum_{k=1}^{n} m_k\gamma_{\mathbf{x}\oplus\mathbf{v}_k}}{\gamma_{\mathbf{x}\oplus\mathbf{v}_0}} \tag{10.205}$$

But, we also have from (10.198)

$$m = \frac{\sum_{k=1}^{n} m_k\gamma_{\mathbf{v}_k}}{\gamma_{\mathbf{v}_0}} \tag{10.206}$$

implying that the positive scalar, (4.95),

$$\begin{aligned} m &= m(\mathbf{v}_1, \dots, \mathbf{v}_n; m_1, \dots, m_n) \\ &= m(\mathbf{x}\oplus\mathbf{v}_1, \dots, \mathbf{x}\oplus\mathbf{v}_n; m_1, \dots, m_n) \end{aligned} \tag{10.207}$$

in (10.205) and (10.206) is invariant under any left gyrotranslation of the points $\mathbf{v}_k \in \mathbb{R}^3_c$, $k = 1, \dots, n$. Clearly, it is also invariant under any rotation of its generating points $\mathbf{v}_k$. Being invariant under the rigid motions of the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$, the gyrovector space scalar m possesses gyrogeometric significance. It forms a scalar field for any n-tuples $(m_1, \dots, m_n)$ of rest masses.

To determine the CM velocity of the system S, that is, the velocity of a frame where the total momentum of the system S vanishes, we substitute $\mathbf{x} = \ominus\mathbf{v}_0$ in (10.202) obtaining

$$\sum_{k=1}^{n} m_k \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_k} (\ominus\mathbf{v}_0\oplus\mathbf{v}_k) = 0 \tag{10.208}$$

The resulting identity (10.208) demonstrates that the relativistic momentum vanishes in the CM frame $\Sigma_{\mathbf{v}_0}$ thus justifying the definition of the CM velocity $\mathbf{v}_0$ in (10.195). The point $\mathbf{v}_0 \in \mathbb{R}^3_c$, therefore, represents the covariant CM velocity of the system S.

Substituting $\mathbf{x} = \ominus\mathbf{v}_0$ in (10.205) we have

$$m = \sum_{k=1}^{n} m_k \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_k} \tag{10.209}$$

revealing the relativistic interpretation of the gyrogeometric scalar field m. It represents the CM frame rest mass of the system S. It is the sum of the relativistically corrected masses $m_k \gamma_{\ominus\mathbf{v}_0\oplus\mathbf{v}_k}$, $k = 1, \dots, n$, relative to the CM frame $\Sigma_{\mathbf{v}_0}$. This justifies the definition in (10.196) of the CM frame rest mass.

10.21 Barycentric Coordinates

In 1827 Möbius published a book whose title, *Der Barycentrische Calcul*, translates as *The Barycentric Calculus.* The word *barycentric* means center of gravity, but the book is entirely geometrical and, hence, called by Jeremy Gray [Gray (1993)], *Möbius's Geometrical Mechanics.* The 1827 Möbius book is best remembered for introducing a new system of coordinates, the *barycentric coordinates*.

The Möbius idea, for a triangle as an illustrative example, is to attach masses, $m_{\mathbf{a}}$, $m_{\mathbf{b}}$, $m_{\mathbf{c}}$, respectively, to three non-collinear points, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, in the Euclidean plane $\mathbb{R}^2$, and consider their center of mass CM, called *barycenter*, given by the equation

$$CM = \frac{m_{\mathbf{a}}\mathbf{a} + m_{\mathbf{b}}\mathbf{b} + m_{\mathbf{c}}\mathbf{c}}{m_{\mathbf{a}} + m_{\mathbf{b}} + m_{\mathbf{c}}} \tag{10.210}$$

Following Hocking and Young [Hocking and Young (1988), pp. 195–200], a set of $h+1$ vectors $\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h\}$ in $\mathbb{R}^n$ is *pointwise independent* if the h vectors $-\mathbf{a}_0 + \mathbf{a}_k$, $k = 1, \ldots, h$, are linearly independent.

Let $A = \{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h\}$ be a pointwise independent set of $h+1$ vectors in $\mathbb{R}^n$. Then, the real numbers $m_0, m_1, \ldots, m_h$ normalized by the condition

$$\sum_{k=0}^{h} m_k = 1 \tag{10.211}$$

are the *barycentric coordinates* of a vector $\mathbf{a} \in \mathbb{R}^n$ with respect to A if

$$\mathbf{a} = \frac{\sum_{k=0}^{h} m_k \mathbf{a}_k}{\sum_{k=0}^{h} m_k} \tag{10.212}$$

It is easy to show that the barycentric coordinates are independent of the choice of the origin of their vector space, that is,

$$-\mathbf{x} + \mathbf{a} = \frac{\sum_{k=0}^{h} m_k(-\mathbf{x} + \mathbf{a}_k)}{\sum_{k=0}^{h} m_k} \tag{10.213}$$

for all $\mathbf{x} \in \mathbb{R}^n$. The analogy that (10.213) shares with (10.202) is remarkable.

When the normalization condition (10.211) is relaxed to the weaker condition

$$\sum_{k=0}^{h} m_k \neq 0 \tag{10.214}$$

the barycentric coordinates become the so called *homogeneous barycentric coordinates*. They are homogeneous in the sense that the homogeneous barycentric coordinates $(m_0, m_1, \ldots, m_h)$ of $\mathbf{a}$ in (10.212) are equivalent to the homogeneous barycentric coordinates $(\lambda m_0, \lambda m_1, \ldots, \lambda m_h)$ for any $\lambda \neq 0$. Since in homogeneous barycentric coordinates only ratios of coordinates are relevant, the homogeneous barycentric coordinates $(m_0, m_1, \ldots, m_h)$ are also written as $(m_0 : m_1 : \ldots : m_h)$.

The set of all points in $\mathbb{R}^n$ for which the barycentric coordinates with respect to A are all positive form an open convex subset of $\mathbb{R}^n$, called the open *h-simplex* with the $h+1$ vertices $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h$. Following Hocking and Young [Hocking and Young (1988), p. 199], the h-simplex with vertices $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h$ is denoted by the symbol $\langle \mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_h \rangle$. If the positive number m_k is viewed as the mass of a massive object situated at the point $\mathbf{a}_k$, $0 \le k \le h$, the point $\mathbf{a}$ in (10.212) turns out to be the center of mass of the $h+1$ masses m_k, $0 \le k \le h$. If, furthermore, all the masses are equal, the center of mass turns out to be the *centroid* of the h-simplex. Three illustrative examples follow.

(1) The 2-simplex $\langle \mathbf{u}, \mathbf{v} \rangle$ in $\mathbb{R}^3$ is the Euclidean segment $\mathbf{uv}$ with endpoints $\mathbf{u}$ and $\mathbf{v}$ and midpoint, Fig. 10.2,

$$\mathbf{m}_{\mathbf{uv}} = \frac{\mathbf{u}+\mathbf{v}}{2} \tag{10.215}$$

The barycentric coordinates of the endpoints $\mathbf{u}$ and $\mathbf{v}$ of the segment $\mathbf{uv}$ with respect to $A = \{\mathbf{u}, \mathbf{v}\}$ are, respectively, $(1, 0)$ and $(0, 1)$ and the barycentric coordinates of the midpoint $\mathbf{m}_{\mathbf{uv}}$ of the segment are $(1/2, 1/2)$.

(2) The 3-simplex $\langle \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ in $\mathbb{R}^3$ is the Euclidean triangle $\mathbf{uvw}$ with vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and its centroid is the point

$$\mathbb{C}_{\mathbf{uvw}} = \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}}{3} \tag{10.216}$$

inside triangle $\mathbf{uvw}$ shown in Fig. 10.2. The barycentric coordinates of the vertices $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ of triangle $\mathbf{uvw}$ with respect to $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ are, respectively, $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ and the barycentric coordinates of the centroid of the triangle are $(1/3, 1/3, 1/3)$.

(3) The 4-simplex $\langle \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p} \rangle$ in $\mathbb{R}^3$ is a Euclidean tetrahedron $\mathbf{uvwp}$ with vertices $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$, and centroid at the point

$$\mathbb{C}_{\mathbf{uvwp}} = \frac{\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{p}}{4} \tag{10.217}$$

of the tetrahedron. The barycentric coordinates of the vertices $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{p}$ of the tetrahedron with respect to the set $A = \{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{p}\}$ are, respectively, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. As we see from (10.217), the barycentric coordinates of the centroid of the tetrahedron are $(1/4, 1/4, 1/4, 1/4)$.

10.22 Einsteinian Gyrobarycentric Coordinates

The concept of barycentric coordinates, studied in Sec. 10.21, can readily be extended to its gyro-counterpart in Einstein gyrovector spaces.

As we see from Def. 10.8, the gyromidpoint $\mathbf{m}_{\mathbf{u},\mathbf{v}}$ in Fig. 10.1 is the $(1:1)$-gyromidpoint of the set $\{\mathbf{u}, \mathbf{v}\}$ or, equivalently, the $(1:1:0)$-gyromidpoint of the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in the Einstein gyrovector space $(\mathbb{R}_c^3, \oplus, \otimes)$. Its gyrobarycentric coordinates with respect to the set $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbb{R}_c^3$ are, therefore, $(1/2 : 1/2)$, and its gyrobarycentric coordinates with respect to the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}_c^3$ are $(1/2:1/2:0)$.

The centroid of the Euclidean triangle $\mathbf{uvw}$ in Fig. 10.2 is the Euclidean $(1:1:1)$–midpoint of the triangle. Its barycentric coordinates with respect to the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}^3$ are, therefore, $(1/3:1/3:1/3)$.

The gyrocentroid of the gyrotriangle $\mathbf{uvw}$ in Fig. 10.3 is the $(1 : 1 : 1)$-gyromidpoint of the gyrotriangle. Its gyrobarycentric coordinates with respect to the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ in $\mathbb{R}_c^3$ are, therefore, $(1/3:1/3:1/3)$.

Finally, the $(m_1 : m_2 : m_3)$-gyromidpoint of the gyrotriangle $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ in Fig. 10.19 has gyrobarycentric coordinates $(m_1/m : m_2/m : m_3/m)$ with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbb{R}_c^3$, where $m_1, m_2, m_3 \geq 0$, and $m = m_1 + m_2 + m_3$.

The various gyromidpoints and their homogeneous gyrobarycentric coordinates in Fig. 10.19 are listed below.

By Def. 10.8, the generalized $(m_1 : m_2)$-gyromidpoint

$$\mathbf{m} = \mathbf{m}(\mathbf{v}_1, \mathbf{v}_2; m_1, m_2) = \frac{m_1\gamma_{\mathbf{v}_1}\mathbf{v}_1 + m_2\gamma_{\mathbf{v}_2}\mathbf{v}_2}{m_1\gamma_{\mathbf{v}_1} + m_2\gamma_{\mathbf{v}_2}} \tag{10.218}$$

has homogeneous gyrobarycentric coordinates $(m_1 : m_2)$ with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ or, equivalently, homogeneous gyrobarycentric coordinates $(m_1 : m_2 : 0)$ with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It represents the CM velocity of the system of two particles with rest masses m_1 and m_2, moving with respective relativistically admissible velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ relative to a rest frame Σ_0. The $(m_1 : m_3)$-gyromidpoint and $(m_2 : m_3)$-gyromidpoint in Fig. 10.19 are similar.

Under the normalization condition $m_1 + m_2 = 1$ we have some sim-

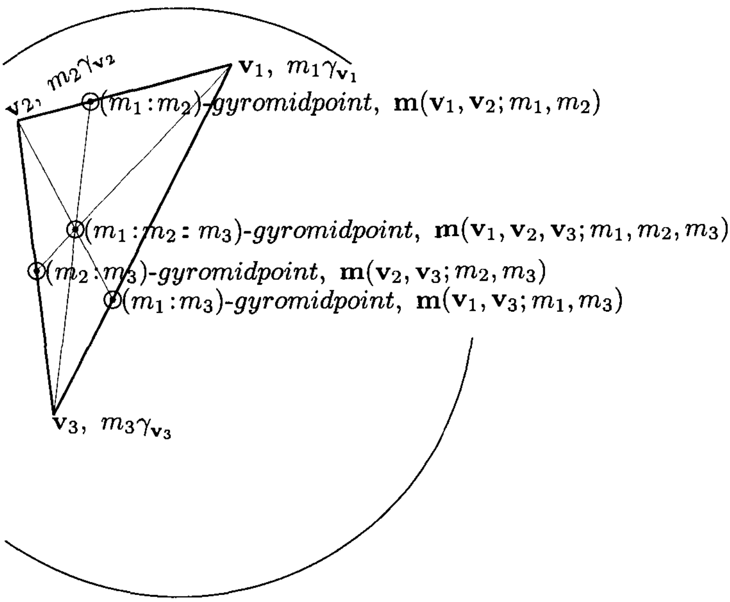


Fig. 10.19 Generalized Gyromidpoints and Gyrobarycentric Coordinates in the Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. Three particles with rest masses $m_1, m_2, m_3 \geq 0$ that move with respective velocities $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ relative to a rest frame Σ_0 are shown together with various generalized gyromidpoints and their homogeneous gyrobarycentric coordinates. The generalized gyromidpoints represent relativistic CM velocities.

plification. The $(m_1 : m_2)$-gyromidpoint (10.218) with homogeneous gyrobarycentric coordinates $(m_1 : m_2)$, $m_1, m_2 \in \mathbb{R}^{\geq 0}$, with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ becomes the $(m_1 : 1 - m_1)$-gyromidpoint

$$\mathbf{m} = \mathbf{m}(\mathbf{v}_1, \mathbf{v}_2; m_1, 1 - m_1) \tag{10.219}$$

with gyrobarycentric coordinates m_1, $1 - m_2 \in \mathbb{R}^{\geq 0}$ with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2\}$. When the gyrobarycentric coordinate m_1 in (10.219) moves along the closed unit interval $[0, 1]$, its corresponding gyromidpoint (10.219) moves along the gyrosegment $\mathbf{v}_1\mathbf{v}_2$ from $\mathbf{v}_1$ to $\mathbf{v}_2$, Fig. 10.19. The gyromidpoint (10.219) coincides, respectively, with $\mathbf{v}_1$, with $\mathbf{m}_{\mathbf{v}_1\mathbf{v}_2}$, (10.24)–(10.28), and with $\mathbf{v}_2$ when $m_1 = \mathbf{0}$, $m_1 = \frac{1}{2}$, and when $m_1 = 1$. Hence, the gyrobarycentric coordinate m_1 acts as a coordinate for points on the gyrosegment $\mathbf{v}_1\mathbf{v}_2$.

To enable the gyrobarycentric coordinate m_1 in (10.219) to act as a coordinate for points on the whole gyroline containing the gyrosegment $\mathbf{v}_1\mathbf{v}_2$

we allow m_1 to take on negative values, $m_1 \in \mathbb{R}$, subject to the condition $\|\mathbf{m}(\mathbf{v}_1, \mathbf{v}_2; m_1, 1-m_1)\| < c$ in (10.219). Of course, the interpretation as relativistic masses is lost for negative gyrobarycentric coordinates.

The generalized $(m_1\!:\!m_2\!:\!m_3)$-gyromidpoint in Fig. 10.19,

$$\mathbf{m}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3; m_1, m_2, m_3) = \frac{m_1\gamma_{\mathbf{v}_1}\mathbf{v}_1 + m_2\gamma_{\mathbf{v}_2}\mathbf{v}_2 + m_3\gamma_{\mathbf{v}_3}\mathbf{v}_3}{m_1\gamma_{\mathbf{v}_1} + m_2\gamma_{\mathbf{v}_2} + m_3\gamma_{\mathbf{v}_3}} \qquad (10.220)$$

has homogeneous gyrobarycentric coordinates $(m_1\!:\!m_2\!:\!m_3)$ with respect to the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in an Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$. It represents the CM velocity of the system of three particles with masses m_1, m_2 and m_3, moving with respective relativistically admissible velocities $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ relative to a rest frame Σ_0.

Analogously to (10.213), the generalized $(m_1 : m_2 : m_3)$-gyromidpoint (10.220) satisfies the identity

$$\begin{aligned} &\ominus\mathbf{x}\oplus\mathbf{m}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3; m_1, m_2, m_3) \\ &\quad = \mathbf{m}(\ominus\mathbf{x}\oplus\mathbf{v}_1, \ominus\mathbf{x}\oplus\mathbf{v}_2, \ominus\mathbf{x}\oplus\mathbf{v}_3; m_1, m_2, m_3) \\ &\quad = \frac{m_1\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_1}(\ominus\mathbf{x}\oplus\mathbf{v}_1) + m_2\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_2}(\ominus\mathbf{x}\oplus\mathbf{v}_2) + m_3\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_3}(\ominus\mathbf{x}\oplus\mathbf{v}_3)}{m_1\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_1} + m_2\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_2} + m_3\gamma_{\ominus\mathbf{x}\oplus\mathbf{v}_3}} \end{aligned} \qquad (10.221)$$

as we see from (10.202). Hence, like in (10.213), the generalized $(m_1\!:\!m_2\!:\!m_3)$-gyromidpoint in (10.220) is independent of the choice of its gyrovector space origin.

The gyroline connecting a generalized gyromidpoint of two vertices with the opposite vertex of a gyrotriangle is called a *generalized gyromedian.* In particular, the $(m_1\!:\!m_2\!:\!0)$-gyromedian of gyrotriangle $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ in Fig. 10.19 is the gyrosegment connecting the $(m_1\!:\!m_2\!:\!0)$-gyromidpoint of gyrotriangle $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$ with its opposite vertex $\mathbf{v}_3$. The $(m_1 : m_2 : 0)$-gyromidpoint of gyrotriangle $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$, in turn, coincides with the $(m_1\!:\!m_2)$-gyromidpoint of gyrosegment $\mathbf{v}_1\mathbf{v}_2$.

Interestingly, the three generalized gyromedians of gyrotriangle $\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3$, that is, the (i) $(m_1\!:\!m_2\!:\!0)$-gyromedian, the (ii) $(m_1\!:\!0\!:\!m_3)$-gyromedian, and the (iii) $(0 : m_2 : m_3)$-gyromedian, are concurrent, the point of concurrency being the gyrotriangle $(m_1 : m_2 : m_3)$-gyromedian as shown in Fig. 10.19.

The concurrency of the three gyrotriangle generalized gyromedians in Fig. 10.19 is not accidental. In its most general form it ensures that in calculating the relativistic CM velocity and mass of a system frame of noninteracting particles, one can replace any subsystem of particles by a single fictitious frame that possesses the CM velocity and mass of the subsystem frame.

An interesting example of gyrobarycentric coordinates that arise naturally in gyroparallelograms is presented in the following theorem.

Theorem 10.9 *The Einstein homogeneous gyrobarycentric coordinates of a vertex* $\mathbf{d}$ *of gyroparallelogram* $\mathbf{abdc}$, *Fig. 10.6, with respect to the set* $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ *of the remaining vertices are*

$$(1 + \gamma_{\mathbf{b}\ominus\mathbf{c}} : -\gamma_{\mathbf{a}\ominus\mathbf{b}} - \gamma_{\mathbf{a}\ominus\mathbf{c}} : -\gamma_{\mathbf{a}\ominus\mathbf{b}} - \gamma_{\mathbf{a}\ominus\mathbf{c}}) \tag{10.222}$$

Proof. The proof follows immediately from Identity (10.52) of Theorem 10.1, p. 379, and from Def. 10.8, p. 421, of Einstein homogeneous gyrobarycentric coordinates. □

10.23 Gyrobarycentric Coordinates for the Universe

The geometry of the universe is non-Euclidean [Penrose (1978)]. Accordingly, coordinates for the universe must be non-Euclidean. Since gyroscopic precessions are sensitive to spacetime geometry, gyroscopes in space are used to test the geometry of the universe.

Gravity Probe B is a NASA–Stanford University project led by C.W. Francis Everitt aimed at the measurement of the gyroscopic precession of gyroscopes of unprecedented accuracy in Earth orbit in order to test Einstein's general theory of relativity. NASA's Gravity Probe B (GP-B) [Everitt, Fairbank and Schiff (1969); Taub (1997)], initiated by William M. Fairbank (1917–1989) [Fairbank (1989)], is a drag-free satellite carrying gyroscopes around Earth program.

On April 20, 2004, the Gravity Probe B spacecraft was launched from Vandenberg Air Force Base in South-central California. The ultra-precise science telescope in the spacecraft must be locked onto the Gravity Probe B guide star, IM Pegasi, for fine attitude control. The need, and the efforts made, to measure the motion of the guide star, IM Pegasi, relative to quasars in the distant universe, are described in the following citation that appeared soon after the launch.

> IM PEGASI (HR 8703) is the guide star for the Gravity Probe B Mission. The motions or precessions of the gy-

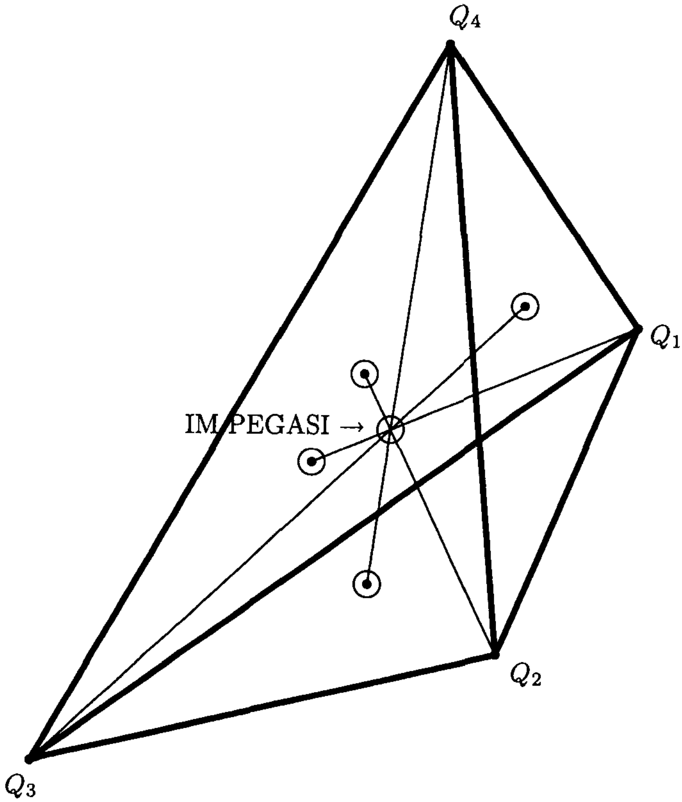


Fig. 10.20 Gyrobarycentric Coordinates for Relative Velocities in the Universe. Selecting a rest frame, the velocities Q_k, $k = 1, 2, 3, 4$, of four distant quasars relative to the rest frame form the four vertices of a gyrotetrahedron in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ of all relativistically admissible velocities. The gyrobarycentric coordinates of the GP-B guide star, IM PEGASI, or any object in the universe, can be calculated covariantly relative to the gyrotetrahedron $Q_1Q_2Q_3Q_4$.

roscopes are measured with respect to this star. The star is, however, not stationary in the universe but moves with other stars around the galactic center. This motion needs to be known in order to correct the precession measurements of the gyroscopes. The motion is measured relative to quasars in the distant universe. To measure this motion we use radio astronomical technique of very-long-baseline interferometry (VLBI) which is at present the most accurate technique to measure such motions.

The Harvard-Smithsonian Center for Astrophysical and York University, with contributions from the Observatoire de Paris, have been determining the motion of the star from observations spanning more than a decade. This will be the most accurate motion determination of a star ever made (http://www.yorku.ca/bartel/guidestar/).

The need to accurately measure the velocity of a guide star relative to quasars in the universe indicates the need to introduce coordinates for all velocities in the universe. Cartesian coordinates work well for Newtonian velocities and their Euclidean geometry. However, they are not suitable for the hyperbolic geometry of Einsteinian velocities. Similarly, barycentric coordinates are well suited for vector spaces since they are origin independent (10.213). For gyrovector spaces, however, the notion of gyrobarycentric coordinates is the shoe that fits the foot. In the same way that barycentric coordinates are vector space origin independent, (10.213), gyrobarycentric coordinates are gyrovector space origin independent, (10.221).

In order to introduce gyrobarycentric coordinates for all relativistically admissible relative velocities in the universe one may select a rest frame, and four distant quasars with known velocities relative to the rest frame. The four quasar velocities relative to the rest frame, Q_1, Q_2, Q_3, Q_4, form the four vertices of a gyrotetrahedron in the Einstein gyrovector space $(\mathbb{R}^3_c, \oplus, \otimes)$ of all relativistically admissible relative velocities in the universe, Fig. 10.20. Relative to the quasar velocity gyrotetrahedron $Q_1Q_2Q_3Q_4$ in Fig. 10.20 one can calculate covariantly the gyrobarycentric coordinates of the relative velocity of any object in the universe.

10.24 The Proper Velocity Lorentz Group

Coordinate time, or observer's time, is the time t of a moving object measured by an observer at rest. Accordingly, special relativity theory is formulated in terms of coordinate time. Contrasting coordinate time, Proper time, or traveler's time, is the time τ of a moving object measured by a co-moving observer. Proper time is useful, for instance, in the understanding of the twin paradox [Hlavatý (1960)], and the mean life time of unstable moving particles like muons.

The mean lifetime of muons between creation, in the upper atmosphere, and disintegration is $2.2\mu s$ (proper lifetime) measured by their proper time. This proper time of the moving muon, measured by the muon own clock, is several orders of magnitude shorter than the time the muon is seen traveling through the atmosphere by Earth observers. Of course, there is no need to attach a co-moving observer to the moving muon. Observers at rest measure the coordinate mean lifetime of the moving muon that, owing to time dilation, is observer dependent. Each observer, however, can translate his measure of the muon coordinate mean lifetime into the muon proper mean lifetime, which is an intrinsic property of the muon and hence observer independent [Frisch and Smith (1963)].

The need to reformulate relativity physics in terms of proper time instead of coordinate time arises from time to time [Yamaleev and Osorio (2001); Montanus (1999); Hall and Anderson (1995)]. Since 1993 T.L. Gill, J. Lindesay, and W.W. Zachary have been emphasizing the need by developing a proper time formulation and studying its consequences in order to gain new insights [Lindesay and Gill (2004)].

Gill and Zachary inform in [Gill and Zachary (1997)] that their proper time formulation is related to the work of M. Wegener in [Wegener (1995)], who showed that the use of the proper time allows the construction of Galilean transformations from Lorentz transformations. Indeed, Wegener claims in [Wegener (1995)] that "proper time being invariant, the transformations of coordinates must be Galilean." Wegener, accordingly, proposes "a classical alternative to special relativity" which is experimentally slightly different from Einstein's special relativity.

Previous attempts to uncover the proper time formulation of special relativity result in pseudo Lorentz transformations that, experimentally, are not equivalent to the standard ones. In contrast, special relativity gyroformalism suggests a most natural proper time formulation that remains gyrogeometrically and, hence, experimentally equivalent to Einstein's formulation of special relativity. The passage to the proper-time Lorentz transformation group by gyroformalism considerations is natural and unique, leaving no room for ambiguities. Geometrically, this is merely the passage from one model to another model of the same hyperbolic geometry of Bolyai and Lobachevsky, which underlies special relativity.

A detailed presentation of the proper time is found in [Woodhouse (2003), Sec. 6.2]. The coordinate time t and the proper time τ of a uniformly moving object with relative coordinate velocity $\mathbf{v} \in \mathbb{R}^3_c$ are related by the equation

$$t = \gamma_{\mathbf{v}}\tau \tag{10.223}$$

Accordingly, the relative coordinate velocities $\mathbf{v} \in \mathbb{R}^3_c$ and proper velocities $\mathbf{w} \in \mathbb{R}^3$ of an object measured by its coordinate time and proper time, respectively, are related by the equations

$$\begin{aligned} \mathbf{w} &= \gamma_{\mathbf{v}}\mathbf{v} \in \mathbb{R}^3 \\ \mathbf{v} &= \beta_{\mathbf{w}}\mathbf{w} \in \mathbb{R}^3_c \end{aligned} \tag{10.224}$$

where $\gamma_{\mathbf{v}}$ is the gamma factor (3.129),

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}} \tag{10.225}$$

and $\beta_{\mathbf{w}}$ is the *beta factor* (3.159),

$$\beta_{\mathbf{w}} = \frac{1}{\sqrt{1 + \frac{\|\mathbf{w}\|^2}{c^2}}} \tag{10.226}$$

Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3_c$ be the bijective map that (10.224) suggests,

$$\phi\mathbf{w} = \beta_{\mathbf{w}}\mathbf{w} = \mathbf{v} \tag{10.227}$$

with inverse $\phi^{-1} : \mathbb{R}^3_c \to \mathbb{R}^3$,

$$\phi^{-1}\mathbf{v} = \gamma_{\mathbf{v}}\mathbf{v} = \mathbf{w} \tag{10.228}$$

Then Einstein addition $\oplus_{\text{E}}$ in $\mathbb{R}^3_c$ induces the binary operation $\oplus_{\text{U}}$ in $\mathbb{R}^3$,

$$\mathbf{w}_1 \oplus_{\text{U}} \mathbf{w}_2 = \phi^{-1}(\phi\mathbf{w}_1 \oplus_{\text{E}} \phi\mathbf{w}_2) \tag{10.229}$$

$\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$, thus uncovering the proper velocity composition law in $\mathbb{R}^3$ in terms of PV addition $\oplus_{\text{U}}$. Using software for symbolic manipulation, it can be shown that the binary operation $\oplus_{\text{U}}$, (10.229), in $\mathbb{R}^3$ is given by the equation

$$\mathbf{u} \oplus_{\text{U}} \mathbf{v} = \mathbf{u} + \mathbf{v} + \left\{ \frac{\beta_{\mathbf{u}}}{1 + \beta_{\mathbf{u}}} \frac{\mathbf{u}\cdot\mathbf{v}}{c^2} + \frac{1 - \beta_{\mathbf{v}}}{\beta_{\mathbf{v}}} \right\} \mathbf{u} \tag{10.230}$$

where $\beta_{\mathbf{v}}$ is the beta factor, satisfying the beta identity

$$\beta_{\mathbf{u} \oplus_{\text{U}} \mathbf{v}} = \frac{\beta_{\mathbf{u}}\beta_{\mathbf{v}}}{1 + \beta_{\mathbf{u}}\beta_{\mathbf{v}}\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \tag{10.231}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.

It follows from (10.229) that

$$\beta_{\mathbf{w}_1 \oplus_U \mathbf{w}_2}(\mathbf{w}_1 \oplus_U \mathbf{w}_2) = \beta_{\mathbf{w}_1}\mathbf{w}_1 \oplus_E \beta_{\mathbf{w}_2}\mathbf{w}_2 \tag{10.232}$$

and, similarly,

$$\gamma_{\mathbf{v}_1 \oplus_E \mathbf{v}_2}(\mathbf{v}_1 \oplus_E \mathbf{v}_2) = \gamma_{\mathbf{v}_1}\mathbf{v}_1 \oplus_U \gamma_{\mathbf{v}_2}\mathbf{v}_2 \tag{10.233}$$

for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3_c$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$.

Owing to the isomorphism ϕ, the groupoid $(\mathbb{R}^3, \oplus_U)$ forms a gyrocommutative gyrogroup of proper velocities, isomorphic to Einstein gyrogroup $(\mathbb{R}^3_c, \oplus_E)$ of coordinate velocities. The proper velocity gyrogroup $(\mathbb{R}^3, \oplus_U)$ turns out to be the PV gyrogroup studied in Sec. 3.8, p. 82. Having the proper velocity gyrogroup in hand, we are now in a position to realize the abstract Lorentz transformation group by a proper velocity Lorentz group.

To uncover the proper velocity Lorentz group from the abstract Lorentz group

(1) we realize the abstract gyrocommutative gyrogroup $(G, \oplus)$ by the proper velocity gyrogroup $(\mathbb{R}^3, \oplus_U)$, where PV addition (that is, the proper velocity addition) $\oplus_U$ in the Euclidean 3-space $\mathbb{R}^3$ is given by (10.230). Furthermore,
(2) we realize the abstract spacetime gyronorm (4.8), p. 91, by the positive valued proper time τ

$$\left\| \begin{pmatrix} \tau \\ \mathbf{v} \end{pmatrix} \right\| = \tau \tag{10.234}$$

$\mathbf{v} \in \mathbb{R}^3, \tau > 0$, since we seek a proper velocity Lorentz transformation that keeps the proper time invariant. Selecting any different norm would break the convention made in step (1) to replace coordinate time by proper time.

Rewriting (10.234) in space, rather than velocity, representation (4.48), p. 104, it takes the form

$$\left\| \begin{pmatrix} \tau \\ \mathbf{x} \end{pmatrix} \right\| = \tau \tag{10.235}$$

where $\mathbf{x} = \mathbf{v}\tau \in \mathbb{R}^3, \tau > 0$.

Comparing (10.234) with (4.8), p. 91, we see that our choice of the spacetime norm (10.234) determines the gyrofactor

$$\rho(\mathbf{v}) = 1 \tag{10.236}$$

for all $\mathbf{v}\in\mathbb{R}^3$. The resulting trivial gyrofactor is legitimate since it, trivially, satisfies the conditions in Def. 4.1.

(*i*) Realizing the gyrocommutative gyrogroup binary operation $\oplus$ by the proper velocity addition $\oplus_{\text{U}}$ in step (1), and
(*ii*) selecting a spacetime norm that realizes the abstract gyrofactor $\rho(\mathbf{v})$, $\mathbf{v}\in G$, by the concrete, legitimate gyrofactor $\rho(\mathbf{v}) = 1, \mathbf{v}\in\mathbb{R}^3$, in step (2),

we can now realize the boost application to spacetime, (4.15), p. 92, obtaining

$$B_p(\mathbf{u})\begin{pmatrix}\tau\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}\tau\\ \mathbf{u}\oplus_{\text{U}}\mathbf{v}\end{pmatrix} \tag{10.237}$$

Translating (10.237) from velocity to space representation (4.48), p. 104, of spacetime, and noting $\mathbf{x} = \mathbf{v}\tau$, we have

$$\begin{aligned}
B_p(\mathbf{u})\begin{pmatrix}\tau\\ \mathbf{x}\end{pmatrix} &= \begin{pmatrix}\tau\\ (\mathbf{u}\oplus_{\text{U}}\mathbf{v})\tau\end{pmatrix}\\
&= \begin{pmatrix}\tau\\ (\mathbf{u}+\mathbf{v})\tau + \left(\frac{\beta_{\mathbf{u}}}{1+\beta_{\mathbf{u}}}\frac{\mathbf{u}\cdot\mathbf{v}\tau}{c^2} + \frac{1-\beta_{\mathbf{v}}}{\beta_{\mathbf{v}}}\tau\right)\mathbf{u}\end{pmatrix}\\
&= \begin{pmatrix}\tau\\ \mathbf{x} + \frac{1}{1+\sqrt{1+\mathbf{u}^2/c^2}}\frac{\mathbf{u}\cdot\mathbf{x}}{c^2}\mathbf{u} + \sqrt{1+\mathbf{v}^2/c^2}\,\mathbf{u}\tau\end{pmatrix}\\
&= \begin{pmatrix}\tau\\ \mathbf{x} + \frac{1}{1+\sqrt{1+\mathbf{u}^2/c^2}}\frac{\mathbf{u}\cdot\mathbf{x}}{c^2}\mathbf{u} + \sqrt{\tau^2+\|\mathbf{x}\|^2/c^2}\,\mathbf{u}\end{pmatrix}\\
&= \begin{pmatrix}\tau'\\ \mathbf{x}'\end{pmatrix}
\end{aligned} \tag{10.238}$$

thus obtaining the proper velocity Lorentz boost of special relativity, which takes spacetime coordinates $(\tau, \mathbf{x})^t$ into spacetime coordinates $(\tau', \mathbf{x}')^t$. Unlike the standard, coordinate-time Lorentz boost, (10.96), the proper velocity Lorentz boost, (10.238), is nonlinear.

In order to extend the proper velocity Lorentz boost to the proper velocity Lorentz transformation we note that the group $SO(3)$ of all rotations of the Euclidean 3-space $\mathbb{R}^3$ about its origin forms a subgroup of the automorphism group $Aut(\mathbb{R}^3, \oplus_U)$ that contains all the gyroautomorphisms of the proper velocity gyrogroup $Aut(\mathbb{R}^3, \oplus_U)$. We, accordingly, realize the abstract automorphism subgroup $Aut_0(G, \oplus)$ by $Aut_0(\mathbb{R}^3, \oplus_U) = SO(3)$. This realization of the abstract Lorentz boost and the abstract automorphism subgroup of the abstract Lorentz group gives the proper velocity Lorentz group which, in space representation, takes the form

$$
\begin{aligned}
L_p(\mathbf{u}, U)\begin{pmatrix}\tau\\ \mathbf{x}\end{pmatrix} &= B_p(\mathbf{u})E(U)\begin{pmatrix}\tau\\ \mathbf{x}\end{pmatrix}\\
&= \begin{pmatrix}\tau\\ (\mathbf{u}\oplus_U U\mathbf{v})\tau\end{pmatrix}\\
&= \begin{pmatrix}\tau\\ U\mathbf{x} + \frac{1}{1+\sqrt{1+\mathbf{u}^2/c^2}}\frac{\mathbf{u}\cdot U\mathbf{x}}{c^2}\mathbf{u} + \sqrt{\tau^2 + \|\mathbf{x}\|^2/c^2}\,\mathbf{u}\end{pmatrix}
\end{aligned}
\tag{10.239}
$$

$\mathbf{u}, \mathbf{x}\in\mathbb{R}^3, U\in SO(3), \tau > 0$.

Finally, being a realization of the abstract Lorentz transformation (4.39), p. 102, the proper velocity Lorentz transformation (10.239) possesses the group composition law (4.41), p. 102,

$$
L_p(\mathbf{u}, U)L_p(\mathbf{v}, V) = L_p(\mathbf{u}\oplus_U U\mathbf{v}, \mathrm{Gyr}[\mathbf{u}, U\mathbf{v}]UV) \tag{10.240}
$$

for all $\mathbf{u}, \mathbf{v}\in\mathbb{R}^3$ and $U, V\in SO(3)$.

This composition law represents the proper velocity Lorentz group as the gyrosemidirect product of (i) a gyrogroup of boosts $B_p(\mathbf{v}), \mathbf{v}\in\mathbb{R}^3$, isomorphic to the proper velocity gyrogroup $(\mathbb{R}^3, \oplus_U)$ of relativistically admissible proper velocities, and (ii) the group $SO(3)$.

10.25 Demystifying the Proper Velocity Lorentz Group

Demystifying the proper velocity Lorentz group by an ad hoc approach, rather than the natural, gyroformalism approach in Sec. 10.24, we find in an obvious way that the proper velocity Lorentz group is equivalent to the standard, coordinate velocity Lorentz group.

Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3_c$ be two relativistically admissible coordinate velocities, and let $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{R}^3$ be their corresponding proper velocities,

$$\begin{aligned} \mathbf{w}_1 &= \gamma_{\mathbf{v}_1}\mathbf{v}_1 \\ \mathbf{w}_2 &= \gamma_{\mathbf{v}_2}\mathbf{v}_2 \end{aligned} \tag{10.241}$$

The standard Lorentz boost B_L takes the form, (10.121), p. 404,

$$B_L : \begin{pmatrix} t \\ \mathbf{v}_2 t \end{pmatrix} \longrightarrow \begin{pmatrix} t' \\ (\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2)t' \end{pmatrix} \tag{10.242}$$

so that it is equivalent to the system of two basic transformations

$$\begin{aligned} t &\longrightarrow t' \\ \mathbf{v}_2 &\longrightarrow \mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2 \end{aligned} \tag{10.243}$$

where

$$t' = \frac{\gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}}{\gamma_{\mathbf{v}_2}} t \tag{10.244}$$

Parametrizing the Lorentz boost B_L by $\mathbf{v}_1$, (10.242) and (10.244) give the standard Lorentz boost transformation, (10.96), p. 397,

$$B_L(\mathbf{v}_1) \begin{pmatrix} t \\ \mathbf{v}_2 t \end{pmatrix} = \begin{pmatrix} \frac{\gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}}{\gamma_{\mathbf{v}_2}} t \\ (\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2) \frac{\gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}}{\gamma_{\mathbf{v}_2}} t \end{pmatrix} \tag{10.245}$$

Hence, the Lorentz boost (10.245) is nothing else but an arrangement of the two basic transformations in (10.243).

Let us now rearrange the basic transformations (10.243) in a different way,

$$B_P : \begin{pmatrix} \frac{t}{\gamma_{\mathbf{v}_2}} \\ \gamma_{\mathbf{v}_2}\mathbf{v}_2 \frac{t}{\gamma_{\mathbf{v}_2}} \end{pmatrix} \longrightarrow \begin{pmatrix} \frac{t'}{\gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}} \\ \gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}(\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2) \frac{t'}{\gamma_{\mathbf{v}_1 \oplus_{\mathrm{E}} \mathbf{v}_2}} \end{pmatrix} \tag{10.246}$$

Clearly, (10.245) and (10.246) are equivalent, being just different arrangements of the basic two transformations (10.243).

Parametrizing B_P by $\gamma_{\mathbf{v}_1}\mathbf{v}_1$ we have

$$B_P(\gamma_{\mathbf{v}_1}\mathbf{v}_1)\begin{pmatrix}\frac{t}{\gamma_{\mathbf{v}_2}}\\ \gamma_{\mathbf{v}_2}\mathbf{v}_2\frac{t}{\gamma_{\mathbf{v}_2}}\end{pmatrix}=\begin{pmatrix}\frac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}}\\ \gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}(\mathbf{v}_1\oplus_E\mathbf{v}_2)\frac{t'}{\gamma_{\mathbf{v}_1}\oplus_E\gamma_{\mathbf{v}_2}}\end{pmatrix} \tag{10.247}$$

Let us now express the rearranged Lorentz boost (10.247) in terms of proper times τ and τ' and proper velocities $\mathbf{w}_1$ and $\mathbf{w}_2$.

By (10.223),

$$\tau=\frac{t}{\gamma_{\mathbf{v}_2}} \tag{10.248}$$

and, accordingly,

$$\tau'=\frac{t'}{\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}} \tag{10.249}$$

so that, by (10.244), the proper time remains invariant,

$$\tau'=\tau \tag{10.250}$$

as it should. Moreover, it follows from (10.241) and (10.233) that

$$\gamma_{\mathbf{v}_1\oplus_E\mathbf{v}_2}(\mathbf{v}_1\oplus_E\mathbf{v}_2)=\mathbf{w}_1\oplus_U\mathbf{w}_2 \tag{10.251}$$

Substituting (10.241), (10.248) – (10.251) in (10.247) we have

$$B_P(\mathbf{w}_1)\begin{pmatrix}\tau\\ \mathbf{w}_2\tau\end{pmatrix}=\begin{pmatrix}\tau\\ (\mathbf{w}_1\oplus_U\mathbf{w}_2)\tau\end{pmatrix} \tag{10.252}$$

thus recovering the proper velocity Lorentz boost (10.238).

The ad hoc approach to recover the proper velocity Lorentz boost clearly shows that both (i) the coordinate velocity Lorentz boost (that is, the standard Lorentz boost of special relativity), and (ii) the proper velocity Lorentz boost are just two different arrangements of the same two basic transformations (10.243). Hence, both the standard, coordinate velocity Lorentz group and the proper velocity Lorentz group are experimentally indistinguishable, as they should.

10.26 Exercises

(1) Verify identities (10.9) - (10.14).
Hint: The identities in (10.9) - (10.14) are verified by lengthy but straightforward algebra that one can readily calculate by using a computer software for symbolic manipulation like MATHEMATICA or MAPLE, as demonstrated in [Ungar (2001), pp. 20 - 27]. For readers who wish to verify (10.11), which is an identity between maps, we may note that the automorphism identity in (10.11) is equivalent to the vector identity

$$\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} = \mathbf{w} \tag{10.253}$$

in the ball $\mathbb{R}_c^3$, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_c^3$, which is an identity between vectors.
A similar remark applies to the identities in (10.14) as well. For instance, to verify the first identity in (10.14), which is an identity between maps, one has to verify the equivalent identity

$$\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}]\mathbf{w} = \mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} \tag{10.254}$$

in the ball $\mathbb{R}_c^3$, for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}_c^3$, which is an identity between vectors. Noting the definition of $\oplus$ and gyr in (10.3) and (10.6), one can verify (10.253) and (10.254) by straightforward algebra with the help of a computer software for symbolic manipulation.

(2) Verify the three equations in (10.47), and show algebraically that any two of the three equations in (10.47) imply the third.

(3) Explain why (10.48) - (10.49) imply that the four gyrotriangle gyrocentroids, $\mathbf{m_{abm}}$, $\mathbf{m_{bdm}}$, $\mathbf{m_{dcm}}$ and $\mathbf{m_{cam}}$, in Fig. 10.7 form a gyroparallelogram that shares its gyrocenter with its generating gyroparallelogram **abdc**.

(4) Verify the second identity in (10.60).

(5) Show that (10.68) and (10.69) follow from the condition imposed on (10.66) exemplified by (10.67).

(6) Verify Identity (10.94) (computer algebra is needed).

(7) Show that the second Identity in (6.102) of Theorem 6.36, in Einstein gyrovector spaces, is a special case of Identity (10.128).
Hint: Consider (10.128) with $t_1 = t_2 = 1$ and use the identity in (10.27) - (10.28).

(8) Show that (10.186) follows from (10.174).

Hint: Note that $\gamma_{\mathrm{gyr}[\mathbf{a}_1,\ominus\mathbf{a}_2]\mathbf{v}} = \gamma_{\mathbf{v}}$

(9) Verify the last identity in (10.175), and show that (10.187) follows from (10.175).

(10) Show that (10.188) follows from (10.176).

(11) The Einstein Three-Quarter. Prove the identity

$$\frac{3}{4}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + \frac{1}{1+\sqrt{2(1+\gamma_{\mathbf{v}})}}}\mathbf{v} \tag{10.255}$$

Hint: Employ (6.325), using (10.190).

(12) Verify the relation (10.224) between coordinate velocities $\mathbf{v}$ and proper velocities $\mathbf{w}$. This relation is traced back to a 1915 work of G.A. Schott [Schott (1915)].

(13) Show that (10.230) follows from (10.229).

(14) Show that (10.231) follows from (10.226) and (10.230).

(15) (An open problem) The gyroparallelogram addition law (10.65), Fig. 10.10, is expressible in terms of Einstein addition $\oplus$. Similarly, express the gyroparallelepiped addition law (10.66) in terms of Einstein addition.

(16) James Bradley (1693–1762) was an English astronomer most famous for the discovery of the aberration of starlight, known as *stellar aberration*, around 1728. He interpreted the stellar aberration classically by classical velocity addition or, equivalently, by the parallelogram addition law of classical velocities.

The effect of stellar aberration is a simple phenomenon in astronomical observations. Yet, attempts to understand it relativistically are not clear. Replacing classical velocity addition by relativistic velocity addition, W. Rindler studies the stellar aberration in his book *"Essential Relativity: Special, General, and Cosmological"*. Specifically, he employs the Einstein velocity sum

$$(-v,\ 0,\ 0)^t \oplus_{\mathrm{E}} (u_1,\ u_2,\ u_3)^t \tag{10.256}$$

$(u_1,\ u_2,\ u_3)$, $(-v,\ 0,\ 0) \in \mathbb{R}^3_c$, giving no explanation why the relativistic velocity sum (10.256) that he selected is preferable over the similar, but different, relativistic velocity sum

$$(u_1,\ u_2,\ u_3)^t \oplus_{\mathrm{E}} (-v,\ 0,\ 0)^t \tag{10.257}$$

Calculate the relativistic velocity sums (10.256) and (10.257) and show that the one selected by Rindler, (10.256), is simpler than the other one, (10.257).

(17) Explain the stellar aberration relativistically by a new way that analytic hyperbolic geometry offers. Rather than relativizing the classical interpretation of stellar aberration by the traditional way of replacing classical velocity addition by relativistic velocity addition, replace the parallelogram addition law of classical velocities by the gyroparallelogram addition law of relativistic velocities.

Notation And Special Symbols

$\oplus$ Gyroaddition, Gyrogroup operation.
$\ominus$ Gyrosubtraction, Inverse gyrogroup operation.
$\boxplus$ Cogyroaddition, Gyrogroup cooperation.
$\boxminus$ Cogyrosubtraction, Inverse gyrogroup cooperation.
$\oplus_{\text{E}}$ Einstein addition (of relativistically admissible coordinate velocities, and generalizations).
$\ominus_{\text{E}}$ Einstein subtraction.
$\boxplus_{\text{E}}$ Einstein coaddition.
$\boxminus_{\text{E}}$ Einstein cosubtraction.
$\oplus_{\text{M}}$ Möbius addition.
$\ominus_{\text{M}}$ Möbius subtraction.
$\boxplus_{\text{M}}$ Möbius coaddition.
$\boxminus_{\text{M}}$ Möbius cosubtraction.
$\oplus_{\text{U}}$ PV addition (of relativistically admissible proper velocities, and generalizations).
$\ominus_{\text{U}}$ PV subtraction.
$\boxplus_{\text{U}}$ PV coaddition.
$\boxminus_{\text{U}}$ PV cosubtraction.
$\otimes$ Scalar multiplication (scalar gyromultiplication) in a gyrovector space.
$\otimes_{\text{E}}$ Einstein scalar multiplication.
$\otimes_{\text{M}}$ Möbius scalar multiplication.
$\otimes_{\text{U}}$ PV scalar multiplication.
$\diamondplus$ Gyropolygonal gyroaddition, Definition 2.13.
ab A segment with distinct endpoints **a** and **b** of (i) a gyroline (gyrosegment), or (ii) a cogyroline (cogyrosegment). A gyroline (cogyroline) containing the distinct points **a** and **b**.

$\|\mathbf{ab}\|$	Length of (i) a gyrosegment (gyrolength), or (ii) a cogyrosegment (cogyrolength).
$\mathbf{abc}$	A gyrotriangle with vertices $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
a	$a = \|\mathbf{a}\|$ is the gyrolength of gyrovector $\mathbf{a}$.
a_s	$a_s = a/s$ in a gyrovector space $(\mathbb{V}_s, \oplus, \otimes)$.
Aut	An automorphism group.
Aut_0	A subgroup of an automorphism group.
CM	Center of momentum.
c	The vacuum speed of light.
gyr	Gyrator. $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ the gyration (gyroautomorphism) generated by $\mathbf{a}$ and $\mathbf{b}$.
s	Gyrovector space analogue of the vacuum speed of light c.
$\mathbf{u}\times\mathbf{v}$	Vector product of $\mathbf{u}, \mathbf{v}\in\mathbb{R}^3$ with components $(\mathbf{u}\times\mathbf{v})_1$ *etc.*
$\gamma_{\mathbf{v}}$	The gamma factor, $\gamma_{\mathbf{v}} = (1 - \|\mathbf{v}\|^2/s^2)^{1/2}$ in the ball $\mathbb{V}_s$.
$\beta_{\mathbf{v}}$	The beta factor, $\beta_{\mathbf{v}} = (1 + \|\mathbf{v}\|^2/s^2)^{1/2}$ in the ball $\mathbb{V}_s$.
$\mathbb{B}^3$	$\mathbb{B}^3 = \mathbb{R}^3_{s=1}$ is the unit ball of the Euclidean 3-space.
I	Identity automorphism.
i	$\sqrt{-1}$
$\mathbb{R}$	The real line.
$\mathbb{R}^{>0}$	The positive ray of the real line $\mathbb{R}$.
$\mathbb{R}^{\geq 0}$	The nonnegative ray of the real line $\mathbb{R}$.
$\mathbb{R}^n$	The Euclidean n-space.
$\mathbb{R}^n_s$	The s-ball of the Euclidean n-space.
$\mathbb{S}_s$	The cone $\mathbb{S}_s = \{(t, \mathbf{x})^t : t \in \mathbb{R}^{>0},\ \mathbf{x} \in \mathbb{V},\ \text{and}\ \mathbf{v} = \mathbf{x}/t \in \mathbb{V}_s\}$
$(S, +)$	A groupoid, a set S with a binary operation $+$.
$\mathbb{V}$	A real inner product space $\mathbb{V} = (\mathbb{V}, +, \cdot)$ with a binary operation $+$ and an inner product $\cdot$.
$\mathbb{V}_s$	The s-ball of the real inner product space $\mathbb{V}$.

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